One-loop string scattering amplitudes at finite α'

Marco Maria Baccianti

University of Amsterdam

Based on work in collaboration with Cheevan Chandra, Lorenz Eberhardt, Tom Hartman, Sebastian Mizera

[2501.13827,2507.22105,WIP]

Synergies Towards the Future Standard Model 2025

Motivation

Very little is known about higher-loop amplitudes exactly in α'

Main technical difficulty: define contour+hard integrals

Plan of this talk

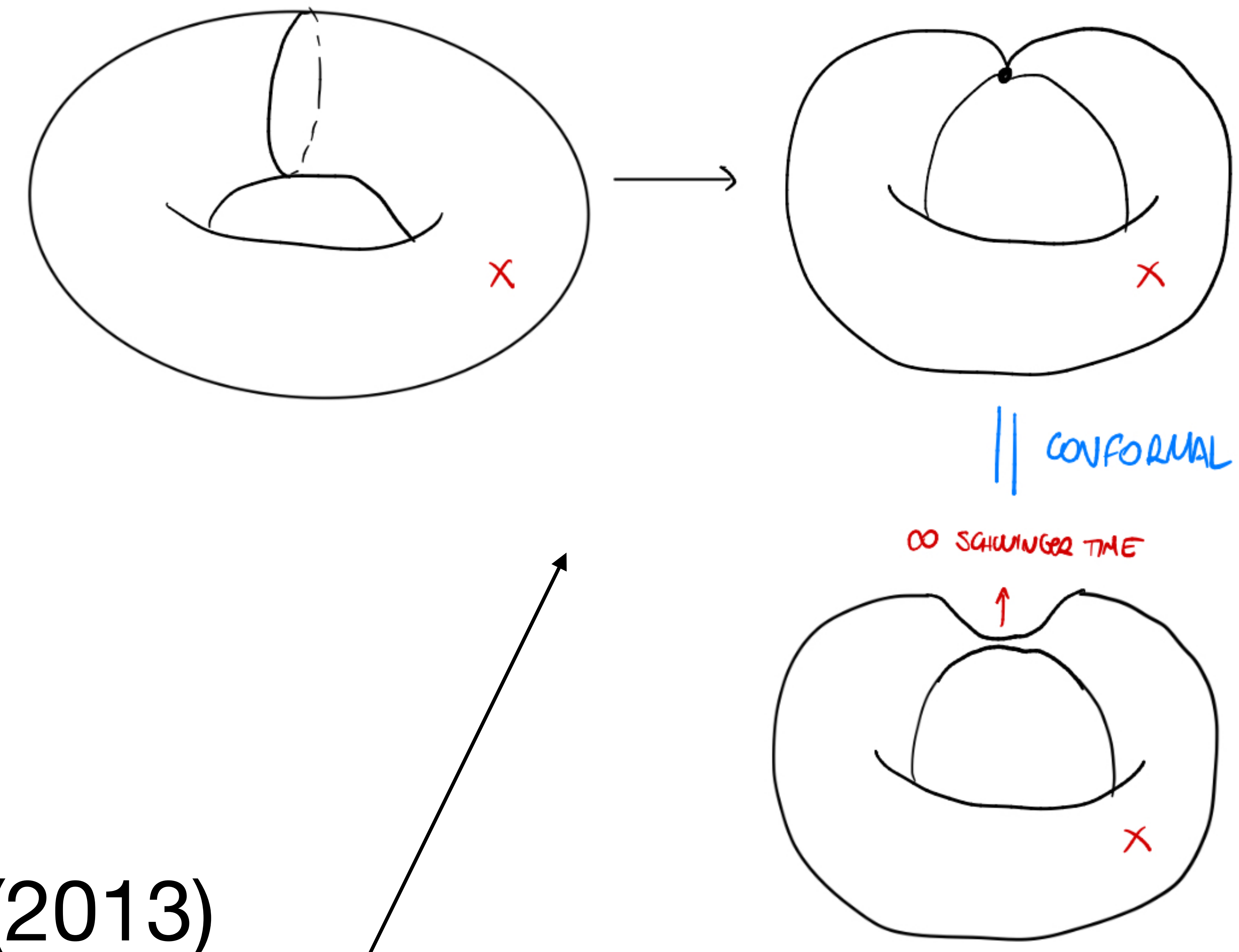
- Part 1: Understanding the contour problem in string amplitudes ($i\varepsilon$ - prescription)
- Part 2: Define contour and evaluate integral in one go: Rademacher evaluation of modular integrals [2501.13827]
- Part 3: Interesting physics [2507.22105,WIP]

Part 1. The problem

String Scattering amplitudes

$$A = \int_{\Gamma \subset \mathfrak{M}^{\mathbb{C}}_{g,n}} (\text{CFT correlator})$$

Contour consistent with Lorentzian signature



How? E. Witten **The Feynman $i\varepsilon$ in string theory** (2013)

Should modify contour when worldsheet degenerates = when intermediate particles go on-shell

Intuition for QFT

Building block of QFT amplitude: propagator

$$\frac{1}{p^2 + m^2 \pm i\varepsilon} = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t(p^2+m^2)\mp i\varepsilon t}$$

t = Schwinger time = length of propagator $\Rightarrow \mathbb{R}_{\geq 0} \sim$ toy $\mathfrak{M}_{g,n}$

Choice of $i\varepsilon$ - prescription = contour choice in $\mathbb{R}_{\geq 0}^{\mathbb{C}}$

No contour choice $\Rightarrow$ integral diverges when $p^2 = -m^2$ (degeneration locus)

Part 2. Rademacher of modular integrals

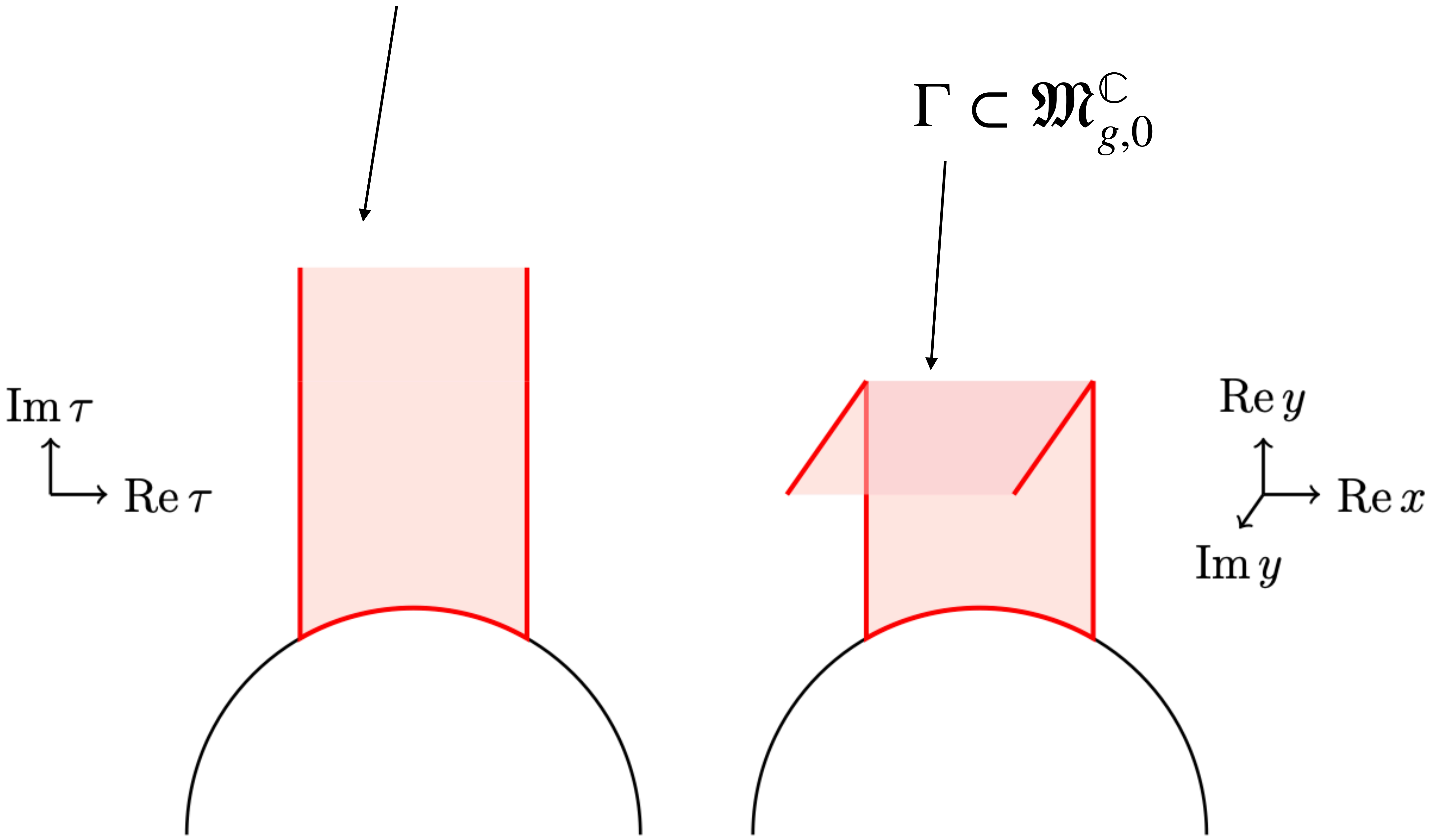
The torus (one loop)

$$\mathfrak{M}_{g,0} = \frac{\mathbb{H}}{SL(2,\mathbb{Z})} \Rightarrow \mathfrak{M}^{\mathbb{C}}_{g,0} = \frac{\mathbb{H}\times\mathbb{H}}{SL(2,\mathbb{Z})}$$

Schwinger time is $\mathrm{Im}(\tau)$

After some contourology, get nice formula!

This formula computes integrals of forms over fundamental domain in general



Not only string theory: ex: 2d CFT Lorentzian inversion formula! [1703.00278]

The Rademacher formula

$f =$ modular form of weight (2,2)

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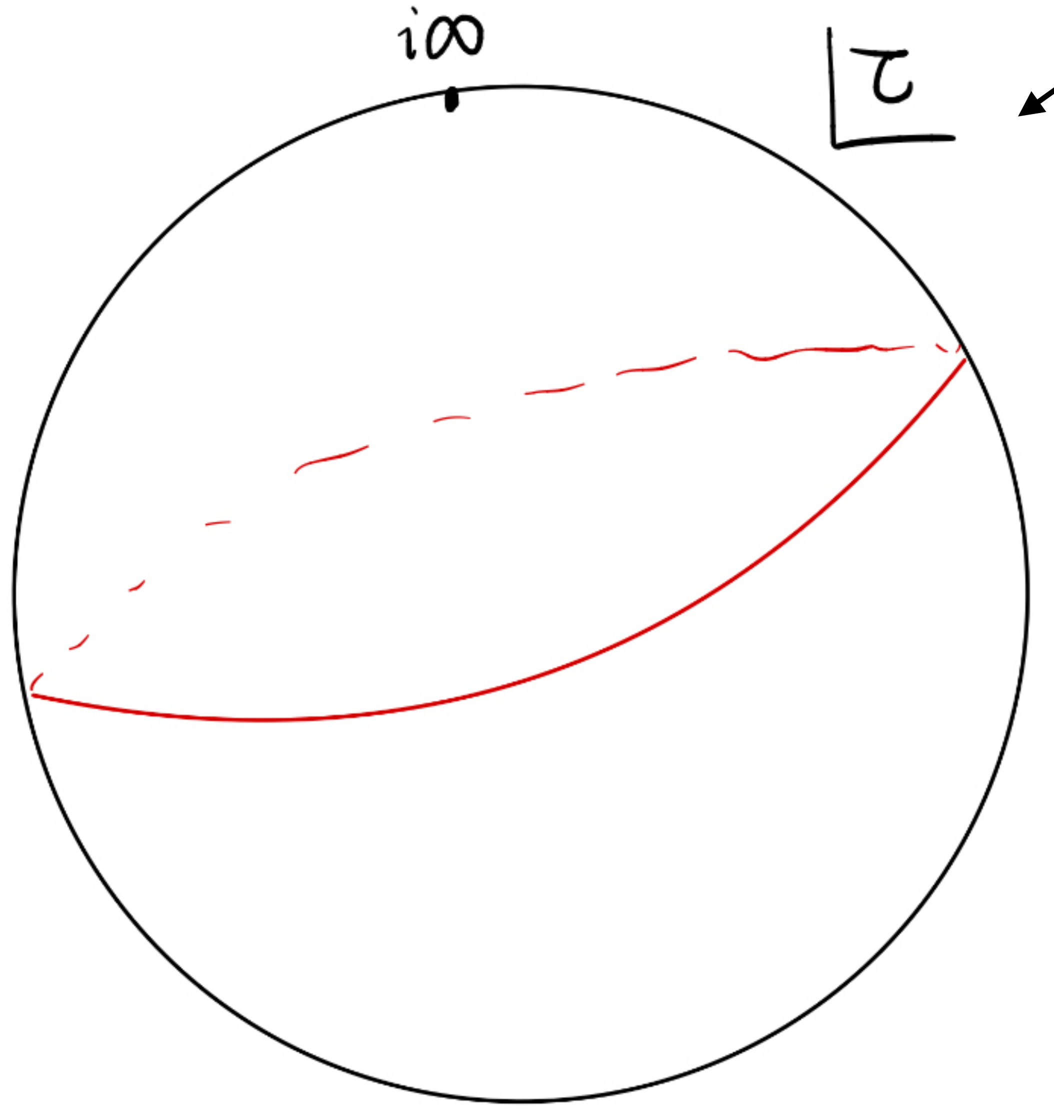
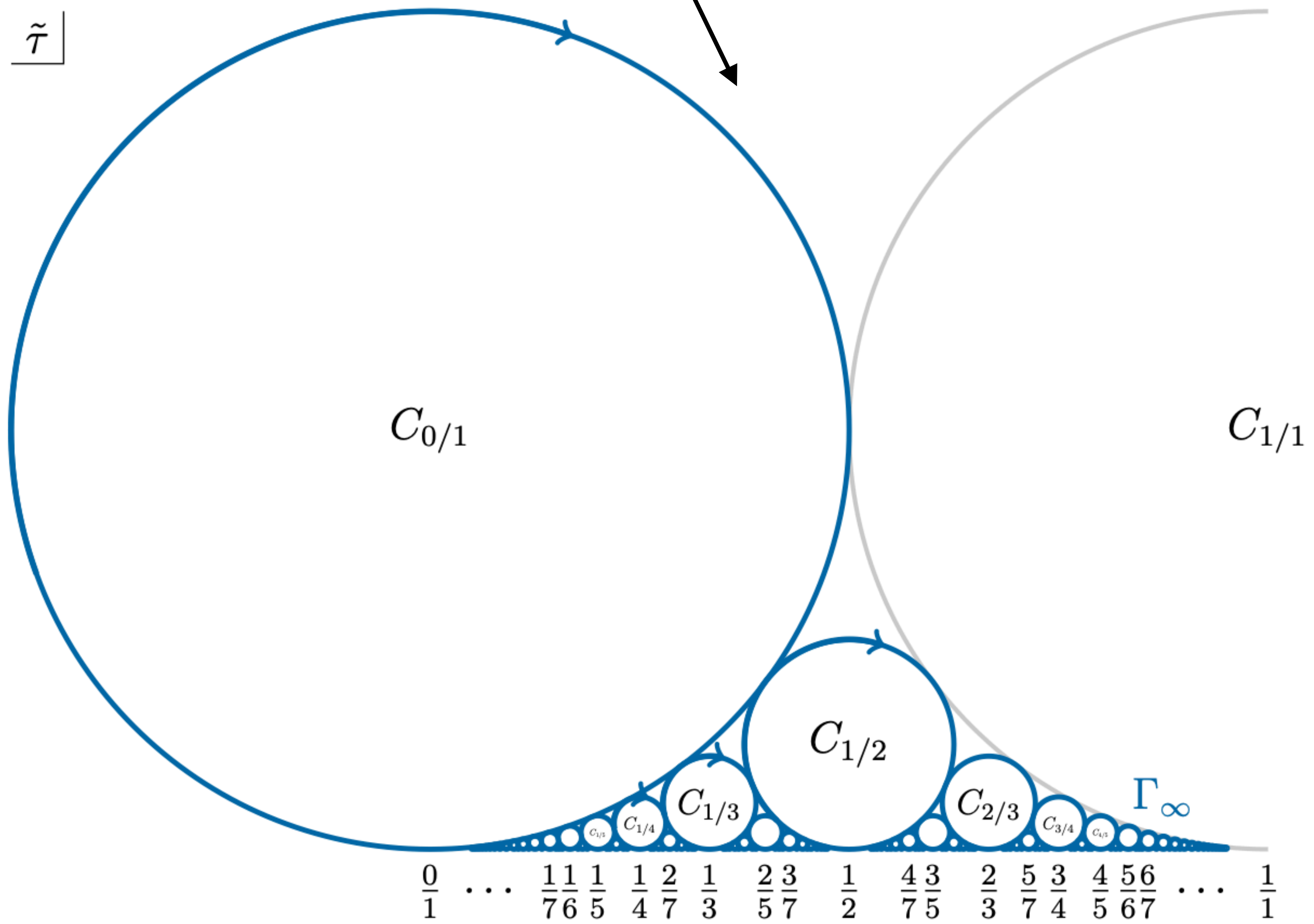
$s(a,c)$ Dedekind sum

$$\int_\Gamma \mathrm{d}^2\tau \, f(\tau,\tilde\tau) = \sum_{c=1}^{\infty} \sum_{\substack{a=0\\(a,c)=1}}^{c-1} \int_{\rightarrow} \mathrm{d}\tau \int_{C_{a/c}} \mathrm{d}\tilde\tau \left[\frac{1}{12i}\left(\tau - \tilde\tau + \frac{2a}{c}\right) + is(a,c) - \frac{1}{2}\right] f(\tau,\tilde\tau)$$





Imaginary part

A modular transformation maps $C_{a/c}$ to $\rightarrow$

$\Rightarrow$ pick polar terms and extract two residues (easy)

Fix ideas: bosonic string partition function

All you need to know:

$$Z = \int_\Gamma \frac{\mathrm{d}^2\tau}{(\mathrm{Im}\,\tau)^{14}\,|\eta(\tau)^{24}|^2}$$

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right)^{24} = (c\tau+d)^{12}\eta(\tau)^{24},$$

$$\eta(\tau)^{-24} = \mathrm{e}^{-2\pi i\tau} + \ldots \quad \text{as} \quad \mathrm{Im}\,\tau \to \infty$$

$\tau = \frac{a\tau'+b}{c\tau'+d}$ Maps $C_{a/c}$ to $\rightarrow$

$$\int_{\rightarrow} \mathrm{d}\tau \int_{C_{a/c}} \frac{\mathrm{d}^2\tau}{(\mathrm{Im}\,\tau)^{14}\,|\eta(\tau)^{24}|^2} = 2^{n-1}\int_{\rightarrow} \frac{\mathrm{d}\tilde{\tau}}{\eta(\tilde{\tau})^{24}} \int_{\rightarrow} \frac{\mathrm{d}\tau}{(-(a\tilde{\tau}+b+\tau(c\tilde{\tau}+d))^{14}\,\eta(\tau)^{24}}$$

$$= 2^{n-1}\int_{\rightarrow} \mathrm{d}\tilde{\tau}\; \mathrm{e}^{2\pi i\tilde{\tau}} \int_{\rightarrow} \frac{\mathrm{d}\tau}{(-(a\tilde{\tau}+b+\tau(c\tilde{\tau}+d))^{14}}\mathrm{e}^{2\pi i\tau}$$

Only polar terms matter!

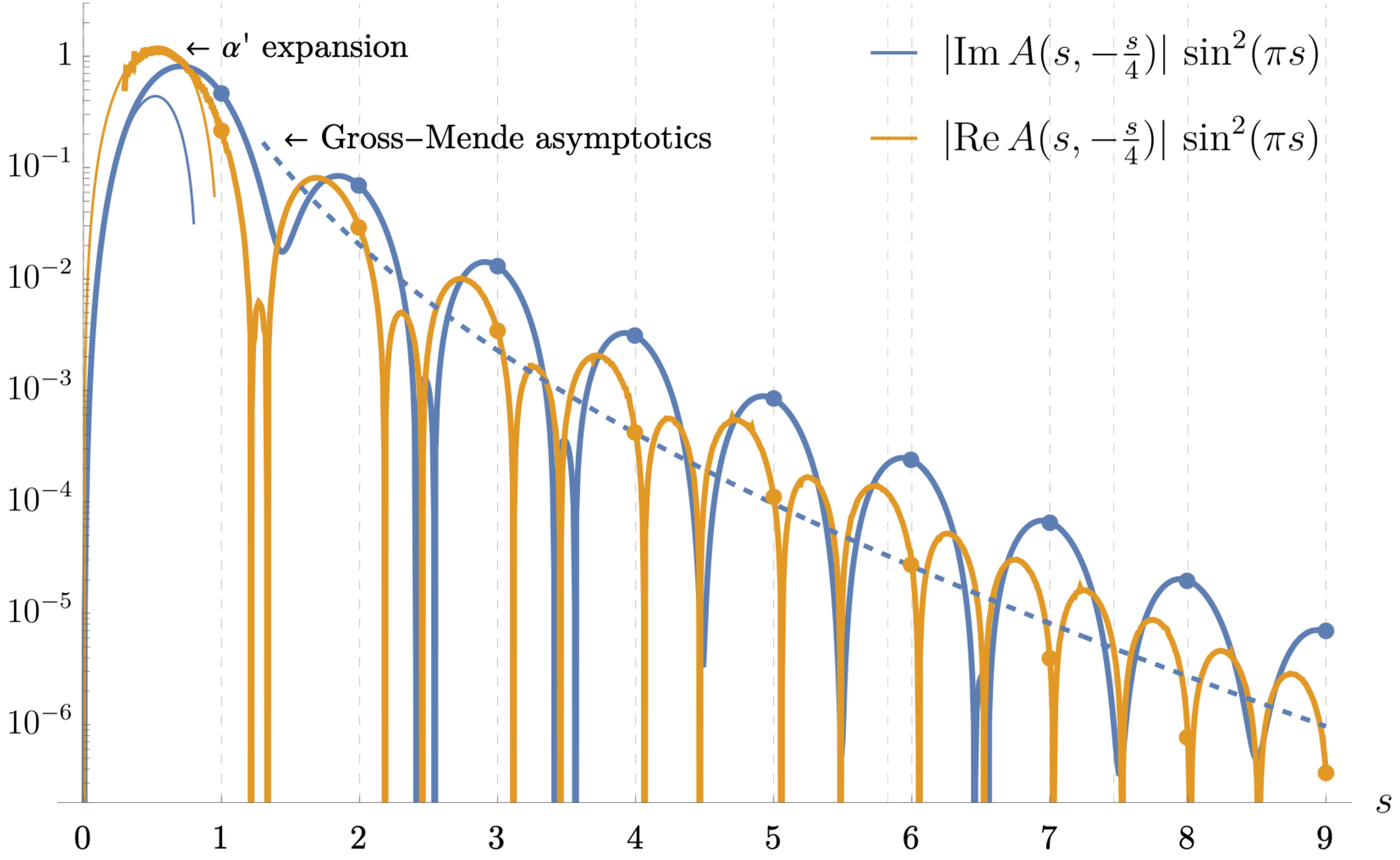
Part 3: Interesting physics

2-2 gravitons in type II
Fixed angle scattering @ 60 degrees

Deacay predicted, but never proved: Gross (Mañes)-Mende ('89)

We see oscillations on top of it

Rademacher allows to include Lorentzian saddles! (WIP)



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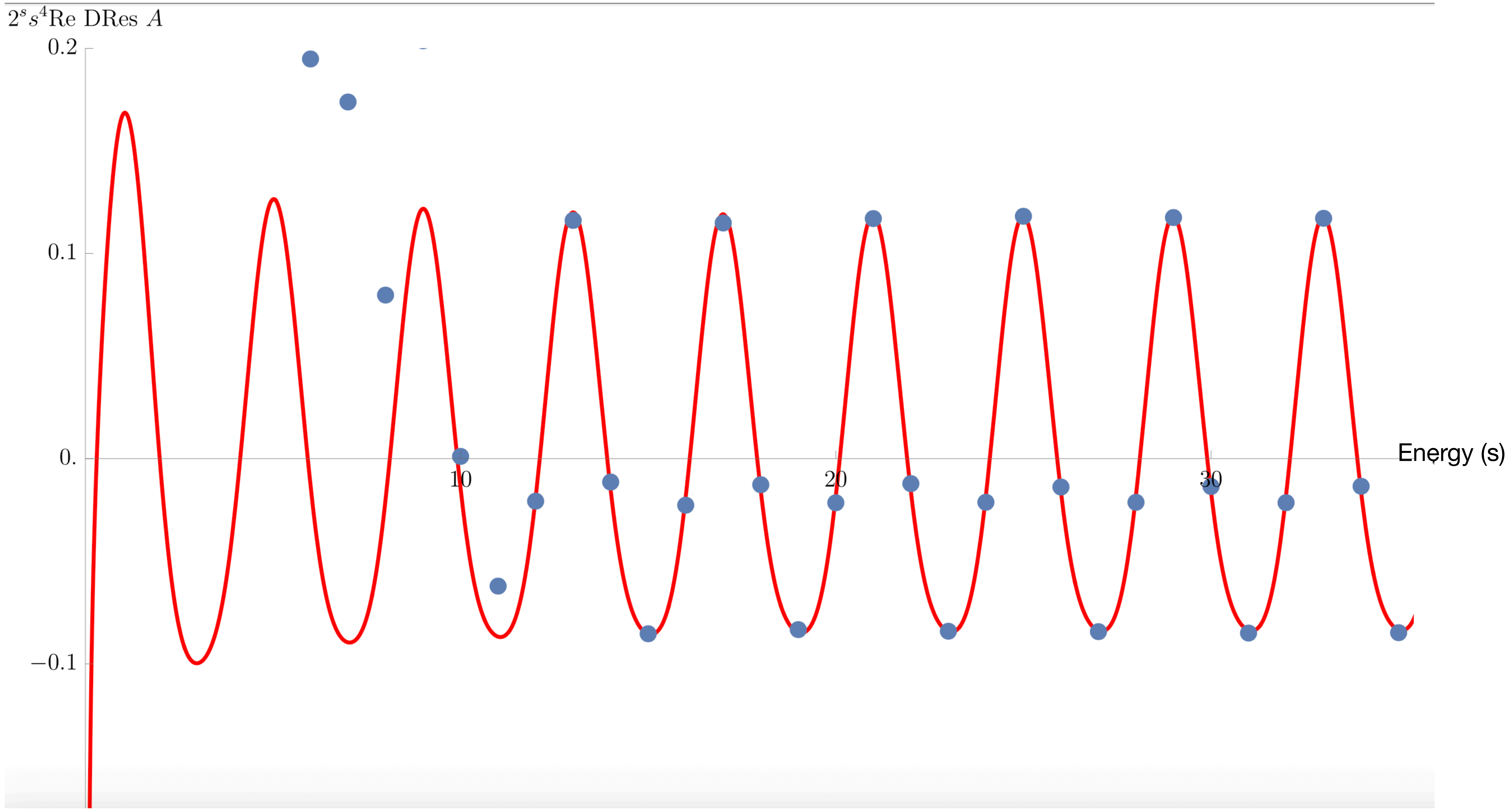
Teaser: Lorenzian saddles (WIP)

Blue is data (high energy)

Red is saddle point with Lorentzian saddles



Conclusions

- Rademacher formula makes available new computations (ex. Bosonic string partition function, 2-2 gravitons at one-loop in type IIB superstring theory)

- Rademacher is Lorenzian

- Allows to explore UV of string theory (WIP: Gross and Mende reloaded)

Backups

Backup: String Scattering amplitudes (naive)

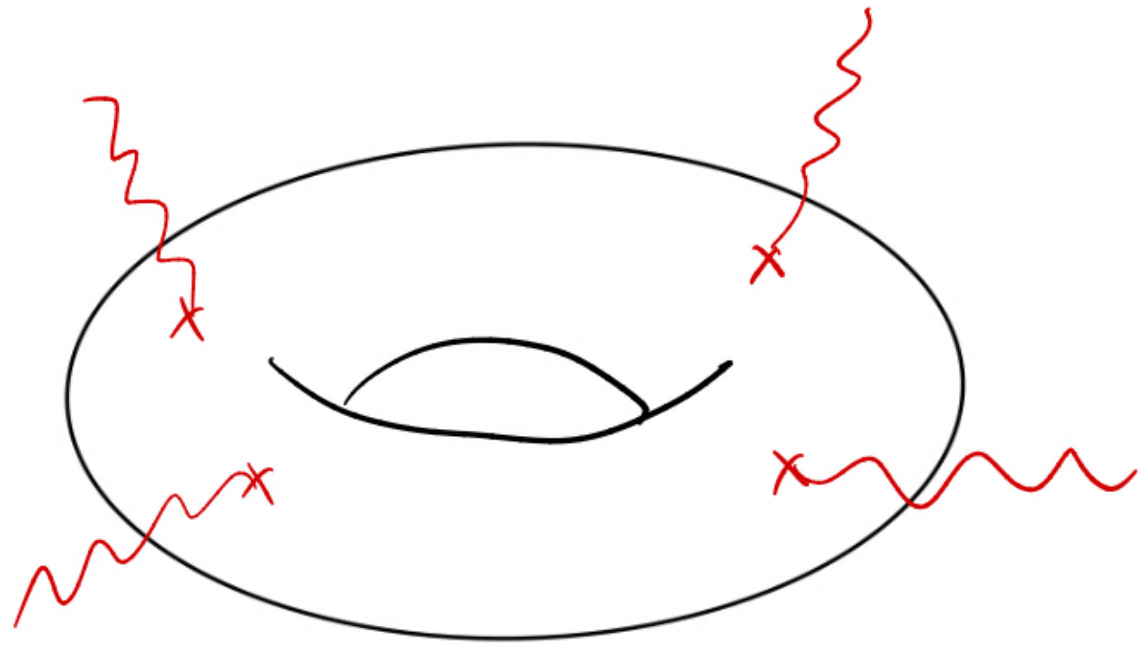
$$A = \int_{\mathfrak{M}_{g,n}} (\text{CFT correlator})$$

g= #loops (genus)

n= #external legs (punctures)

Moduli space of Riemann surfaces

Ex: $g = 1, n = 4$ →



Problem: formal expression! (diverges) Need to define the integration contour

Recap: why is Rademacher useful

- General formula ready to be used!
- Allows to compute integrals over fundamental domain ANALYTICALLY
- Do string theory EXACTLY in α'
- Lorentzian vs Euclidean string theory computation

Backup: some things computed

- Bosonic string theory partition function (both in $\mathbb{R}^{1,25}$ and $\mathbb{R}^{1,24} \times S^1$)
- $SO(16) \times SO(16)$ string partition function
- First mass-shift of type II strings (= at level $\alpha' m^2 = 1$)

$$\operatorname{Re} Z = \frac{(4\pi)^{15}}{24 \cdot 13!} \sum_{c=1}^{\infty} \sum_{\substack{a=0 \\ (a,c)=1}}^{c-1} \frac{\mathrm{e}^{\frac{2\pi i(a+a^*)}{c}}}{c^2} \left[12ics(a,c) J_{13}\left(\frac{4\pi}{c}\right) + J_{12}\left(\frac{4\pi}{c}\right) - J_{14}\left(\frac{4\pi}{c}\right) \right]$$

$$\operatorname{Im} Z = \frac{(4\pi)^{13}\pi}{13!}$$

Fast convergence:

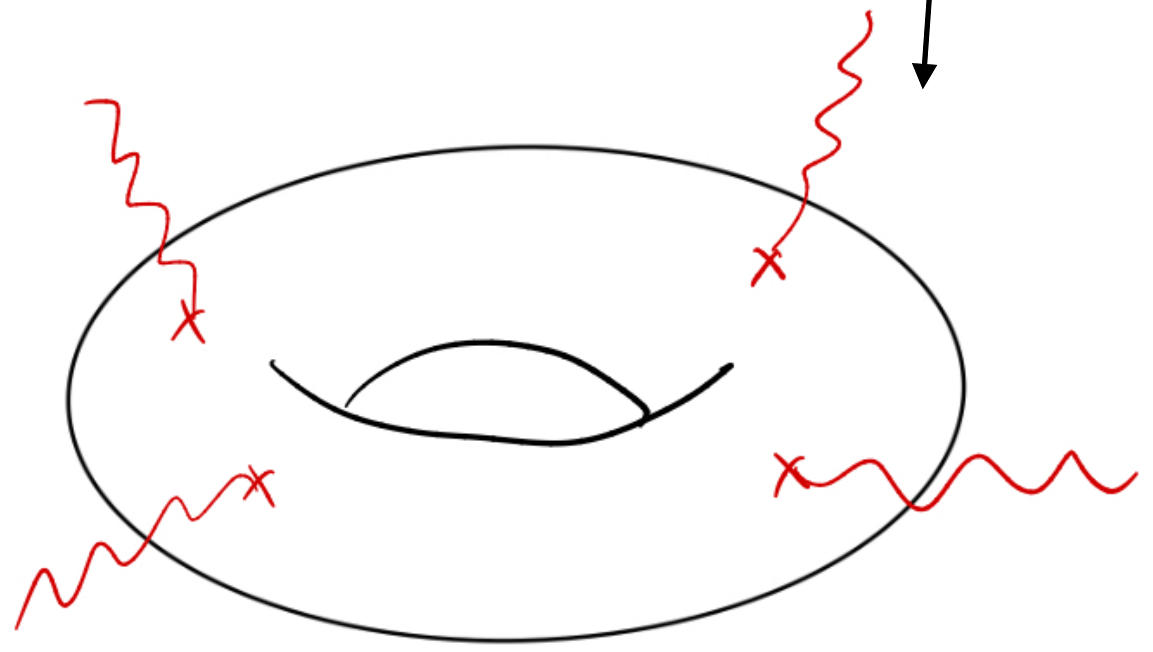
$$J_k\left(\frac{4\pi}{c}\right) \sim \frac{1}{c^k}$$

Backup: 2-2 gravitons in type II

$$A \sim \int_F \frac{\mathrm{d}^2\tau}{(\operatorname{Im}\tau)^5} \int_{\mathbb{T}^2} \prod_{j=1}^{3} \mathrm{d}^2 z_j \prod_{1\leqslant i<j\leqslant 4} \left|\vartheta_1\left(z_{ij} \mid \tau\right)\right|^{-2s_{ij}} \mathrm{e}^{\frac{2\pi s_{ij}(\operatorname{Im} z_{ij})^2}{\operatorname{Im}\tau}}$$

Jacobi theta function

High oscillations near $\operatorname{Im}(\tau) = \infty$; awful for numerical evaluation!



Backup: 2-2 gravitons at 1-loop: Rademacher formula

$$A \sim \sum_{c=1}^{\infty} \sum_{\substack{a=0 \\ \gcd(a,c)=1}}^{c-1} \sum_{n_{\rm L}, n_{\rm D}, n_{\rm R}=0}^{c-1} \mathrm{e}^{i(\text{phase})} A_{a/c}^{n_{\rm L}, n_{\rm D}, n_{\rm R}}$$

Windings come from z_i integrals

Complicated expression

Very good for numerics: can fix angle, push at high s and explore the UV of string theory!

Backup: 2-2 scattering expression

$$
\begin{aligned}
&A_{a/c}^{n_\mathrm{L},n_\mathrm{D},n_\mathrm{R}} \\
&\quad = \sum_{\substack{\sqrt{m_\mathrm{D}}+\sqrt{m_\mathrm{U}}\leqslant\sqrt{s}\\ \sqrt{\tilde{m}_\mathrm{D}}+\sqrt{\tilde{m}_\mathrm{U}}\leqslant\sqrt{s}}} \frac{(\Delta\tilde{\Delta})^{\frac{9}{4}}}{720\sqrt{2}c^{\frac{7}{2}}s^{\frac{11}{2}}}\,\mathrm{e}^{\frac{2\pi i}{c}(n_\mathrm{D}(m_\mathrm{D}+d\tilde{m}_\mathrm{D})+n_\mathrm{U}(m_\mathrm{U}+d\tilde{m}_\mathrm{U}))-\frac{\pi i}{cs}\Delta^{(2)}} \\
&\qquad \times \int_{\mathbb{D}} \mathrm{d}x\,\mathrm{d}y \int_{\mathbb{D}} \mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y}\, Q\tilde{Q}\,\mathrm{e}^{\frac{\pi i\sqrt{\Delta\tilde{\Delta}}}{sc}(x\tilde{x}+y\tilde{y})}\big((1-x^2-y^2)(1-\tilde{x}^2-\tilde{y}^2)\big)^{\frac{5}{4}} \\
&\qquad \times \left[\sqrt{\frac{\tilde{P}}{P}}\Big(J_{\frac{3}{2}}\big(\tfrac{4\pi\sqrt{P\tilde{P}}}{c}\big)-J_{\frac{7}{2}}\big(\tfrac{4\pi\sqrt{P\tilde{P}}}{c}\big)\Big)+12ic\,s(a,c)J_{\frac{5}{2}}\big(\tfrac{4\pi\sqrt{P\tilde{P}}}{c}\big)\right]\Bigg|_{\substack{P\to\frac{\Delta}{4s}(1-x^2-y^2)\\ \tilde{P}\to\frac{\tilde{\Delta}}{4s}(1-\tilde{x}^2-\tilde{y}^2)}} \\
&\qquad \times \Bigg(\frac{\Gamma(-t_\mathrm{L})\Gamma(s+t_\mathrm{L}-m_\mathrm{D}-m_\mathrm{U})\Gamma(-\tilde{t}_\mathrm{L})\Gamma(s+\tilde{t}_\mathrm{L}-\tilde{m}_\mathrm{D}-\tilde{m}_\mathrm{U})}{\Gamma(s)^2} \\
&\qquad\qquad \times \begin{cases} \mathrm{e}^{2\pi i t_\mathrm{L}\left(\left(\frac{n_\mathrm{L}}{c}\right)\right)+2\pi i\tilde{t}_\mathrm{L}\left(\left(\frac{dn_\mathrm{L}}{c}\right)\right)}\,, & n_\mathrm{L}\neq 0 \\ \frac{\mathrm{e}^{\pi i(t_\mathrm{L}-\tilde{t}_\mathrm{L})}+\mathrm{e}^{\pi i(\tilde{t}_\mathrm{L}-t_\mathrm{L})}-\mathrm{e}^{\pi i(t_\mathrm{L}+\tilde{t}_\mathrm{L})}-\mathrm{e}^{-\pi i(2s+t_\mathrm{L}+\tilde{t}_\mathrm{L})}}{1-\mathrm{e}^{-2\pi i s}}\,, & n_\mathrm{L}=0 \end{cases}\Bigg) \\
&\qquad \times (\mathrm{L}\leftrightarrow\mathrm{R})\,.
\end{aligned}
$$