Superconformal Invariants for 5-point correlators in $\mathcal{N} = 4$ SYM

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Correlation functions of half-BPS operators

n-point correlation functions $\langle \mathcal{O}_1^{(k_1)} \dots \mathcal{O}_n^{(k_n)} \rangle$ of scalar half-BPS single-trace operators of R-charge k

$$\mathcal{O}^{(k)}(x,y) = \mathrm{Tr}[\phi(x,y)^k] + O(1/N_c)$$

Lagrangian insertion procedure for loop corrections [Intriligator '98, He '24]

$$\langle \mathcal{O}_1^{(k_1)} \dots \mathcal{O}_n^{(k_n)} \rangle^{(\ell)} = \frac{1}{\ell!} \int d^4x_{n+1} \cdots d^4x_{n+\ell} \left\langle \mathcal{O}_1^{(k_1)} \cdots \mathcal{O}_n^{(k_1)} \mathcal{L}_{n+1} \cdots \mathcal{L}_{n+\ell} \right\rangle$$

$5+\ell$-tree-level correlator

$$\left\langle \mathcal{O}_1^{(2)} \cdots \mathcal{O}_5^{(2)} \mathcal{L}_{5+1} \cdots \mathcal{L}_{5+\ell} \right\rangle = \left\langle \bullet \bullet \bullet \bullet \bullet \blacksquare \dots \blacksquare \right\rangle$$

The charge-2 superoperator contains the chiral on-shell Lagrangian

$$\hat{\mathcal{O}}^{(2)}(x, y, \rho) = \mathcal{O}^{(2)}(x, y) + \cdots + \rho^4 \mathcal{L}(x)\,.$$

Relate $5 + \ell$-tree-level-correlator and $5 + \ell$ supercorrelator

$$\Big\langle \bullet \bullet \bullet \bullet \bullet \blacksquare \ \dots \blacksquare \Big\rangle = \frac{\partial^4}{\partial \rho_{5+1}^4} \cdots \frac{\partial^4}{\partial \rho_{5+\ell}^4} \langle \hat{\mathcal{O}}_1^{(2)} \dots \hat{\mathcal{O}}_5^{(2)}\, \hat{\mathcal{O}}_{5+1}^{(2)} \dots \hat{\mathcal{O}}_{5+\ell}^{(2)} \rangle \Big|_{\rho_{1\dots5}=0}$$

➔ The supercorrelator has full $S_{5+\ell}$ permutation symmetry.

Supercorrelators $\langle \hat{\mathcal{O}}_1^{(2)} \dots \hat{\mathcal{O}}_{n+\ell}^{(2)} \rangle$

$$\langle \hat{\mathcal{O}}_1 \dots \hat{\mathcal{O}}_{n+\ell} \rangle = \sum_{m=0}^{n+\ell-4} \langle \hat{\mathcal{O}}_1 \dots \hat{\mathcal{O}}_{n+\ell} \rangle |_{\rho^{4m}}$$

Example: 5-point 1-loop

$$\langle \hat{\mathcal{O}}_1 \dots \hat{\mathcal{O}}_{5+1} \rangle = \sum_{m=0}^{2} \langle \mathcal{O}_1^{(k_1)} \dots \mathcal{O}_6^{(k_6)} \rangle |_{\rho^{4m}} = \langle \hat{\mathcal{O}}_1 \dots \hat{\mathcal{O}}_6 \rangle |_{\rho^0} + \langle \hat{\mathcal{O}}_1 \dots \hat{\mathcal{O}}_6 \rangle |_{\rho^4} + \langle \hat{\mathcal{O}}_1 \dots \hat{\mathcal{O}}_6 \rangle |_{\rho^8}$$

Maximally nilpotent sector:

$$\langle \hat{\mathcal{O}}_1 \dots \hat{\mathcal{O}}_6 \rangle |_{\rho^8} \ni \langle \mathcal{O}_1 \dots \mathcal{O}_4 \ \rho_5^4 \mathcal{L} \ \rho_6^4 \mathcal{L} \rangle$$

Next-to-maximally nilpotent sector:

$$\langle \hat{\mathcal{O}}_1 \dots \hat{\mathcal{O}}_6 \rangle |_{\rho^4} \ni \langle \mathcal{O}_1 \dots \mathcal{O}_5 \ \rho_6^4 \mathcal{L} \rangle$$

Our supercorrelator $\quad G_5^{(\ell)} := \langle \hat{\mathcal{O}}_1 \cdots \hat{\mathcal{O}}_{5+\ell} \rangle \Big|_{\rho^{4\ell}}$

$$\Big\langle \bullet \bullet \bullet \bullet \bullet \blacksquare \dots \blacksquare \Big\rangle = \frac{\partial^4}{\partial \rho_{5+1}^4} \dots \frac{\partial^4}{\partial \rho_{5+\ell}^4} G_5^{(\ell)}$$

Next-to-maximally nilpotent scalar invariants

Decompose the supercorrelator into a finite basis of next-to-maximally scalar nilpotent invariants

$$G_5^{(\ell)} = \sum_I R_I(x_i, y_i)\mathcal{I}_I(x_i, y_i, \rho_i)$$

OPE considerations:

- $R_I(x_i, y_i) = \dfrac{P_I(x_{ij}^2, g_{ij})}{\prod_{i<j=1}^{n+\ell} x_{ij}^2} \qquad g_{ij} = \dfrac{y_{ij}^2}{x_{ij}^2}$
- OPE of $\left\langle \bullet\;\bullet \right\rangle$ is regular

➔ **OPE condition**

$$\lim_{x_{ij}^2 \to 0} \frac{\partial^4}{\partial \rho_{5+1}^4} \cdots \frac{\partial^4}{\partial \rho_{5+\ell}^4} P_I(x_{ij}^2, g_{ij})\mathcal{I}_I(x_{ij}^2, g_{ij}) = 0 \qquad i, j \in \{1, \ldots 5\}$$

Lorentz scalars $\qquad P_I(x_i) \ni x_{ij}^2 \wedge \det(x_i, x_j, x_k, x_m, x_p, x_q)$

Construction of invariants

[Chicherin '15, Heslop '22]

$$\mathcal{I}_{\ell,i_1i_2i_3i_4}^{\alpha_1\alpha_2\alpha_3\alpha_4,c_1c_2c_3c_4} = \int \mathrm{d}^{16}\Xi \, \frac{\partial}{\partial\hat{\rho}_{i_1\alpha_1c_1}} \frac{\partial}{\partial\hat{\rho}_{i_2\alpha_2c_2}} \frac{\partial}{\partial\hat{\rho}_{i_3\alpha_3c_3}} \frac{\partial}{\partial\hat{\rho}_{i_4\alpha_4c_4}} \, \hat{\rho}_1^4\hat{\rho}_2^4\dots\hat{\rho}_{5+\ell}^4$$

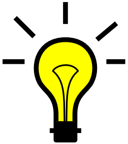
$$\text{with } \hat{\rho}_{i,\alpha c} = \rho_{i,\alpha c} + x_{i,\alpha}^A \Xi_A{}^B y_{i,Bc}^{\perp}$$

scalar invariants

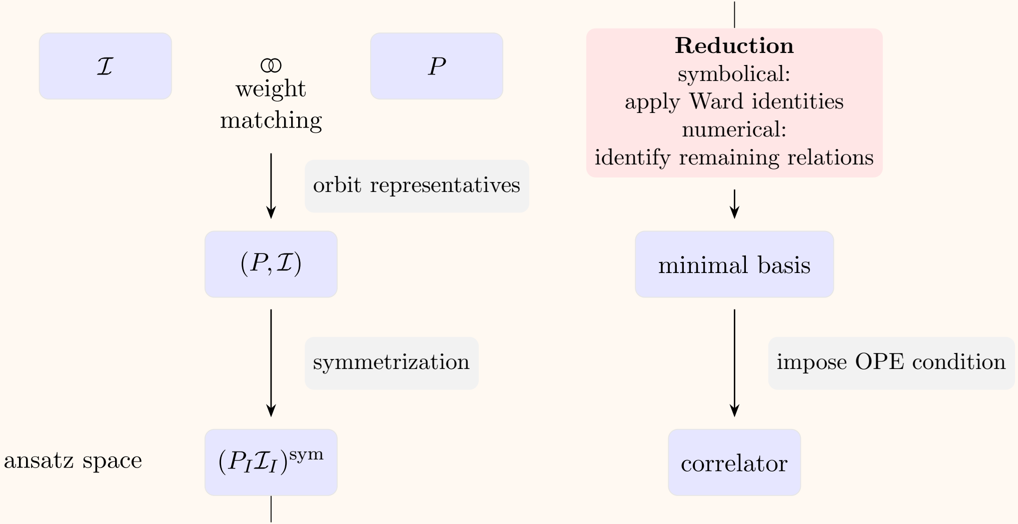
$$\mathcal{I}_{i_1\dots i_4}(\boldsymbol{x},\boldsymbol{y}) \equiv \boldsymbol{x}_{\alpha_1\dots\alpha_4}\mathcal{I}_{i_1\dots i_4}^{\alpha_1\dots\alpha_4c_1\dots c_4}\boldsymbol{y}_{c_1\dots c_4}$$

Challenges:

- need tensor basis of $\boldsymbol{x}$ and $\boldsymbol{y}$
- set of invariants is complete but redundant $\to$ not a basis
- need numerical implementation



Bootstrap


$\mathcal{I}$
$\oplus$
weight matching
P
orbit representatives
$(P, \mathcal{I})$
symmetrization
ansatz space
$(P_I \mathcal{I}_I)^{\text{sym}}$
Reduction
symbolical:
apply Ward identities
numerical:
identify remaining relations
minimal basis
impose OPE condition
correlator

5-point 1-loop correlator $\left\langle \bullet\bullet\bullet\bullet\bullet\,\blacksquare \right\rangle$

With our bootstrap we can decompose the 5-point 1-loop supercorrelator as

$$G_5^{(1)} \sim \frac{1}{x_{16}^2 x_{26}^2 x_{36}^2 x_{46}^2 x_{56}^2}\left(8x_{16}^2 x_{23}^2 \mathcal{I}_{142,153} + \left\{2x_{15}^2 x_{26}^2 - x_{12}^2 x_{56}^2\right\}\mathcal{I}_{132,142}\right)$$

The projection to the $n+\ell$-tree-level correlator is

$$\left\langle \bullet\bullet\bullet\bullet\bullet\,\blacksquare \right\rangle \sim g_{12}g_{13}g_{24}g_{35}g_{45}\frac{x_{16}^2 x_{25}^2 x_{34}^2 - x_{16}^2 x_{24}^2 x_{35}^2 - x_{14}^2 x_{26}^2 x_{35}^2 - \frac{1}{5}\varepsilon_{123456}}{x_{16}^2 x_{26}^2 x_{36}^2 x_{46}^2 x_{56}^2} + g_{12}^2 g_{34}g_{35}g_{45}\frac{x_{12}^2 x_{45}^2}{x_{16}^2 x_{26}^2 x_{46}^2 x_{56}^2} + S_5 \text{ permutations}$$

Take aways

- We classified all next-to-maximally nilpotent scalar superconformal invariants
- The OPE condition and the $S_{5+\ell}$ symmetry determine the 5-point correlator for $\ell = 1, 2$ completely
- ? $\ell = 3$ is a technical not a conceptual problem
- ? arbitrary charges

Thank you!