Bubbles in AdS

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AdS/CFT Framework

- Weakly coupled gravity theories in AdS $\leftrightarrow$ strongly coupled CFT on the boundary with many degrees of freedom (Large N) and large gap in the spectrum.
- Bulk dynamics affects the CFT spectrum.
- Scattering information can be extracted from CFT correlators.
- Mellin transform of correlators $\rightarrow$ AdS amplitudes
- Loop expansion in AdS $\leftrightarrow$ $1/N^2$ expansion in CFT

Goals

- Extract a specific contribution to the anomalous dimension of CFT double-trace operators from bulk $\lambda\phi^4$
- Compute the corresponding Mellin amplitude
- Relate it to unitarity cuts of the amplitude

Setup

The **four-point function** of identical scalar operators $\mathcal{O}$ is fixed by conformal symmetry to have the form:

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \frac{\mathcal{G}(u,v)}{x_{12}^{2\Delta_{\mathcal{O}}} x_{34}^{2\Delta_{\mathcal{O}}}} \tag{1}$$

with cross-ratios:

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar{z}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1-z)(1-\bar{z}) \tag{2}$$

- The correlator satisfies **Crossing symmetry**:

$$\mathcal{G}(u,v) = \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} \mathcal{G}(v,u) \tag{3}$$

- It admits a **partial wave expansion**:

$$\mathcal{G}(u,v) = \sum_{\Delta,\ell} a_{\Delta,\ell}\, u^{\frac{\Delta-\ell}{2}}\, G_{\Delta,\ell}(u,v) \tag{4}$$

where $G_{\Delta,\ell}(u,v)$ are the conformal blocks in 4d.

Setup

- The data $\{\Delta, \ell; a_{\Delta,\ell}\}$ from the expansion is known as the **OPE data**.
- At leading order in the large N expansion:

$$\mathcal{O} \times \mathcal{O} = 1 + [\mathcal{O}\mathcal{O}]_{n,\ell} \tag{5}$$

  (assuming a $\mathbb{Z}_2$ symmetry to forbid $\mathcal{O}$ in the OPE)
- $[\mathcal{O}\mathcal{O}]_{n,\ell} \equiv \mathcal{O}\Box^n \partial_{\mu_1} \dots \partial_{\mu_\ell} \mathcal{O}$ are **double-trace operators**. With:

$$\Delta = 2\Delta_{\mathcal{O}} + 2n + \ell + \frac{\gamma^{(1)}_{n,\ell}}{N^2} + \frac{\gamma^{(2)}_{n,\ell}}{N^4} + \dots \tag{6}$$

$$a_{n,\ell} = a^{(0)}_{n,\ell} + \frac{a^{(1)}_{n,\ell}}{N^2} + \frac{a^{(2)}_{n,\ell}}{N^4} + \dots \tag{7}$$

- Bulk interactions induce anomalous dimensions in the CFT.
- For a ϕ^4 theory (no derivatives) in the bulk, crossing symmetry implies:

$$\gamma^{(1)}_{n,\ell} \propto \delta_{\ell,0}$$

Method

$$\mathcal{G}(u,v) = \mathcal{G}^{(0)}(u,v) + \frac{\mathcal{G}^{(1)}(u,v)}{N^2} + \frac{\mathcal{G}^{(2)}(u,v)}{N^4} + \dots \tag{8}$$

- Plugging OPE data expansion into the partial-wave expansion gives

$$\mathcal{G}^{(k)}(u,v) \supset \sum_n \frac{u^{\Delta_{\mathcal{O}}+n}}{2^k k!} a^{(0)}_{n,0} (\gamma^{(1)}_{n,0})^k \log^k u \, G_{2\Delta_{\mathcal{O}}+2n,0}(u,v) \tag{9}$$

- This is the **leading** $\log u$ **term** at order $\frac{1}{N^{2k}}$.
- By crossing symmetry, we also get **leading** $\log v$ **term**:

$$\mathcal{G}^{(k)}(u,v) \supset \sum_n \frac{u^{\Delta_{\mathcal{O}}} v^n}{2^k k!} a^{(0)}_{n,0} (\gamma^{(1)}_{n,0})^k \log^k v \, G_{2\Delta_{\mathcal{O}}+2n,0}(v,u) \tag{10}$$

- **Goal**: Extract contribution of this piece to anomalous dimensions of double trace operators.

Method

Lorentzian Inversion Formula

- Extracts OPE data from the correlator:

$$c_{\Delta,\ell} \propto \int_0^1 dz\, d\bar{z}\, \frac{(z-\bar{z})^2}{z^4\bar{z}^4}\, G_{\ell+3,\Delta-3}(z,\bar{z})\, \mathrm{dDisc}[\mathcal{G}(z,\bar{z})] \tag{11}$$

- Double discontinuity:

$$\mathrm{dDisc}[\mathcal{G}(z,\bar{z})] = \mathcal{G}(z,\bar{z}) - \tfrac{1}{2}\mathcal{G}^{\circlearrowright}(z,\bar{z}) - \tfrac{1}{2}\mathcal{G}^{\circlearrowleft}(z,\bar{z}) \tag{12}$$

- Singularities of $c_{\Delta,\ell}$ encode OPE data:

$$c_{\Delta,\ell} \xrightarrow[\Delta\to\Delta_*]{} \frac{a_{\Delta_*,\ell}}{\Delta-\Delta_*} \tag{13}$$

Anomalous Dimensions

- Fix $\Delta_{\mathcal{O}} = 2$.
- Inserting (10) into the inversion formula (11), and using (13) to read singularities yields anomalous dimensions order by order in $1/N^2$:

$$\gamma^{(2)}_{0,\ell} = -\frac{6}{J^4} - \frac{108}{5J^6} + \dots \tag{14}$$

$$\hat{\gamma}^{(3)}_{0,\ell} = \frac{6(\log J + \gamma)}{J^4} + \frac{924 \log J + 924\gamma - 899}{25J^6} + \dots \tag{15}$$

- $\hat{\gamma}$ denotes the part coming from (10).
- Conformal spin is defined as:

$$J^2 = (\ell + 2)(\ell + 1)$$

Anomalous Dimensions

- At large J:

$$\gamma^{(k)}_{n,\ell} \sim \frac{\log^{k-2} J}{J^4} \tag{16}$$

- Leading log behavior of $\hat{\gamma}$ matches the full anomalous dimension:

$$\hat{\gamma}^{(k)}_{n,\ell}\big|_{\log^{k-2} J} = \gamma^{(k)}_{n,\ell}\big|_{\log^{k-2} J} \tag{17}$$

Resummation

- Expansion in $1/N^2$:

$$\hat{\gamma}_{0,\ell} = \frac{\gamma^{(1)}_{0,\ell}}{N^2} + \frac{\gamma^{(2)}_{0,\ell}}{N^4} + \frac{\hat{\gamma}^{(3)}_{0,\ell}}{N^6} + \dots \tag{18}$$

- Reorganize the expansion by powers of J:

$$\hat{\gamma}_{0,\ell} = \frac{\gamma^{(1)}_{0,\ell}}{N^2} + \frac{\delta^{(1)}_{0,\ell,N}}{J^4} + \frac{\delta^{(2)}_{0,\ell,N}}{J^6} + \dots \tag{19}$$

Anomalous Dimensions

- The $\delta^{(i)}$ pieces can be resummed. For example:

$$\delta^{(1)}_{0,\ell,N} = -\frac{6}{N^4} J^{-1/N^2} \tag{20}$$

$$\delta^{(2)}_{0,\ell,N} = \frac{1}{N^4}\left(4J^{-1/N^2} - \frac{128}{5} J^{-8/(5N^2)}\right) \tag{21}$$

- Suggests a general resummed structure:

$$\hat{\gamma}_{0,\ell} = \frac{\gamma^{(1)}_{0,\ell}}{N^2} + \sum_{n,m} \frac{f_{nm}}{J^{4+2n+2m+\frac{\gamma^{(1)}_{n,0}}{N^2}}} \tag{22}$$

for a set of coefficients f_{nm}.

- So far, we studied the leading $\log v$ term in the CFT correlator and its contribution to leading $\log J$ piece of double-trace anomalous dimensions.
- These contributions are naturally encoded in the **Mellin amplitude** $M(s,t)$, which can be thought of as the AdS scattering amplitude.

Mellin Space

- The inverse Mellin transform of the 4-pt function:

$$\mathcal{G}(u,v) = \frac{1}{(4\pi i)^2} \int_{-i\infty}^{+i\infty} ds\, dt\, M(s,t)\, u^{t/2} v^{(\hat{u}-2\Delta_{\mathcal{O}})/2} \times \Gamma^2\left(\frac{2\Delta_{\mathcal{O}}-t}{2}\right)\Gamma^2\left(\frac{2\Delta_{\mathcal{O}}-s}{2}\right)\Gamma^2\left(\frac{2\Delta_{\mathcal{O}}-\hat{u}}{2}\right) \tag{23}$$

- Poles in t generate leading $\log u$ terms:

$$M(s,t) \supset \sum_{m=0}^{\infty} \frac{R_m^{(q)}(s)}{(t-(2\Delta_{\mathcal{O}}+2m))^q} := M^R_{q\text{-loop}}. \tag{24}$$

Mellin Space

- Coefficients $R_m^{(q)}$ are fixed by matching tree-level data:

$$R_m^{(1)} = -\frac{9(3m+4)4^m(m+1)!^2}{(2m+3)!} \quad \text{(one loop)} \tag{25}$$

- We can find $\gamma_{0,l}^{(k)}$ from $R_m^{(k-1)}$ and vice versa.
- At large m, higher-loop coefficients scale as

$$R_m^{(k)} \propto (R_m^{(1)})^k, \quad m \gg 1 \tag{26}$$

Flat-space limit

The flat space amplitude $A(S_{ij})$ relates to Mellin amplitude by

$$A(S_{ij}) \propto \lim_{R_{\text{AdS}}\to\infty} \int_{-i\infty}^{i\infty} \frac{d\alpha}{2\pi i} \alpha^{\frac{d}{2}-\frac{1}{2}\sum_i \Delta_i} e^{\alpha} \frac{M\left(s_{ij} = \frac{R_{\text{AdS}}^2}{2\alpha} S_{ij}\right)}{R_{\text{AdS}}^{n(1-d)/2+d+1}}. \tag{27}$$

Mellin Space

- At one-loop gives:

$$A_{1-\text{loop}}(T) = \frac{9\pi^3 R_{\text{AdS}}^2}{N^4}\sqrt{-T} \tag{28}$$

  which match the dimensional regularization result for flat space amplitude.
- *n*-**bubble diagram** amplitudes in flat space satisfy:

$$A_{n-\text{bubble}} = \frac{(A_{1-\text{bubble}})^n}{\lambda^{n-1}} \tag{29}$$

- Implying in Mellin space:

$$M_{n-\text{bubble}}(t) \propto \left(M_{1-\text{bubble}}(t)\right)^n, \quad t \gg 1 \tag{30}$$

- So (26) suggests that R-terms are related to **bubble diagrams**.

Mellin Space

- **Unitarity cuts** split diagrams and put cut propagators on-shell.
- The n-bubble diagram is the only diagram at n-loop thay admits n simultaneous **unitarity cuts** in a given channel.
- Proposing that unitarity cuts relate to discontinuities in the 4-pt correlator:

$$\mathrm{Disc}_u \mathcal{G}(u,v) \quad \longleftrightarrow \quad (1 - e^{i\pi t}) M(s,t)$$

- The R-term of the Mellin amplitude corresponds to iterated unitarity cuts:

$$(1 - e^{i\pi t})^{n-1} M^{R}_{n-\mathrm{loop}} \xrightarrow[\text{pole contribution}]{\text{flat space limit}} \mathrm{Disc}_t^{-1}\left(A^{(n)-\mathrm{cut}}_{n-\mathrm{loop}}\right)$$

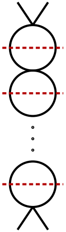
Summary

- We found a **triangular relationship** between:

$$\begin{cases} \text{Leading } \log v \;/\; \log u \text{ terms in the correlator} \\ \text{Leading } \log J \text{ part of double-trace anomalous dimensions} \\ R\text{-terms of Mellin amplitudes} \end{cases}$$

- All these connect to **simultaneous unitarity cuts** of the amplitudes.
- This suggests providing a diagrammatic understanding of the leading $\log u$ term in the correlator.

$$\frac{1}{2^\kappa \kappa!} \log^\kappa u \sum_n u^{\Delta_{\mathcal{O}}+n} a^{(0)}_{n,0} (\gamma^{(1)}_{n,0})^\kappa G_{2\Delta_{\mathcal{O}}+2n}(u,v) \longleftrightarrow R_m^{(\kappa-1)} \longleftrightarrow$$



Thank You!