The Four-Point Non-Planar Integrand in $\mathcal{N} = 4$ Super Yang–Mills Theory at Five Loops

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Upcoming work together with Till Bargheer

Four-Point Correlator

$$\mathcal{O}(x,y) = \mathrm{tr}(y\cdot\phi(x))^2$$

Only consider integrand

$$G_4 = \langle \mathcal{O}(x_1,y_1)\mathcal{O}(x_2,y_2)\mathcal{O}(x_3,y_3)\mathcal{O}(x_4,y_4)\rangle$$

What has been done?

- Found up to 12 loops in the planar theory [Bourjaily, He, Shi, Tang '25]
- Non-planar integrand at 4 loops [Fleury, Pereira '19]

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Strategy

Constrain integrand as much as possible → Fix remaining freedom with twistor calculation

Superconformal Constraints

Lagrangian insertion method:

Integrand: $G_4^{(\ell)}(x_i, y_i) = \langle \mathcal{O}(x_1, y_1) \ldots \mathcal{O}(x_4, y_4) \, \mathcal{L}(x_5) \ldots \mathcal{L}(x_{4+\ell}) \rangle_{\text{tree}}$

$$G_4 = \sum_{\ell \geq 0} a^\ell \int \frac{d^4 x_5 \ldots d^4 x_{4+\ell}}{(-1)^\ell \, \ell!} \, G_4^{(\ell)}(x_i, y_i)$$

Same supermultiplet: $\mathcal{T}(x, y, \theta) = \mathcal{O}(x, y) + \cdots + \theta^4 \mathcal{L}(x)$

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Partial non-renormalization theorem: [Eden, Petkou, Schubert, Sokatchev '00]

$$G_4^{(\ell>0)}(x_i, y_i) \sim R_4(x_i, y_i) \times F(x_i)$$

Unique 4-point superconformal invariant

Only depends on x_i
$i = 1, \ldots, 4 + \ell$

f-Graphs: Construction

ℓ	$\mathcal{N}$
1	1
2	1
3	4
4	32
5	930
6	$189\,341$

Properties of $F^{(\ell)}(x_i)$:

- At most simple poles in x_{ij}^2
- Conformal weight of $+4$ in each point
- Hidden $S_{4+\ell}$ symmetry

[Eden, Heslop, Korchemsky, Sokatchev '12]

Example ($\ell = 1$):

$$\frac{1}{x_{12}^2 x_{13}^2 x_{14}^2 x_{15}^2 x_{23}^2 x_{24}^2 x_{25}^2 x_{34}^2 x_{35}^2 x_{45}^2}$$

Example ($\ell = 2$):

$$\frac{1}{x_{12}^2 x_{13}^2 x_{14}^2 x_{15}^2 x_{23}^2 x_{24}^2 x_{26}^2 x_{35}^2 x_{36}^2 x_{45}^2 x_{46}^2 x_{56}^2}$$

f-Graphs: Construction

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1	1
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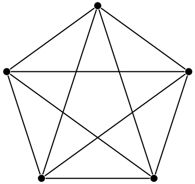
$$F^{(\ell)}(x_i) = \sum_{k=1}^{\mathcal{N}_\ell} c_k^{(\ell)} f_k^{(\ell)}$$



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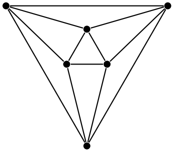

f-Graphs: Genus Expansion

ℓ	$\mathcal{N}$	genus 0	genus 1	genus 2
1	1	–	1	–
2	1	1	–	–
3	4	1	3	–
4	32	3	29	–
5	930	7	833	90
6	189 341	36	...	...

$$F^{(\ell)}(x_i) = \sum_{g\geq 0} \frac{1}{N_c^{2g}} \sum_{k=1}^{\mathcal{N}_\ell} c_k^{(g,\ell)} f_k^{(\ell)}$$

The $(1/N_c^{2g})$-correction to $G_4^{(\ell)}$ only depend on f-graphs up to genus g:

$$c_k^{(g,\ell)} = 0 \quad \text{if} \quad \mathrm{genus}(f_k^{(\ell)}) > g$$

f-Graphs: Gram Identities

ℓ	$\mathcal{N}$	genus 0	genus 1	genus 2	Gram
1	1	–	1	–	–
2	1	1	–	–	–
3	4	1	3	–	1
4	32	3	29	–	4
5	930	7	833	90	208
6	$189\,341$	36	$\ldots$	$\ldots$	$\ldots$

f-graphs are not independent of each other, they satisfy **conformal Gram identities**.

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Further Constraints

Planar integrand: [Bourjaily, Heslop, Tran '16] [He, Shi, Tang, Zhang '24] [Bourjaily, He, Shi, Tang '25]

- Euclidean/Minkowskian OPE constraints (triangle rule/double-triangle rule)
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$\rightarrow$ Implementable graph by graph! Fixed up to 12 loops!

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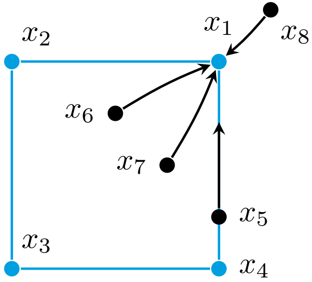
Full integrand: [Eden, Heslop, Korchemsky, Sokatchev '12]

$$\log\left(1+2\sum_{\ell\geq 1} a^\ell \int d^4x_5 \dots d^4x_{4+\ell}\, \mathcal{F}^{(\ell)}(x_i)\right)\Bigg|_{a^\ell} \sim (\log)^\ell$$



- No non-planar corrections up to 3 loops
- For $\ell = 4$: $29\,(+3) \longrightarrow 4\,(+3)$

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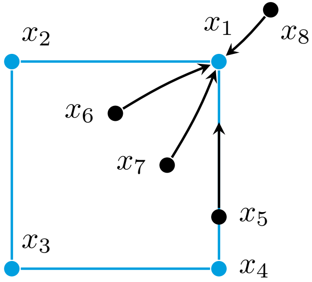

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- For $\ell = 4$: $29\,(+3) \longrightarrow 4\,(+3)$
- For $\ell = 5$: $722\,(+208) \longrightarrow 23\,(+208)$

Perturbative Twistor Computation

Twistor reformulation of $\mathcal{N} = 4$ SYM: [Boels, Mason, Skinner '06]

- Twistor Feynman rules for correlators of the stress-tensor multiplet
  [Chicherin, Doobary, Eden, Heslop, Korchemsky, Mason, Sokatchev '15]
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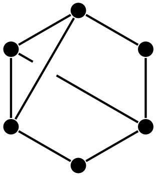

$$\text{(color factor)} \times \text{(propagators)} \times \sum_{k=1}^{2\ell} R_k$$

Only genus-1 correction
to five-loop correlator!

$R_k \sim \theta^2$
In total $\sim \theta^{20}$

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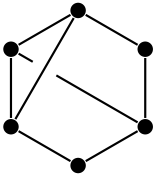

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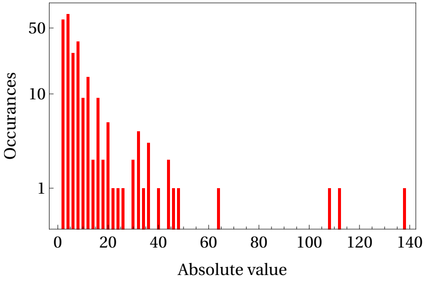
Translate Grassmann contraction to **sparse tensor multiplications**
$\longrightarrow$ Efficient implementation on GPUs ($\sim$ 17 hours / point)

Solution & Outlook

- Fix 23 coefficient by numerical comparison to twistor result
- Solution is not unique due to Gram identities
- Solution with **many zeros** and remaining **small integer values** exists



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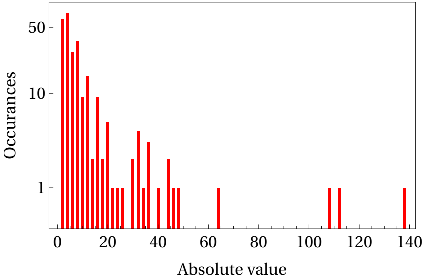

Obtain non-planar data at 5 loops:

- Konishi anomalous dimension
- Cusp anomalous dimension
- Twist two anomalous dimension

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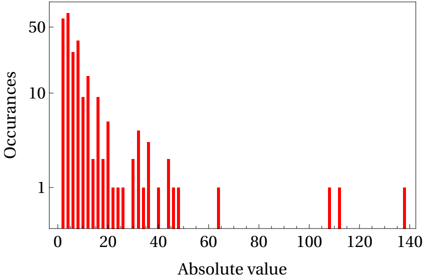

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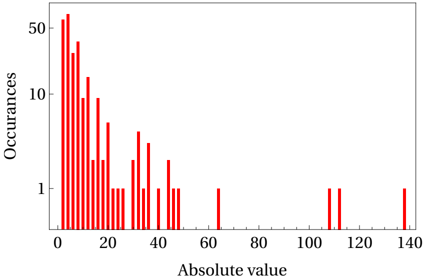

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