Critical Points and Syzygies for Feynman Integrals

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Motivation

- In multiloop amplitude calculation, an important step is to reduce tensor integrals into a linear combination of MIs through IBP reduction.
- The classical approach of IBP reduction is to generate a large set of IBP relations through total derivative, and solve for MI coefficients with linear algebra, so-called *Laporta algorithm*.
  [S. Laporta]
- This method has been very successful and implemented into many public packages, such as *Kira*, *FIRE*, *LiteRed*, *Blade* ...
  [P. Maierhöfer, J. Usovitsch and P. Uwer]
  [J. Klappert, F. Lange, P. Maierhöfer and J. Usovitsch]
  [A. V. Smirnov,]
  [A. V. Smirnov and F. S. Chukharev]
  [A. V. Smirnov and M. Zeng]
  [R. N. Lee]
  [X. Guan, X. Liu, Y. Q. Ma and W. H. Wu]

Motivation

- The main bottleneck of IBP reduction is the memory usage and time-consuming due to huge size of IBP relations.

- One way to improve IBP reduction is to target on smaller but complete set of IBP relations, which can be realized by searching for block-triangular form of IBPs, machine-learning technique and etc.

  [X. Guan,X. Liu, Y. Q. Ma and W. H. Wu]
  [Z. Y. Song, T. Z. Yang, Q. H. Cao, M. X. Luo and H. X. Zhu]
  [M. von Hippel and M. Wilhelm]
  [M. Zeng]

- Syzygy-based approaches lead to IBP relations between Feynman integrals without raising propagator powers and without shifted space-time dimension, which can improve efficiency of IBP reduction process

  [J. Gluza, K. Kajda, D. A. Kosower]
  [K. J. Larsen, Y. Zhang]
  [R. Schabinger]
  [D. Cabarcas, J. Ding]
  [B. Agarwal, S. P. Jones, A. von Manteuffel]
  [S. Abreu, G. De Laurentis, H. Ita, M. Klinkert, B. Page, V. Sotnikov]
  [Z. Wu, J. Boehm, R. Ma, H. Xu and Y. Zhang]

Content

In this talk,

- I will discuss that for certain type of Feynman integrals, syzygy equation under on-shell and large-ϵ limit has a deep connection with Feynman integrals
- This connection further improves IBP reduction, i.e. a subset of syzygy solutions is sufficient to reduce all tensor integrals into MIs.
- We will demonstrate the sufficiency of IBPs using a cutting-edge example, two-loop leading-colour light-fermion correction to $pp \to t\bar{t}H$

Syzygy Equation

- Total derivative in Baikov representation, [P. A. Baikov]

$$0 = \int_{\mathcal{C}} \mathrm{d}^N \vec{z}\, \partial_k \left[\frac{B^\gamma a_k}{\prod_{i=1}^N z_i^{\nu_i}} \right]$$
$$= \int_{\mathcal{C}} \mathrm{d}^N \vec{z} \frac{B^\gamma}{\prod_{i=1}^N z_i^{\nu_i}} \left[\partial_k a_k + \gamma a_k \partial_k \log B - \nu_k \frac{a_k}{z_k} \right]$$

  $\vec{z}$: Baikov variables, B :Baikov polynomial, $\vec{\nu}$: integer exponent
  N :number of Baikov Variables, $\gamma = (D - E - L - 1)/2$, and
  a_k: IBP generating vectors, which are polynomials in Baikov variables.

- $a_k \partial_k \log B$: dimension-shift Feynman Integrals;
  a_k / z_k: increase propagator powers.

- Eliminating them introduces constraints to a_k, *Syzygy equations*: [K. J. Larsen, Y. Zhang]

$$\left.\begin{aligned} a_k \partial_k B &= a_0 B \\ a_e &= \bar{a}_e z_e \end{aligned}\right\} \Rightarrow a_0 B + \sum_{i \in \text{ISPs}} a_i \partial_i B + \sum_{e \in \text{props}} \bar{a}_e z_e \partial_e B = 0$$

Syzygy Equation

- Normal syzygy equation introduced before [K. J. Larsen, Y. Zhang]

$$0 = a_0 B + \sum_{i \in \text{ISPs}} a_i \partial_i B + \sum_{e \in \text{props}} \bar{a}_e z_e \partial_e B$$

  unknowns $\{a_0, a_i, \bar{a}_e\}$ depend on all Baikov variables.
- We work with the syzygy equation of following form

$$0 = a_0 B + \sum_{i \in \text{ISPs}} a_i \partial_i B + \sum_{e \in \text{props}} \tilde{a}_e z_e B + \sum_{e \in \text{props}} \bar{a}_e z_e \partial_e B$$

  where $\{a_i, \tilde{a}_e, \bar{a}_e\}$ depend on all Baikov variables, but a_0 depends on ISPs.
- a_0 of normal syzygy equation is separated into on-shell (not dependent on propagators) and off-shell parts (proportional to propagators),

$$a_0 = a_0\big|_{z_e=0} + \sum_{e \in \text{props}} z_e \tilde{a}_e$$

Surface Terms

- Each solution to syzygy equation leads to an IBP

$$0 = \int_C \mathrm{d}^N\vec{z} \frac{B^\gamma S_\Gamma(\vec{a},\vec{\nu})}{\prod_{k=1}^N z_k^{\nu_k}} = \int_C \mathrm{d}^N\vec{z} \frac{B^\gamma}{\prod_{k=1}^N z_k^{\nu_k}} \left[a_0 + \sum_{e\in \text{props}} \tilde{a}_e z_e - \frac{1}{\gamma}\left(\sum_{i\in \text{ISPs}} \partial_i a_i + \sum_{e\in\text{props}} (z_e \partial_e \bar{a}_e - (\nu_e - 1)\bar{a}_e \right) \right]$$

  $S_\Gamma(\vec{a},\vec{\nu})$ is the so-called surface term.

- The integrand numerator of loop integrals can be decomposed as a linear combination of propagator structure Γ with a direct sum of surface terms S_Γ and master integrands M_Γ

$$\mathcal{A}^{(l)} = \sum_\Gamma \sum_{i\in M_\Gamma \cup S_\Gamma} \frac{c_{\Gamma,i} m_{\Gamma,i}}{\prod_{j\in\Gamma}\rho_j}$$ [H. Ita]

  Surface terms integrate to zero. This decomposition facilitate numerical unitarity approach.

Geometric Interpretation of a_0

- We consider surface terms under on-shell and large-ϵ limit,

$$\left[\lim_{\epsilon\to\infty} S_\Gamma(\vec{a},\vec{\nu})\right]\Big|_{z_e=0\,:\,e\in\text{props}} = a_0$$

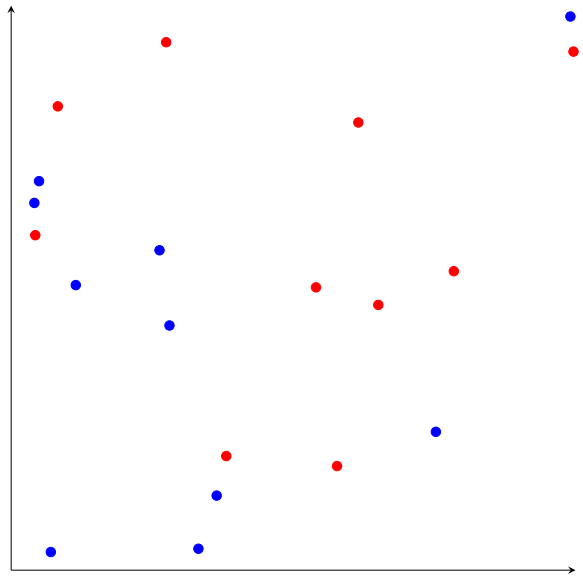
  which contains a single term. This dramatic simplification provide a strong motivation to understand a_0 deeply.
- How to analyze a_0 from syzygy equation? Think about geometric features of points where $a_0 = 0$.
- Judiciously, we set other terms to zero.

$$U^\Gamma_{\text{syz}} = \{\vec{z} : \partial_i B = z_e B = z_e \partial_e B = 0\} \;\Rightarrow\; a_0 B\big|_{U^\Gamma_{\text{syz}}} = 0$$

- Points in U^Γ_{syz} has multiples branches. We can identify two branches where $B = 0$ and $B \neq 0$.

Geometric Interpretation of a_0

- Assume two branches are isolated points. $a_0 = 0$ on red points, but we are unable to exclude the possibility that $a_0 = 0$ on blue points.



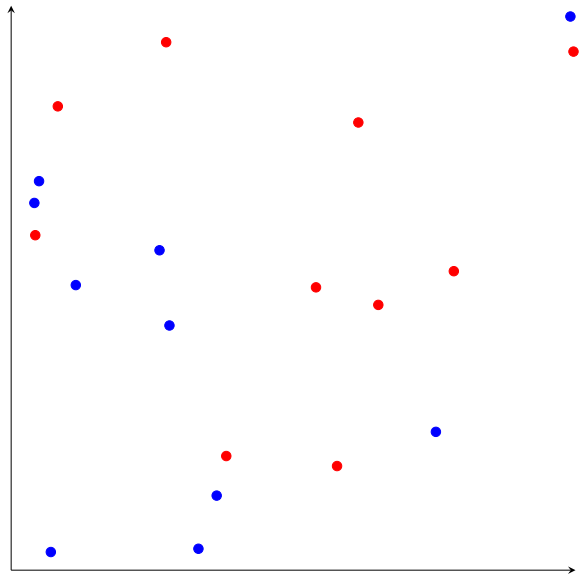
$a_0 B = 0$

$$\begin{aligned} &\bullet : \{\vec{z} : \partial_i B = z_e \partial_e B = z_e B = 0\} \\ &\bullet : \{\vec{z} : \partial_i B = z_e = 0, B \neq 0\} \end{aligned}$$

- This naive analysis have subtleties for finding correct geometric picture of $a_0 = 0$, which can be handled by algebraic geometry.

Geometric Interpretation of a_0

- In the language of algebraic geometry, the solution of $a_0 = 0$ are categorized into the following two branches



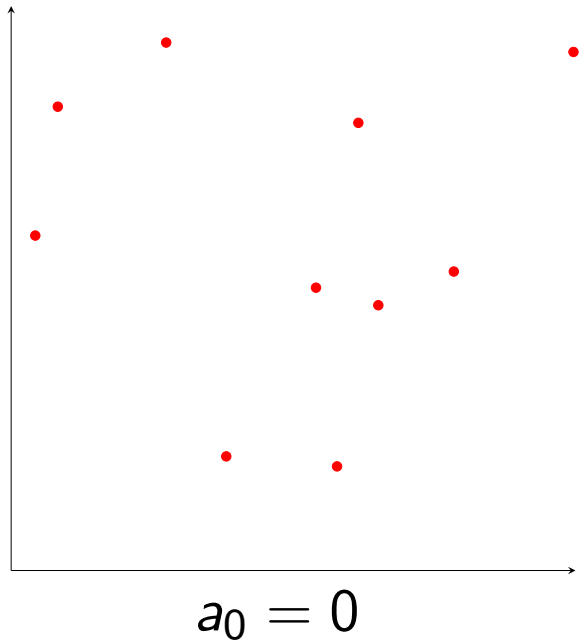
$a_0 = 0$

$$\bullet : \{\vec{z} : \partial_i B = z_e \partial_e B = z_e B = B^\mu = 0\} \setminus \{\vec{z} : B = 0\}, \mu \in \mathbb{Z}^+$$
$$\bullet : \{\vec{z} : \partial_i B = z_e = 0, B \neq 0\}$$

- The value of μ depends on integral family, and can be easily computed with *SingularSat*. [https://www.singular.uni-kl.de]

Geometric Interpretation of a_0

- **In the case of $\mu = 1$, blue points are empty, no solution to $\{B = 0, B \neq 0\}$. Thus $a_0 = 0$ only contains red points.**



$\bullet : \{\vec{z} : \partial_i B = z_e = 0, B \neq 0\}$

- Red points are called *critical points*.
- Lee and Pomeransky found that if number of critical points are finite, the number of those points are the number of MIs. If number of critical points are infinite, we can not use this approach to count MIs.

[R. N. Lee, A. A. Pomeransky]

Geometric Interpretation of a_0

- For some integral family with $\mu = 1$, if number of solutions of $a_0 = 0$ are finite, this number equals the number of MIs, which provides new ways of counting number of MIs.
- In algebraic geometry, counting number of solutions of $a_0 = 0$ does not require solve the polynomial equation system

$$\partial_i B = z_e = 0, \ B \neq 0$$

- Instead, counting number of independent polynomials: two polynomials are considered to be equivalent if they differ by polynomial combinations of a_0

$$\#\mathrm{MIs} = \mathrm{vdim}\left(R/\langle a_0 \rangle\right), \ \langle a_0 \rangle = \langle \partial_i B, z_e \rangle : \langle B \rangle$$

- How does this connect to Feynman integrals?

Isomorphism

- The integrand numerator of Feynman integral can be decomposed into a linear combination of surface terms and master integrand.

$$N = \sum_{i \in M_\Gamma} c_{\Gamma,i} m_{\Gamma,i} + \sum_{i \in S_\Gamma} d_i m_{\Gamma,i}$$

- Integrand numerators also follow equivalence relation: two integrand numerators are equivalent if they differ by surface terms.
- The number of MIs can be considered as number of independent integrand numerators modulo surface terms

$$\#\mathrm{MIs} = \dim(R/S_\Gamma|_{z_e=0})$$

Isomorphism

- Thus in the case of $\mu = 1$ and $\mathrm{vdim}\,(R/\langle a_0\rangle)$ is finite

$$\#\mathrm{MIs} = \dim(R/S_\Gamma|_{z_e=0}) = \mathrm{vdim}\,(R/\langle a_0\rangle)$$

- We can re-read this relation as a statement that the vector-space of a_0 is isomorphic to the space of on-shell surface terms,

$$\langle a_0\rangle = \langle \partial_i B, z_e\rangle : \langle B\rangle \simeq S_\Gamma|_{z_e=0}$$

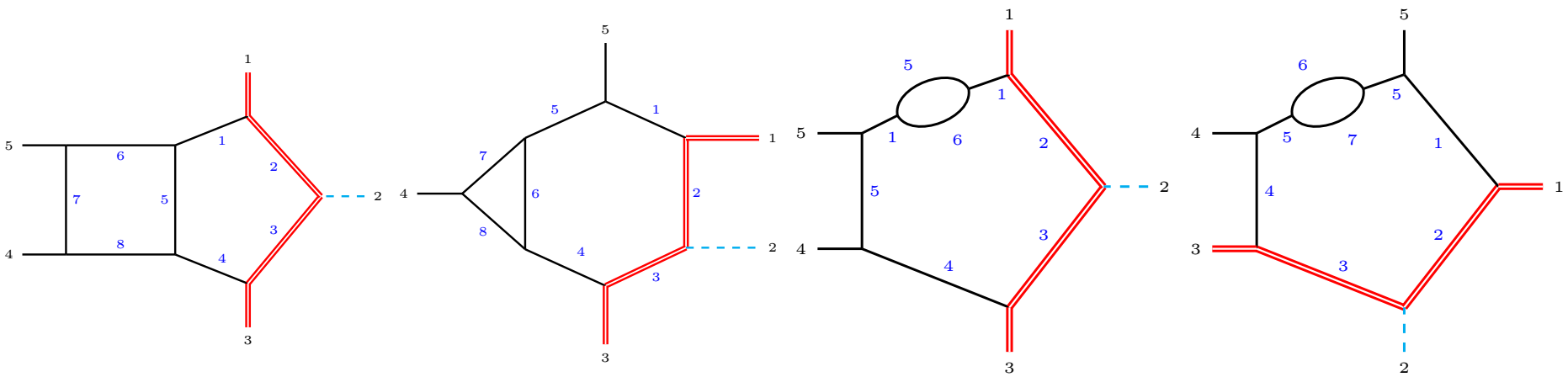
- This isomorphism can improve IBP reduction: syzygy solution can be separated into two subset:

$$\{a_0 = 0\} \cup \{a_0 \neq 0\}$$

The set of solutions $a_0 \neq 0$ are sufficient for IBP reduction.
Moreover, we target on a minimal subset, $\langle a_0\rangle = \langle \partial_i B, z_e\rangle : \langle B\rangle$

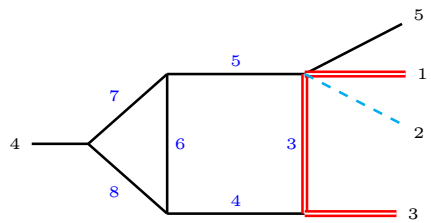
Use Isomorphism in Practice Calculation

- Implement isomophism to generate complete IBPs for two-loop leading colour light-quark loop contributions for $t\bar{t}H$ production.
- There are 123 independent unfactorizable subsectors, in which 122 sectors have finite number of critical points and $\mu = 1$
- We generate complete set of IBPs at numerical phase space points and cross-check with FIRE.



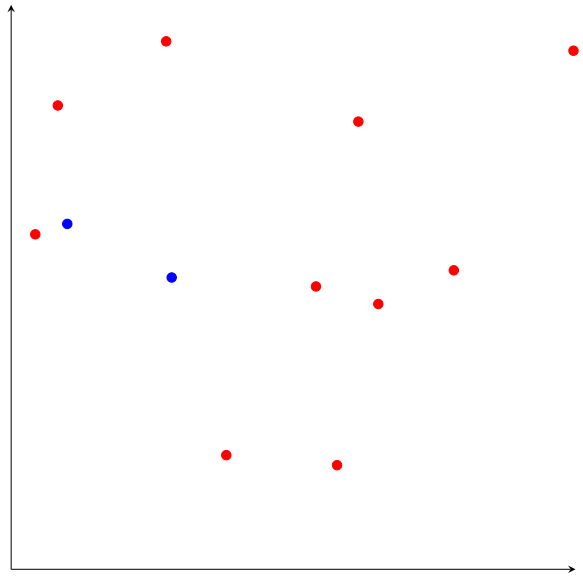
Use Isomorphism in Practice Calculation

- Implement isomophism to generate complete IBPs for two-loop leading colour light-quark loop contributions for $t\bar{t}H$ production.
- There are 123 independent unfactorizable subsectors, in which 122 sectors have finite number of critical points and $\mu = 1$
- We generate complete set of IBPs at numerical phase space points and cross-check with FIRE.
- For left sector $\mu = 2$, isomorphism is not applicable. Using syzygy solutions from $\langle a_0 \rangle = \langle \partial_i B, z_e \rangle : \langle B \rangle$ to compute as many surface terms as possible, and we find that we miss only one surface term, which can be recovered.



$\mu > 1$

- If $\mu > 1$, $a_0 = 0$ contains both red and blue points when $\mu > 1$. Lee-Pomeransky approach only counts red points as number of MIs.



$a_0 = 0$

$$\begin{aligned} &\bullet : \{\vec{z} : \partial_i B = z_e \partial_e B = z_e B = B^\mu = 0\} \\ &\quad \backslash \{\vec{z} : B = 0\}, \mu \in \mathbb{Z}^+ \\ &\bullet : \{\vec{z} : \partial_i B = z_e = 0, B \neq 0\} \end{aligned}$$

- Thus consider extra constraints in order to remove blue points.

Conclusion

- One way to improve efficiently of IBP reduction is to generate IBP relations between Feynman integrals without raising propagator powers, and such IBP relations can be obtained from solutions to the syzygy equations.
- In the language algebraic geometry, surface terms under on-shell and large-ϵ limits have the structure of an ideal. The geometric interpretation connects to Lee-Pomeransky approach for counting the number of MIs.
- We can re-read this connection as the isomorphism between tensor integrals and surface terms, which can improve IBP reduction, and we implement it to find complete set of IBPs for two-loop leading-colour light-quark loop contributions for $t\bar{t}H$ production.

Thanks for your attention!

Ideal Saturation Example

[D. Cox, D. O'Shea, J. Little]

To understand saturation index better, let us consider a simple example. Consider two ideals in $k[x, y]$, which are defined as

$$I_1 = \langle x^2(y-1)\rangle,\ I_2 = \langle x\rangle$$

Variety of I_1, denotes as V_1, are composed by y-axis and the line $y = 1$. Variety of I_2, denoted as V_2, are composed by y-axis only. The smallest variety of V_1 and V_2 difference, $\overline{V_1 \setminus V_2}$, is y=1 line.
However, $I_1 : I_2 = \langle x(y-1)\rangle$. The variety associated to this quotient ideal are composed by y-axis and the line $y = 1$! Thus variety associated to ideal quotient is not equal to variety difference.
If we consider $I_1 : I_2^2 = \langle y-1\rangle$. The variety associated to this saturation ideal only contains y=1 line. Thus variety difference is equal to variety of ideal saturation. The exponent of I_2^2 is saturation index.