Non-Invertible Duality Defects and Lattice Symmetries in 2D $c = 2$ CFTs

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Work in progress

The $c = 2$ compact boson

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- Non-linear sigma model with target space $T^2 = \mathbb{R}^2/\Lambda,\ \Lambda \subset \mathbb{R}^2$ non-degenerate lattice.
  - $S = \dfrac{1}{4\pi}\displaystyle\int d^2z\big(\delta^{\mu\nu}G_{ij} + i\epsilon^{\mu\nu}B_{ij}\big)\partial_\mu\phi^i\partial_\nu\phi^j, \quad G = R^2\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & \tau_1^2 + \tau_2^2 \end{pmatrix},\ B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}.$
- Parameters: $(\tau, \rho) \in \mathcal{T}^2$, where $\tau = \tau_1 + i\tau_2$ and $\rho = \rho_1 + i\rho_2 = b + i\sqrt{det(G)}$.
  - Moduli space: $\mathcal{T}^2 = \dfrac{\mathbb{H}_\tau \times \mathbb{H}_\rho}{P(SL(2,\mathbb{Z})_\tau \times SL(2,\mathbb{Z})_\rho) \rtimes (\mathbb{Z}_2^M \times \mathbb{Z}_2^I)} = (\mathbb{H}_\tau \times \mathbb{H}_\rho)/O(2,2,\mathbb{Z}).$
- Continuous global symmetry: $U(1)^2_{\mathbf{n}} \times U(1)^2_{\mathbf{w}}$, $\mathbf{n}$ and $\mathbf{w}$ are the momentum and winding numbers.

The $c = 2$ compact boson

- Spectrum: $\Gamma^{2,2} \ni \begin{pmatrix} \boldsymbol{p} \\ \bar{\boldsymbol{p}} \end{pmatrix} = \frac{1}{\sqrt{2\tau_2\rho_2}} \begin{pmatrix} \tau_2 & 0 & \rho_2 & \rho_1\tau_2 + \rho_2\tau_1 \\ -\tau_1 & 1 & -\rho_1 & -\rho_1\tau_1 + \rho_2\tau_2 \\ \tau_2 & 0 & -\rho_2 & \rho_1\tau_2 - \rho_2\tau_1 \\ -\tau_1 & 1 & -\rho_1 & -\rho_1\tau_1 - \rho_2\tau_2 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ w_1 \\ w_2 \end{pmatrix} = L \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{w} \end{pmatrix}.$

- Partition function: $Z(q, \bar{q}) = \frac{1}{\eta^2\bar{\eta}^2} \sum_{(p,\bar{p}) \in \Gamma^{2,2}} q^{\frac{1}{2}p^2} \bar{q}^{\frac{1}{2}\bar{p}^2}.$

- RCFT $\Longleftrightarrow \tau,\ \rho \in \mathbb{Q}\left(\sqrt{D}\right),\ D < 0$, i.e. complex multiplication on the torus. [1]
    - RCFTs are enhanced symmetry points in the moduli space.

- The generalised metric: $\mathscr{E} = \begin{pmatrix} G^{-1} & G^{-1}B \\ -BG^{-1} & G - BG^{-1}B \end{pmatrix}$ s.t. $\left(\boldsymbol{n}^T \quad \boldsymbol{w}^T\right) \mathscr{E} \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{w} \end{pmatrix} = \left(\boldsymbol{p}^T \quad \bar{\boldsymbol{p}}^T\right) \begin{pmatrix} \boldsymbol{p} \\ \bar{\boldsymbol{p}} \end{pmatrix} \implies \mathscr{E} = L^T L$

[1] Gukov-Vafa (2002), arXiv:hep-th/0203213

Non-Invertible symmetries for $c = 2$

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- Gauging of a finite subgroup $G \subset U(1)^2_{\mathbf{n}} \times U(1)^2_{\mathbf{w}}$ can be represented as a matrix $\sigma \in O(2,2,\mathbb{Q})$ and the overall action is: $(\tau, \rho) \longrightarrow (\tau', \rho')$.

    - If the resulting theory is dual to the original one $\Longrightarrow$ combining the gauging of G with a duality transformation (an element of $O(2,2,\mathbb{Z})$) leaves the parameters of the theory invariants; we call this a duality symmetry.

- Half-space gauging $\Longrightarrow$ existence of a topological defect line $\mathscr{D}$ which implements the duality symmetry. In general, it is a non-invertible duality defect, defining a Tambara-Yamagami fusion category.

Non-Invertible symmetries for $c = 2$

- Given a duality symmetry $\mathscr{D} \implies$ there exists $D \in O(2,2,\mathbb{Q})$ s.t.: $D^T \mathscr{E} D = \mathscr{E}$. [2]

- The action on primary operators of $\mathscr{D}$: $\begin{cases} V_{n,w} \to \sqrt{|G|}(-1)^\alpha V_{n',w'} \text{ if } n_i', w_i' \in \mathbb{Z} \\ V_{n,w} \longrightarrow \text{non genuine op. if } n_i', w_i' \notin \mathbb{Z} \end{cases}$, $\begin{pmatrix} \mathbf{n}' \\ \mathbf{w}' \end{pmatrix} = D \begin{pmatrix} \mathbf{n} \\ \mathbf{w} \end{pmatrix}$

- In particular: $\begin{cases} D \in O(2,2,\mathbb{Z}) \implies \text{invertible symmetry} \\ D \in O(2,2,\mathbb{Q}) \backslash O(2,2,\mathbb{Z}) \implies \text{non-invertible symmetry [3]} \end{cases}$

- $\mathscr{E} = L^T L \implies L \to L' = LD$ s.t. $L'^T L' = \mathscr{E}$

[2] Aguilera Damia-Galati-Hulik-Mancani (2024), arXiv:2401.04166

[3] Bharadwaj–Niro–Roumpedakis (2024), arXiv:2408.14556

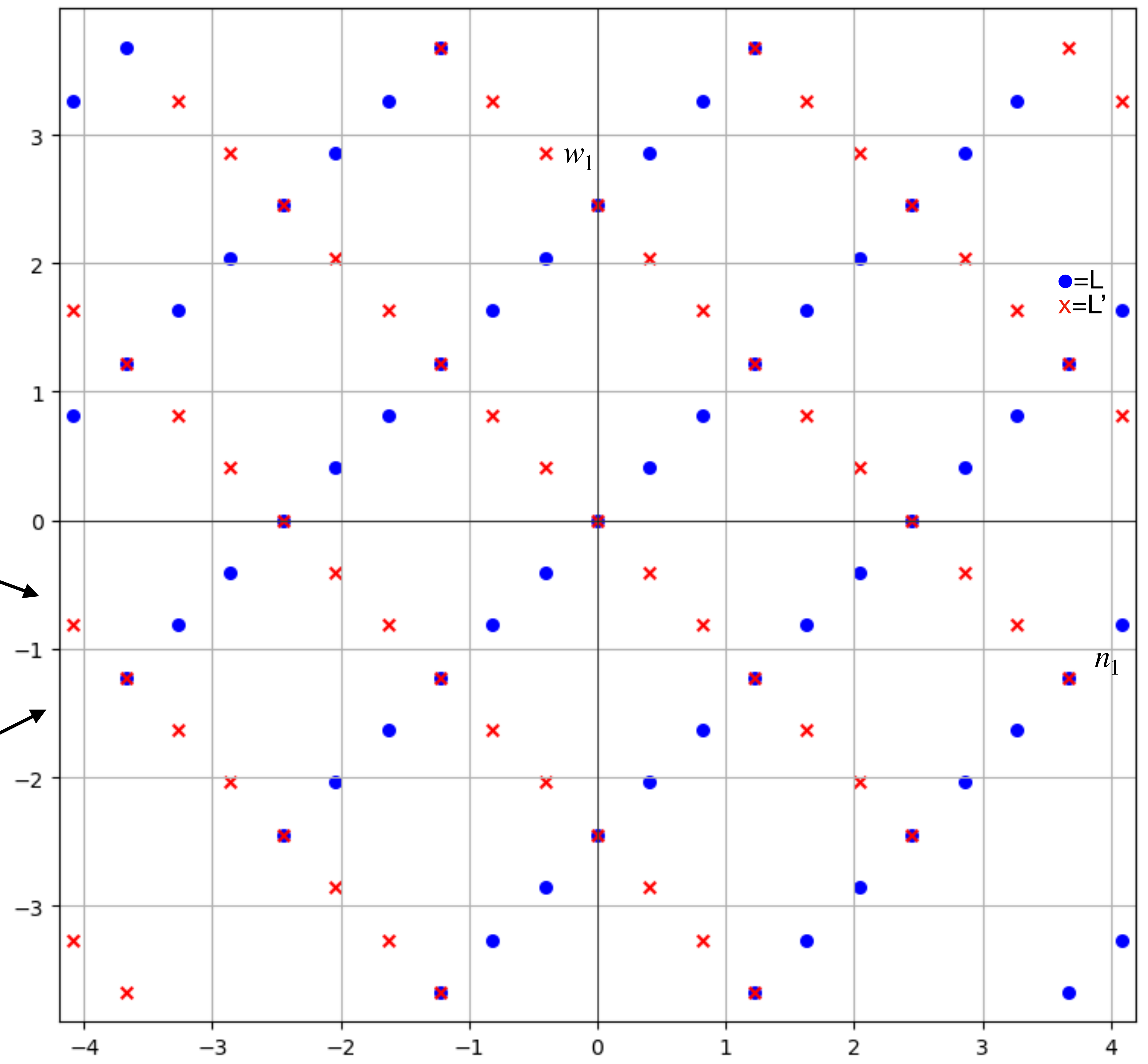
Non-Invertible symmetries for $c = 2$: an example

- $(\tau, \rho) = (i, 3i)$:

$$L = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & -3 \end{pmatrix} \quad \text{and} \quad \mathscr{E} = \begin{pmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

- Gauging $\mathbb{Z}_3 \times \mathbb{Z}_3 \subset U(1)_{n_1} \times U(1)_{n_2}$ takes the form:

$$D = \begin{pmatrix} 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \\ -1/3 & 0 & 0 & 0 \\ 0 & -1/3 & 0 & 0 \end{pmatrix} \Longrightarrow L \to L' = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 & 0 & -3 & 0 \\ 0 & -1 & 0 & -3 \\ 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & -3 \end{pmatrix}$$



- $L' = LD = SL = \begin{pmatrix} R(\pi) & 0 \\ 0 & Id \end{pmatrix} L$. In particular, $R(\pi)$ is a symmetry of Λ, where $T^2 = \mathbb{R}^2/\Lambda$.

The discrete symmetry groups of the torus

- In general: $L' = LD = SL = \begin{pmatrix} R & 0 \\ 0 & Id \end{pmatrix} L$, s.t. R generates G, a symmetry of Λ.

- There are 17 possible crystallographic space groups in two dimensions. [4]

  - Orbifolding procedure $\Longrightarrow$ 28 non-exceptional irreducible components in the $c = 2$ moduli space.

- Some examples:

$$R(\pi) : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix}, \ \forall\ (\tau, \rho) \in \mathcal{T}^2, \ \ R\left(\frac{2\pi}{3}\right) : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} \cos\left(\frac{2\pi}{3}\right) & -\sin\left(\frac{2\pi}{3}\right) \\ \sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \text{ for } \tau = e^{2\pi i/3}, \ \ R_1 : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix}, \text{ for } \tau_1 = 0, \frac{1}{2}.$$

[4] Dulat-Wendland (2000), arXiv: hep-th/0002227

Non-inv. defects from lattice symmetries analysis

- We can reverse the process in order to construct the non-invertible defects:

Consider a $c = 2$ compact boson, with lattice generator matrix $L(\tau, \rho)$, and with a discrete symmetry group G, generated by R.

If $D = L^{-1}\begin{pmatrix} R & 0 \\ 0 & Id \end{pmatrix} L \in O(2,2,\mathbb{Q})\backslash O(2,2,\mathbb{Z}) \implies D$ corresponds to a non-invertible duality symmetry.

- E.g.: $(\tau, \rho) = \left(e^{2\pi i/3}, -1/2 + i3\sqrt{3}/2\right)$. Discrete symmetry: $R\left(\frac{2\pi}{3}\right)$.

$$L = \begin{pmatrix} \frac{\sqrt{6}}{6} & 0 & \frac{\sqrt{6}}{2} & -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{2}}{6} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{6} & \frac{2\sqrt{2}}{3} \\ \frac{\sqrt{6}}{6} & 0 & -\frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{2}}{6} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{6} & -\frac{5\sqrt{2}}{6} \end{pmatrix} \longrightarrow L' = \begin{pmatrix} R & 0 \\ 0 & Id \end{pmatrix} L = \begin{pmatrix} -\frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{3} & -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{2}}{6} & -\frac{\sqrt{2}}{6} & \frac{2\sqrt{2}}{3} & -\frac{5\sqrt{2}}{6} \\ \frac{\sqrt{6}}{6} & 0 & -\frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{2}}{6} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{6} & -\frac{5\sqrt{2}}{6} \end{pmatrix} \implies D = \begin{pmatrix} 0 & -\frac{2}{3} & -\frac{7}{3} & 0 \\ \frac{2}{3} & \frac{2}{3} & \frac{7}{3} & -\frac{7}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

Summary

- 2D $c = 2$ CFT specified by $(\tau, \rho) \implies \Gamma^{2,2} \ni \begin{pmatrix} \mathbf{p} \\ \bar{\mathbf{p}} \end{pmatrix} = L(\tau, \rho) \begin{pmatrix} \mathbf{n} \\ \mathbf{w} \end{pmatrix} \implies \mathscr{E} = L^T L$

- $\mathscr{D}$ is a non-invertible duality defect $\iff D \in O(2,2,\mathbb{Q}) \backslash O(2,2,\mathbb{Z})$ and $L' = LD$ s.t. $L'^T L' = \mathscr{E}$

- $L' = LD = SL = \begin{pmatrix} R & 0 \\ 0 & Id \end{pmatrix} L$, s.t. R generates G, a symmetry of Λ.

- Classification of the crystallographic space groups in 2D $\implies$ infer the duality defects from the lattice symmetries.

- From G we know how the Ishibashi states transform $\implies$ we can obtain the Dp-branes in the orbifolded theory.

Thank you for your attention.