# Self-organized criticality in a relativistic Yukawa theory with Luttinger fermions

**DESY Theory Workshop**
**Synergies towards the future Standard Model**

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based on: arXiv:2506.13441
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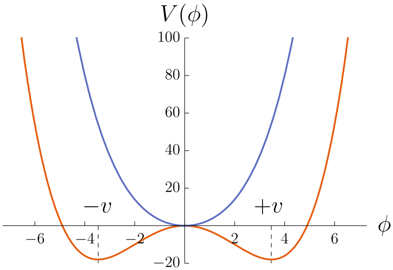
# Introduction

- QFTs with scalars are a framework to describe phase transitions
- Critical phenomena and features of universality




- Shape of the scalar potential → vacuum structure
  - minimum at the origin: symmetric phase $\bar{m}^2 > \bar{m}^2_{\text{crit}}$
  - non-trivial minimum: broken phase $\bar{m}^2 < \bar{m}^2_{\text{crit}}$

In relativistic *4d* spacetimes approaching the critical region requires a fine tuning of the parameters: $\bar{m}^2$ with quadratic precision $\rightarrow \bar{m}^2_{\text{crit}}$.

Relevant parameter → large UV sensitivity.

# Self-organized criticality

- Prominent example: Higgs sector of the Standard Model $\rightarrow$ naturalness problem
- Alternative idea fro a solution from statistical and dynamical systems

**Self-organized criticality**

If a system has a critical point that is an attractor of the evolution, critical phenomena emerge without fine-tuning; the dynamics itself drives the system to criticality.

RG language: existence of an IR attractive fixed point and a slow RG running of the $\langle\phi\rangle$, stabilized by a sufficiently large scalar anomalous dimension.

S. Bornholdt and C. Wetterich, Phys. Lett. B, 282, 399-405 (1992)

In our Yukawa model of scalar fields interacting with relativistic Luttinger fermions self-organized criticality emerges naturally.

# What is a Luttinger fermion?

- Effective degrees of freedom in solid-state physics J. M. Luttinger, Phys. Rev. 102, 1030 (1956)
- Description of Quadratic Band Touching/Crossing (QBT/QBC) points:

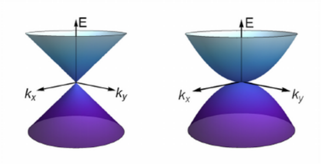



Figure: doi.org/10.1103/PhysRevB.101.161111

$$H = \sum_{i,j=1}^{d} G_{ij} p_i p_j, \qquad H^2 = p^4 \mathbb{1}$$

L. Janssen and I. F. Herbut, Phys. Rev. B 92, 045117 (2015)

Inspired by the diverse set of structures emerging from Luttinger fermions in solid-state physics, we generalized these d.o.f. to the relativistic case.

H. Gies, P. Heinzel, J. Laufkötter, and M. Picciau, Phys. Rev. D, 110, 065001 (2024)

# Relativistic Luttinger fermions

Our model is inspired by the asymptotically free purely fermionic model of self-interacting relativistic Luttinger fermions with classical Euclidean action

$$S_{\mathrm{F}} = \int d^4x[-\bar{\psi}G_{\mu\nu}\partial^\mu\partial^\nu\psi - \frac{\bar{g}}{2}(\bar{\psi}\gamma_{11}\psi)^2]$$

H. Gies and M. Picciau, Phys. Rev. D, 111, 085001 (2025) Mean-field study of this and other self-interacting models

$G_{\mu\nu}$ are $d_\gamma \times d_\gamma$ ($32 \times 32$) matrices which satisfy the relativistic Abrikosov algebra

$$\{G_{\mu\nu}, G_{\kappa\lambda}\} = -\frac{2}{d-1}g_{\mu\nu}g_{\kappa\lambda} + \frac{d}{d-1}(g_{\mu\kappa}g_{\nu\lambda} + g_{\mu\lambda}g_{\nu\kappa})$$

Hubbard-Stratonovich transformation → introduce an auxiliary scalar field to linearize the fermionic self-interaction

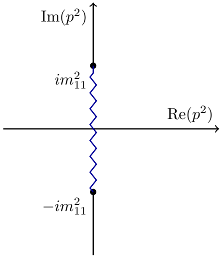
# The $\gamma_{11}$ Yukawa model

Generalize the model by considering a fully dynamical scalar field

$$S = \int d^4x \left[ -\bar{\psi} G_{\mu\nu} \partial^\mu \partial^\nu \psi + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \bar{h} \phi \bar{\psi} \gamma_{11} \psi + \frac{1}{2} \bar{m}^2 \phi^2 + \frac{\bar{\lambda}}{8} \phi^4 \right]$$

$\mathbb{Z}_2$ symmetry broken by $\langle \phi \rangle = v$: the fermions become massive

- from the massive fermionic Lagrangian: $p^2 = \pm i m_{11}^2$




- branch cut in the imaginary axis
- no spectral representation

Luttinger fermions are not asymptotic states. Context of confinement (QCD).

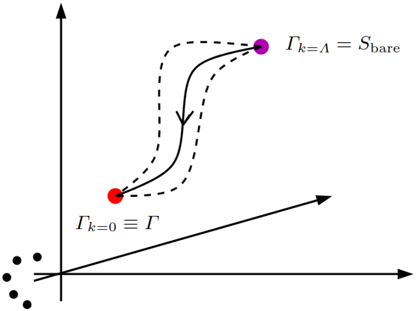

M. Stingl, Phys. Rev. D 34, 3863 (1986)

M. Stingl, Z.Phys.A 353, 423-445 (1996)

# Functional Renormalization Group

Nonperturbative implementation of Wilson's RG idea that physics at different momentum scales is obtained by integrating out fluctuations step by step.

Effective average action $\Gamma_k$ interpolates between the microscopic action $S$ at $k = \Lambda$ and the full quantum effective action $\Gamma$ at $k = 0$, where all fluctuations are integrated out.




The flow of $\Gamma_k$ is governed by the Wetterich equation

$$\partial_t \Gamma_k = \frac{1}{2} \mathrm{STr} \left[ \frac{\partial_t R_k}{\Gamma_k^{(2)} + R_k} \right]$$

C. Wetterich, Phys. Lett. B 301, 90 (1993)

Image source: H. Gies, Lect. Notes Phys. 852, 287-348 (2012)

# RG flow

Ansatz into the Wetterich equation → flow equations for all $k$-dependent quantities:

$$\Gamma_k = \int d^4x \left[ -Z_\psi \bar{\psi} G_{\mu\nu} \partial^\mu \partial^\nu \psi + \frac{Z_\phi}{2} \partial_\mu \phi \partial^\mu \phi - \bar{h} \phi \bar{\psi} \gamma_{11} \psi + U(\phi) \right]$$

Polynomial expansion of the potential:

$$u(\rho) \simeq \begin{cases} \sum_{n=1}^{N_\mathrm{p}} u_n \rho^n & \text{(SYM)} \\ \sum_{n=2}^{N_\mathrm{p}} u_n (\rho - \kappa)^n & \text{(SSB)} \end{cases}$$

$u(\rho)$ dimensionless effective potential, $\rho$ dimensionless renormalized field invariant.

Estimates for the physical observables are determined once the RG flows are computed down to low scales $k$:

$$v = k\sqrt{2\kappa}\big|_{k\to 0}$$

$$m_\sigma = k\sqrt{2\lambda\kappa}\big|_{k\to 0}$$

$$m_\psi = k\sqrt[4]{2\kappa h^2}\big|_{k\to 0}$$

# Leading order

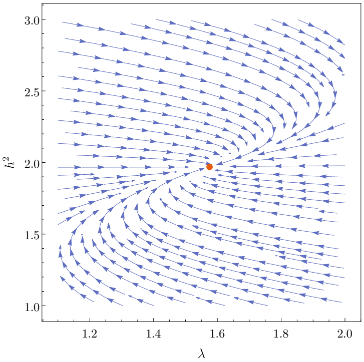
We consider the large-$N_\mathrm{f}$ (Luttinger flavors) limit at leading order and we find

- Gaussian fixed point;
- Interacting fixed point for the Yukawa coupling

$$h_*^2 = \frac{32\pi^2}{5N_\mathrm{f}d_\gamma} \Leftrightarrow \eta_\phi = 2 \quad \text{for } N_\mathrm{f}d_\gamma \to \infty,$$

- and for the scalar self-interaction $\lambda$

$$\lambda_* = \frac{128\pi^2}{25N_\mathrm{f}d_\gamma} \quad \text{for } N_\mathrm{f}d_\gamma \to \infty.$$




Fully IR attractive fixed point.
Critical exponents $\theta_{h^2} = -2$ & $\theta_\lambda = -4$: both couplings are RG irrelevant.

# Self-organized criticality

The previous FP is only a partial fixed point, since the mass parameter flows logarithmically slowly towards zero and to the SSB regime:

$$\partial_t \epsilon = \frac{4}{5}, \quad \text{for } N_{\mathrm{f}} d_\gamma \to \infty,$$

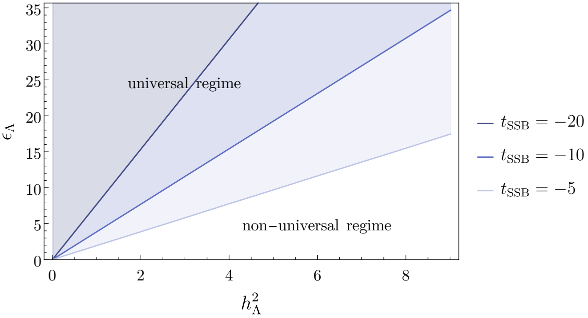
At the partial FP the mass parameter is marginal, since $\eta_\phi \simeq 2$.

- Large scale separation $k_{\mathrm{SSB}} \ll \Lambda$ is obtained for generic initial conditions and without the need for fine-tuning. Moreover, the system always ends up in the broken regime.
- Reminiscent of self-organized criticality: the RG flow drives the system to criticality.
- Same conclusions can be drawn from a perturbative analysis.

# Universal IR observables

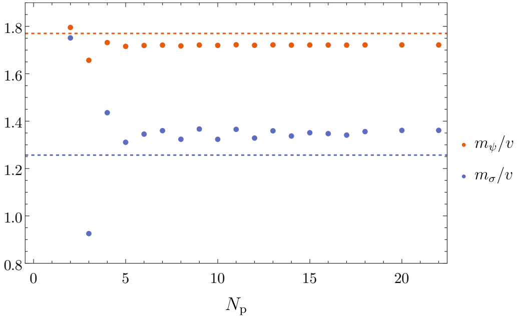
A large region in the space of initial parameters leads to universal long-range physics: no fine-tuning.

- The degree of universality of the long-range observables is governed by the RG time spent at the partial fixed point $\sim t_{\text{SSB}} = \ln \frac{k_{\text{SSB}}}{\Lambda}$.



# Physical masses

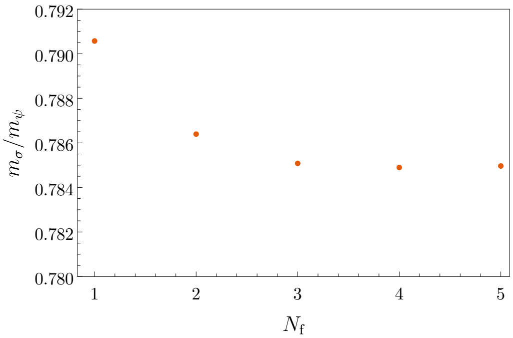
- At $k_{\text{SSB}}$, $\epsilon \to 0$:
  - mass spectrum quantitatively determined by the SSB flow
  - the flow "freezes out"
- For increasing order $N_{\text{p}}$ of the polynomial truncation, we find convergence on the sub-permille level of the values of $m_\sigma/v \simeq 1.36$ and $m_\psi/v \simeq 1.72$.




Small difference between the estimates (fixed point values) and the numerical results ⇒ the properties of the long range observables are governed by the partial fixed point.

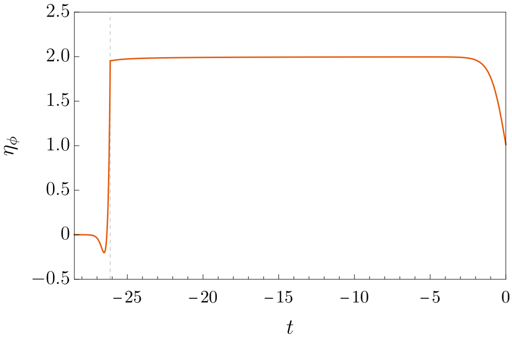
# Mass ratios

- Deviation of the ratio from $m_\sigma \simeq 2m_\psi$ is a measure for the binding energy of the scalar state.
- $m_\sigma/m_\psi \simeq 0.786 < 2$: deeply bound relativistic state.



# Scalar anomalous dimension

- The flow rapidly approaches the fixed point value $\eta_\phi \simeq 2$ and remains there for a wide range of scales.
- The scalar mass parameter becomes marginal at the partial fixed point and all the other couplings ($h^2$, $\lambda$, $u_{n>2}$) become irrelevant.
- When $\epsilon < 0$, the fixed point controlled flow stops and $\eta_\phi \to 0$: massive modes decouple.



# High-energy completion

The irrelevant coupling are repulsive in the UV $\Rightarrow$ the only RG trajectories for which we have UV control are the ones emanating from the partial fixed point.

Study the flow of $g = \frac{h^2}{\epsilon}$

- UV limit: $\epsilon \to +\infty$, same result as the purely fermionic $\gamma_{11}$ model ($g \equiv$ matching condition in HS transformation)
- The beta function is negative and the theory is asymptotically free and UV complete

$$\partial_t g = -\frac{1}{8\pi^2}(N_\mathrm{f} d_\gamma - 2) g^2$$

- Since $\eta_\phi \simeq 2$ near the partial fixed point $\Rightarrow Z_\phi \sim \frac{1}{k^2}$: scalar kinetic term is suppressed in the UV $\rightarrow$ nondynamical auxiliary field.

# Recap I

Yukawa model with relativistic Luttinger fermions features an IR attractive partial fixed point:

- emerges for generic initial conditions: no fine-tuning needed to get universal long-range behavior;
- all couplings ($h^2$, $\lambda$, $u_n$) become RG irrelevant;
- scalar mass parameter $\epsilon$ is RG marginal ($\eta_\phi = 2$) with slow logarithmic running towards low energies;
- large scale separation between UV cutoff and low energy scales emerges naturally.

The system always ends up in the SSB phase.
IR physics characterized by mass gap generation in both scalar and fermionic sector.

# Recap II

Above mentioned properties are reminiscent of the phenomenon of self-organized criticality in dynamical or statistical systems:

- physical time $\Leftrightarrow$ RG time;
- slow driving force $\Leftrightarrow$ slowly running mass parameter;
- the system always evolves to criticality $\Leftrightarrow$ SSB.

Moreover:
The Yukawa model features UV complete extension by virtue of the asymptotically free purely fermionic $\gamma_{11}$ model.

# Outlook

- Include a separate Luttinger fermionic sector to the standard model,
- or embed the SM fermionic content into Luttinger spinors.
- Application of other techniques like Dyson-Schwinger equations or lattice field theory.
- Shed light on existence or not of Luttinger fermions as asymptotic states.
- Spin-statistic or CPT theorem satisfied?

# Properties of the algebra

Span $G_{\mu\nu}$ by an Euclidean Clifford algebra

$$G_{\mu\nu} = a^A_{\mu\nu}\gamma_A, \quad \{\gamma_A, \gamma_B\} = 2\delta_{AB} \qquad a^A_{\mu\nu} \in \mathbb{C} \quad A, B = 1 \dots d_e$$

- Number of linearly independent elements to span the algebra
  $d_e = \frac{1}{2}d(d-1) + d - 1$
- Dimension of the Clifford algebra
  $d_{\gamma,irr} = 2^{\lfloor d_e/2 \rfloor}$

Define the conjugate spinor: $\bar{\psi} = \psi^\dagger h$ $\quad$ $h$ spin metric H. Gies and S. Lippoldt, Phys. Rev. D 89, 064040 (2014)

Reality of the kinetic action implies $h^\dagger = h$, $\{h, G_{0i}\} = 0$, $[h, G_{ij}] = 0$, $[h, G_{\underline{\mu\mu}}] = 0$.

We find a solution for $h$ if $d_e = 11 \rightarrow d_\gamma = 32$ (in $d = 4$).

The spin metric we use is $h = \gamma_1\gamma_2\gamma_3\gamma_{10}$.

# Spin-base invariance

The algebra is invariant under Lorentz transformations

$$G_{\mu\nu} \to G_{\kappa\lambda}\Lambda^{\kappa}_{\mu}\Lambda^{\lambda}{}_{\nu}, \quad \Lambda \in \mathrm{SO}(1, d-1)$$

and under spin-base transformation

$$G_{\mu\nu} \to \mathcal{S} G_{\mu\nu} \mathcal{S}^{-1}, \quad \mathcal{S} \in \mathrm{SL}(d_\gamma, \mathbb{C})$$

which correspond to the similarity transformations of the $G_{\mu\nu}$ matrices.

The action is spin-base invariant $\psi \to \mathcal{S}\psi$, provided $h \to (\mathcal{S}^\dagger)^{-1} h \mathcal{S}$.

Lorentz transformations $\mathcal{S}_{\mathrm{Lor}}$ of Luttinger spinors $\equiv$ subgroup of the spin-base group $\mathrm{SL}(32, \mathbb{C})$ which rotates the Lorentz transformed $G_{\mu\nu}$ matrices back to their original constant forms:

$$\mathcal{S}^{-1}_{\mathrm{Lor}} G_{\mu\nu} \mathcal{S}_{\mathrm{Lor}} = G_{\kappa\lambda}\Lambda^{\kappa}_{\mu}\Lambda^{\lambda}{}_{\nu}.$$

# Mass terms I

In order to classify different possibilities of gap formation, we look at the different mass terms that can be constructed for Luttinger fermions.

## Standard mass term

$$\mathcal{L} = -\bar{\psi} G_{\mu\nu} \partial^\mu \partial^\nu \psi - m^2 \bar{\psi}\psi$$

Equation of motion for $\psi$:

$$(G_{\mu\nu} p^\mu p^\nu - m^2)\psi = 0$$
$$\rightarrow (p^2 - m^2)(p^2 + m^2)\psi = 0$$

- spin-base & Lorentz invariant
- real
- regular mass poles ($p^2 = m^2$)
- tachyons ($p^2 = -m^2$)

## Bilinear with $\gamma_{10}$

$$\mathcal{L} = -\bar{\psi} G_{\mu\nu} \partial^\mu \partial^\nu \psi - i m_{10}^2 \bar{\psi}\gamma_{10}\psi$$

- same kind of solutions (regular mass and tachyons)

# Mass terms II

## Bilinear with $\gamma_{11}$

$$\mathcal{L} = -\bar{\psi} G_{\mu\nu} \partial^\mu \partial^\nu \psi - m_{11}^2 \bar{\psi} \gamma_{11} \psi$$

Equation of motion: $(p^4 + m^4)\psi = 0$

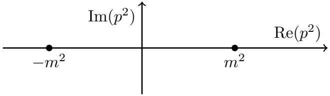
- two complex conjugate zeros ($p^2 = \pm i m_{11}^2$)

## Bilinear with $\gamma_{01}$

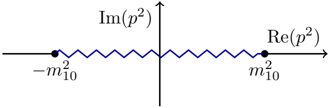
$$\mathcal{L} = -\bar{\psi} G_{\mu\nu} \partial^\mu \partial^\nu \psi - i m_{01}^2 \bar{\psi} \gamma_{01} \psi$$

- $\gamma_{01} := -i\gamma_{10}\gamma_{11} \quad \gamma_{01} = \gamma_{01}^\dagger$
- again two complex conjugate zeros ($p^2 = \pm i m_{01}^2$)

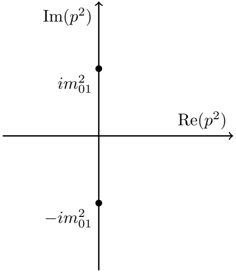
# Structure of the massive propagators

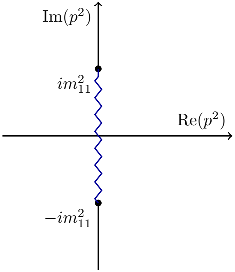



- standard mass & tachyon
- higher derivative theory: the tachyon is also a ghost




- branch cut (square root singularities)
- no Källén–Lehmann spectral representation




- complex poles & complex branch cut
- no spectral representation

Luttinger fermions are not asymptotic states.
Context of confinement (QCD).

M. Stingl, Phys. Rev. D 34, 3863 (1986)

M. Stingl, Z.Phys.A 353, 423-445 (1996)

# Flow equations

$$\partial_t u(\rho) = -d\, u + (d-2+\eta_\phi)\rho u' + 2v_d \Big[ l_0^d\,(u' + 2\rho u''; \eta_\phi) - N_\mathrm{f} d_\gamma l_0^{(\mathrm{L})\,d}\left(2\rho h^2; \eta_\psi\right) \Big]$$

$$\partial_t h^2 = -(2 - 2\eta_\psi - \eta_\phi)h^2 + 8v_d\, h_k^4 l_{1,1}^{(\mathrm{LB})\,d}\,(\omega_1, \omega_2; \eta_\psi, \eta_\phi)\,,$$

$$\eta_\psi = \frac{16}{d(d+2)} v_d h^2 m_{1,2}^{(\mathrm{LB}),d}(\omega_1, \omega_2; \eta_\psi, \eta_\phi),$$

$$\eta_\phi = \frac{8v_d}{d}\kappa(3u'' + 2\kappa u''')^2 m_{2,2}^d(\omega_2; \eta_\phi) + \frac{8v_d}{d} N_\mathrm{f} d_\gamma h^2 \left( m_4^{(\mathrm{L}),d}(\omega_1, \eta_\psi) - 2h^2\kappa m_2^{(\mathrm{L}),d}(\omega_1, \eta_\psi) \right).$$

$l$, $m$ threshold functions encoding the regularization of loop integrals; $\omega_1 = 2\kappa h^2$, $\omega_2 = u'(\kappa) + 2\kappa u''(\kappa)$; anomalous dimensions $\eta_{\psi,\phi} = -\partial_t \ln Z_{\psi,\phi}$.