Cluster-algebraic letters for 5- and 6- point QCD processes

Rigers Aliaj

DESY Theory workshop 2025

Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

Based on : R.A., G. Dian, G. Papathanasiou: to appear

From $\mathcal{N} = 4$ Super Yang Mills to real world

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- F.I. is singular when $\alpha_i \to \infty$.

Letters dictate analytic structure!

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- **Cluster Algebras** [Golden, Goncharov, Spradlin, Vergu, Volovich][Drummond, Foster, Gurdogan]

Grassmannian Cluster Algebras for (planar) $\mathcal{N} = 4$ SYM

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Massless DCI kinematics

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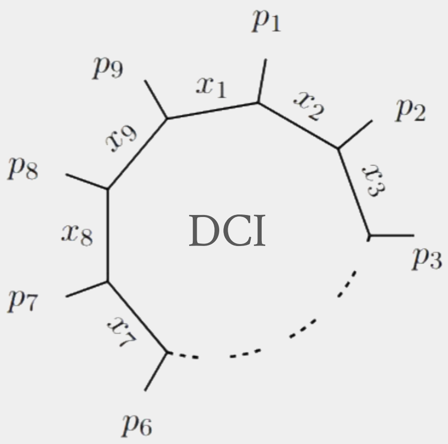

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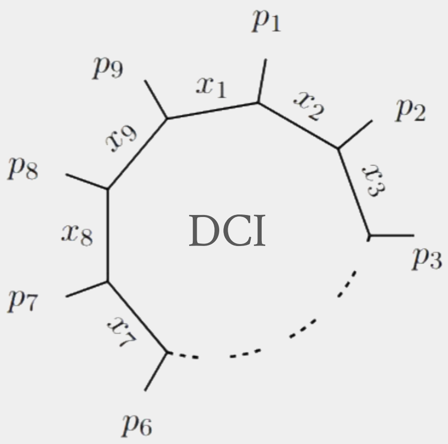

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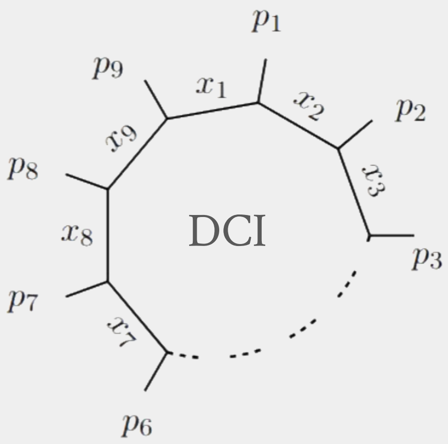

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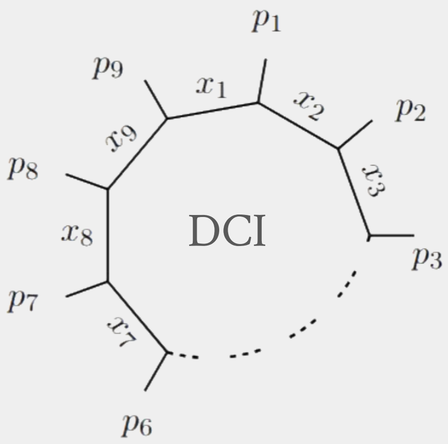

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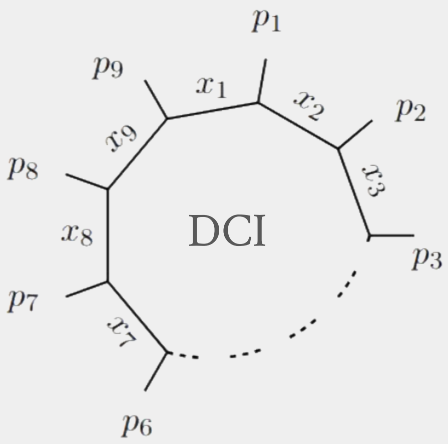

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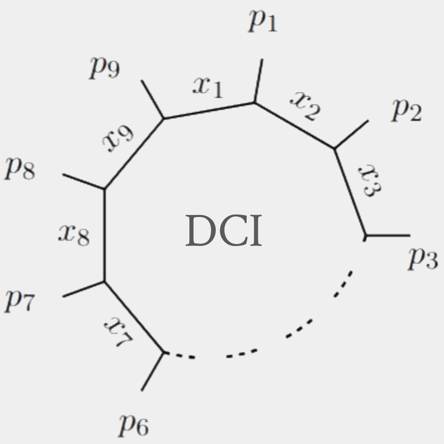

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DCI variables

$$p_i^2 = 0 \to x_{i,i+1}^2 = 0 \qquad i = 1, \ldots, 9$$

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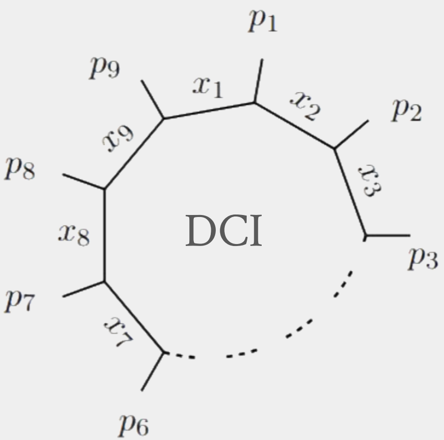

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DCI variables

What is the sub-kinematic space that does not depend on x_1 and x_8 ?

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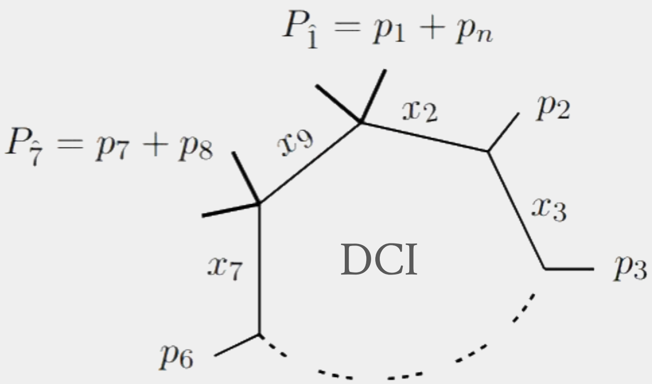
Massive DCI kinematics



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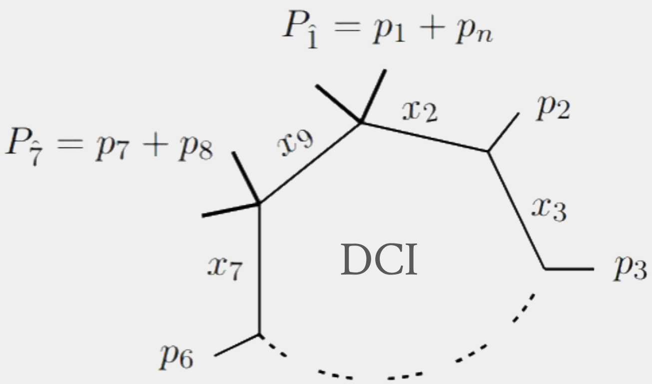

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DCI variables

Can I break DCI using the unconstrained $\boldsymbol{x_9}$?

$$p_i^2 = 0, \quad i \neq 8, 9, \quad P_{\hat{1}}^2, P_{\hat{7}}^2 \neq 0$$

$$(x_9 - x_1)^2 \neq 0$$

$$(x_9 - x_8)^2 \neq 0$$

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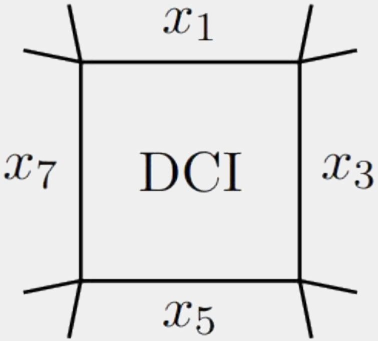
From DCI to Lorentz Invariant kinematics



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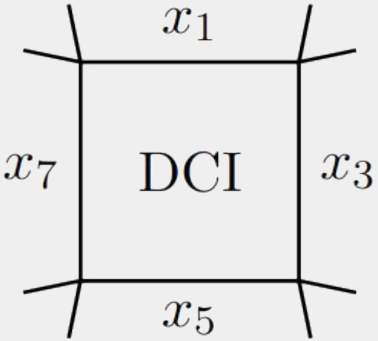

$$\xrightarrow{x_7 \to \infty}$$

[Broadhust 93'][Usyukina, Davydychev '93]

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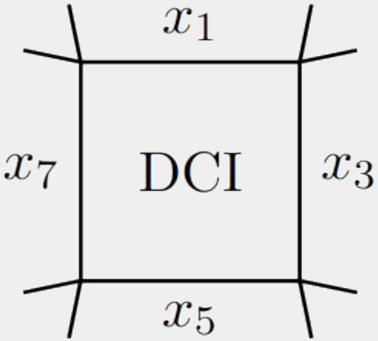

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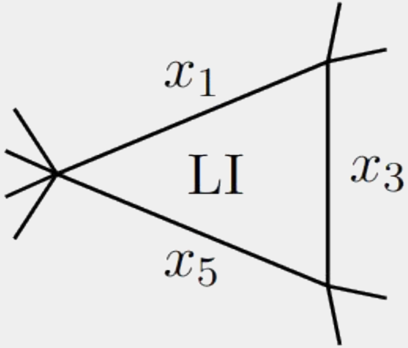

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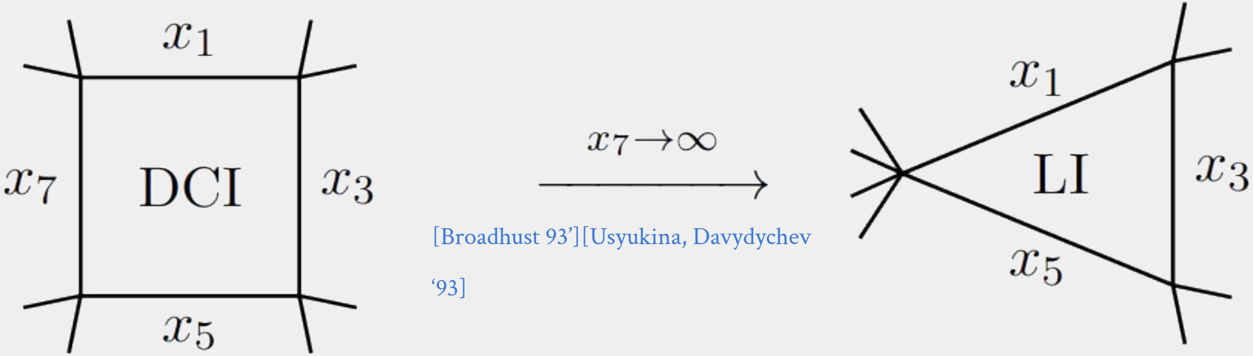

[Broadhust 93'][Usyukina, Davydychev '93]

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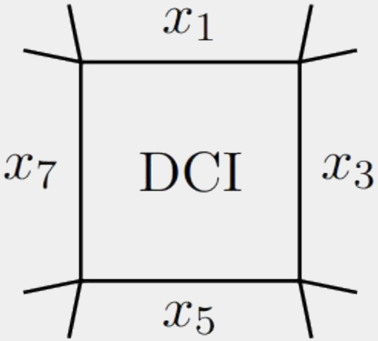

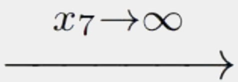

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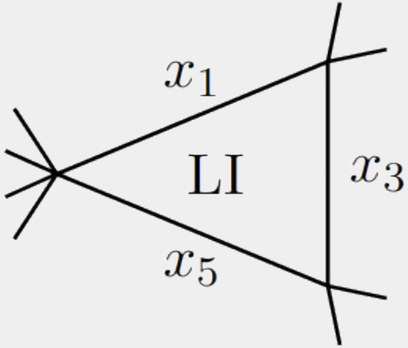

- Due to the adjacent massive legs, the dual variables are of the 4 mass DCI box are not constrained!
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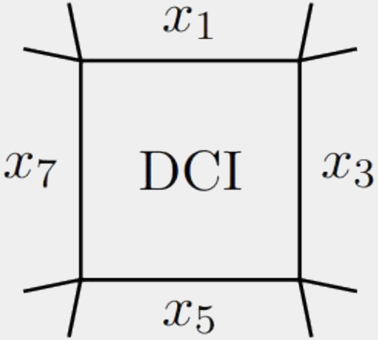

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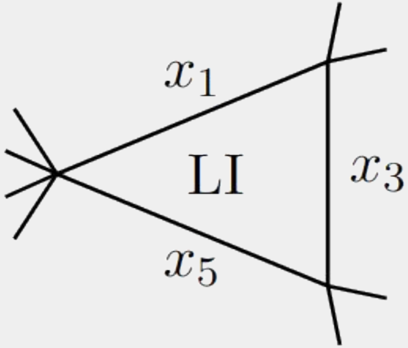

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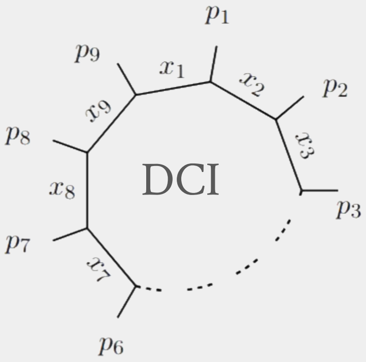
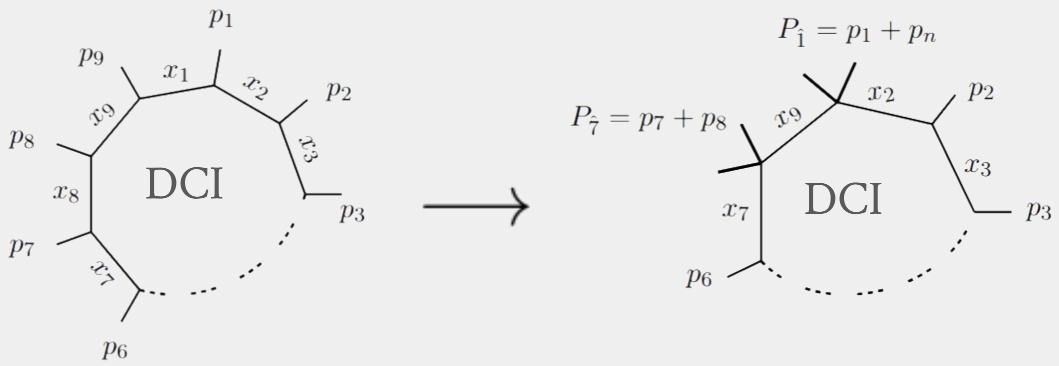
- Naturally the corresponds letters match!

9 particle alphabet reduction



9 point massless DCI

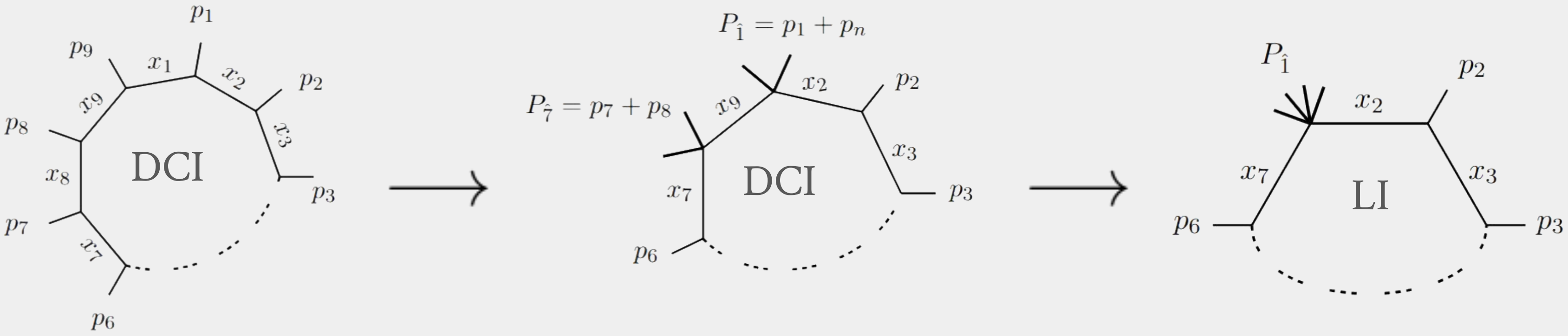
9 particle alphabet reduction



9 point massless
DCI

7 point-2 mass
DCI

9 particle alphabet reduction



9 point massless
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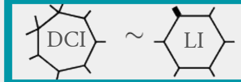
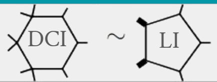
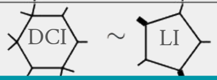
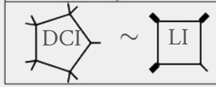
7 point-2 mass
DCI

6 point-1 mass
LI

Results

Graph	Variables	# Rationalised Letters	# Algebraic Letters
	12	3078	2349
-	11	1050	692
	10	692	395
	8	178	68
	8	166	46
	8	192	81
	6	72	46
	6	52	19
	6	36	0
	4	12	8

6 point-1 mass

5 point-2 mass

Results

6 point 1 mass

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$$\boxed{t_\pm = \frac{P(s) \pm Q(s)\epsilon_{2346}}{Tr_\pm(2346)}}$$

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  ✓ We predict 12 new letters (relevant at L=3?)

Conclusion

- ✓ Provide the first genuinely new predictions of Cluster algebras for QCD amplitudes.
- ✓ Predictions involve 6 point massive kinematics, not yet analyzed by Feynman Integral techniques.
- ✓ Unveil novel letter structure only observed in F.I. with massive internal propagators.
- ✓ Correctly predict all amplitude letters for 6 point massless scattering as well as new letters.
- ? Can we retrieve the missing letters from $Gr(4,9)$
- ? Can the letters be embedded in known Cluster Algebras? (Cluster Adjacency)
- ? Is there an underlying geometric object similar to the Amplituhedron for QCD?

Thank you!