High-Precision Analytic Continuation of Multivariable Hypergeometric Functions and Prospects for Feynman Integral Evaluation

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Bezuglov, M. A., Kniehl, B. A., Onishchenko, A. I., & Veretin, O. L. (2025). Computer Physics Communications, 109812.

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Why Hypergeometric Functions?

1) Loop Feynman integrals

$$\int \dots \int \frac{d^4k_1 \dots d^4k_n}{D_1^{j_1} \dots D_l^{j_l}}, \qquad D_r = \sum_{i \geq j \geq 1} A_r^{ij} p_i p_j - m_r^2$$

2) Euler Integrals

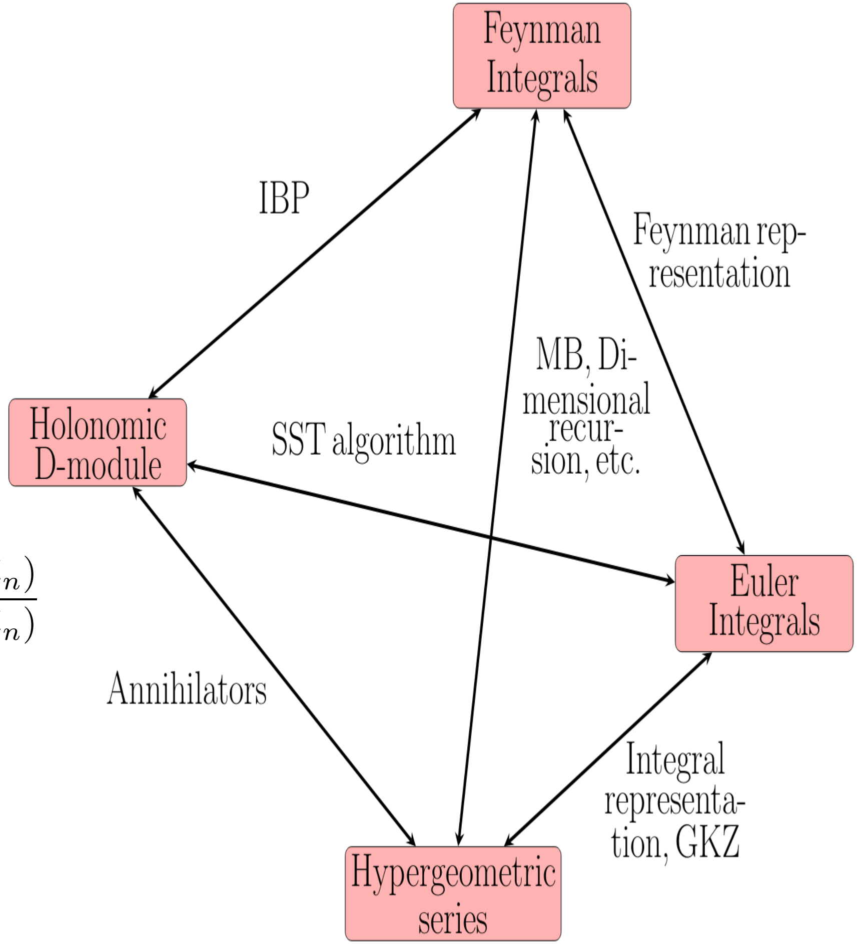
$$\int_\Gamma \frac{x_1^{\nu^1} \cdots x_n^{\nu^n}}{f_1^{s^1} \cdots f_\ell^{s^\ell}} \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n} = \int_\Gamma f^{-s} x^\nu \frac{dx}{x}.$$

3) Hypergeometric series

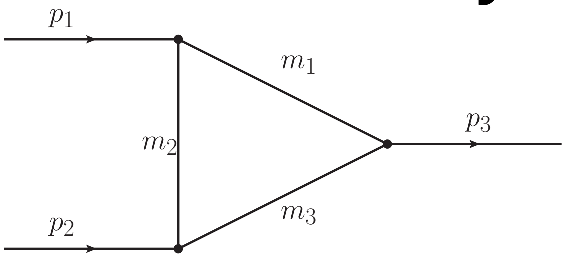
$$\sum A_{i_1,i_2,\dots,i_n} x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}, \qquad \frac{A_{i_1,\dots,i_k+1,\dots,i_n}}{A_{i_1,\dots,i_k,\dots,i_n}} = \frac{P_k(i_1,i_2,\dots,i_n)}{P'_k(i_1,i_2,\dots,i_n)}$$

4) Holonomic D-modules

Parametric systems of partial differential equations with polynomial coefficients



Why Hypergeometric Functions?



$$= \frac{\Gamma\left(2-\frac{d}{2}\right)}{2\left(r_{123}-r_{23}\right)} r_{123}^{\frac{d}{2}-2}\Bigg\{ {}_2F_1\left(1, \tfrac{1}{2}; \tfrac{3}{2}; \frac{r_3-r_{23}}{r_{123}-r_{23}}\right) - \left(\frac{r_{23}}{r_{123}}\right)^{\frac{d}{2}-2} F_1\left(\tfrac{1}{2}, 1, 2-\tfrac{d}{2}; \tfrac{3}{2}; \frac{r_{23}-r_3}{r_{23}-r_{123}}, 1-\frac{r_3}{r_{23}}\right)\Bigg\}.$$

O. V. Tarasov, JHEP, vol. 06, p. 155, 2022

$$F_1(\alpha,\beta_1,\beta_2,\gamma;x,y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta_1)_m(\beta_2)_n}{(\gamma)_{m+n}m!n!} x^m y^n, \qquad |x|<1,\ |y|<1$$

$$F_1(\alpha,\beta_1,\beta_2,\gamma;x,y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-xt)^{-\beta_1}(1-yt)^{-\beta_2}\,\mathrm{d}t, \quad \Re\gamma > \Re\alpha > 0\ .$$

$$\begin{cases} \left[x(1-x)\dfrac{\partial^2}{\partial x^2} + y(1-x)\dfrac{\partial^2}{\partial x\partial y} + [\gamma-(\alpha+\beta_1+1)x]\dfrac{\partial}{\partial x} - \beta_1 y\dfrac{\partial}{\partial y} - \alpha\beta_1\right] F_1 = 0, \\ \left[y(1-y)\dfrac{\partial^2}{\partial y^2} + x(1-y)\dfrac{\partial^2}{\partial x\partial y} + [\gamma-(\alpha+\beta_2+1)y]\dfrac{\partial}{\partial y} - \beta_2 x\dfrac{\partial}{\partial x} - \alpha\beta_2\right] F_1 = 0. \end{cases}$$

Existing methods of numerical calculation of hypergeometric functions

$$ {}_{q+1}F_q\left(\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\middle| z\right) = \sum_{k=1}^{q+1} C_k(-z)^{-a_k}\ {}_{q+2}F_{q+1}\left(\begin{matrix} 1, a_k, 1+a_k-b_1, 1+a_k-b_2,\dots,1+a_k-b_q \\ 1+a_k-a_1, 1+a_k-a_2,\dots,1+a_k-a_{p+1} \end{matrix}\middle| \frac{1}{z}\right) $$

$$ \begin{aligned} &F_1(a,b_1,b_2;c|x,y) \\ &= (-x)^{-b_1}(-y)^{b_1-a}\frac{\Gamma(c)\Gamma(a-b_1)\Gamma(-a+b_1+b_2)}{\Gamma(a)\Gamma(b_2)\Gamma(c-a)} G_2\left(b_1, a-c+1, a-b_1, -a+b_1+b_2\middle| -\frac{y}{x}, -\frac{1}{y}\right) \\ &+ (-x)^{-a}\frac{\Gamma(c)\Gamma(b_1-a)}{\Gamma(b_1)\Gamma(c-a)} F_1\left(a, a-c+1, b_2; a-b_1+1\middle| \frac{1}{x}, \frac{y}{x}\right) \\ &+ (-x)^{-b_1}(-y)^{-b_2}\frac{\Gamma(c)\Gamma(a-b_1-b_2)}{\Gamma(a)\Gamma(c-b_1-b_2)} F_1\left(b_1+b_2-c+1, b_1, b_2; -a+b_1+b_2+1\middle| \frac{1}{x}, \frac{1}{y}\right) \end{aligned} $$

$$ \frac{1}{|x|} < 1 \wedge \frac{1}{|y|} < 1 \wedge \left|\frac{y}{x}\right| < 1. $$

- `AppellF1.wl`: 28
- `AppellF3.wl`: 24
- `LauricellaFD.wl`: 96
- `LauricellaSaranFS.wl`: 102

P. O. Olsson, Journal of Mathematical Physics, vol. 5, no. 3, pp. 420–430, 1964.

S. Bera and T. Pathak, Comput. Phys. Commun., vol. 306, p. 109386, 2025, 2403.02237.

Appell F1 function

$$F_1(\alpha, \beta_1, \beta_2, \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta_1)_m(\beta_2)_n}{(\gamma)_{m+n} m! n!} x^m y^n, \qquad |x| < 1,\ |y| < 1$$

$$\left[x(1-x)\frac{\partial^2}{\partial x^2} + y(1-x)\frac{\partial^2}{\partial x \partial y} + [\gamma - (\alpha + \beta_1 + 1)x]\frac{\partial}{\partial x} - \beta_1 y \frac{\partial}{\partial y} - \alpha\beta_1\right] F_1 = 0,$$

$$\left[y(1-y)\frac{\partial^2}{\partial y^2} + x(1-y)\frac{\partial^2}{\partial x \partial y} + [\gamma - (\alpha + \beta_2 + 1)y]\frac{\partial}{\partial y} - \beta_2 x \frac{\partial}{\partial x} - \alpha\beta_2\right] F_1 = 0.$$

$$J_1 = \left\{F_1, x\frac{\partial}{\partial x}F_1, y\frac{\partial}{\partial y}F_1\right\},$$

$$\frac{\partial}{\partial x}J_1 = \left(\frac{\mathbf{A}_0}{x} + \frac{\mathbf{A}_1}{x-1} + \frac{\mathbf{A}_y}{x-y}\right) J_1$$

$$\frac{\partial}{\partial y}J_1 = \left(\frac{\mathbf{B}_0}{y} + \frac{\mathbf{B}_1}{y-1} + \frac{\mathbf{B}_x}{y-x}\right) J_1$$

Method components

Our goal is to adapt—and actually modify—the Frobenius method for analytic continuation of hypergeometric functions. While Frobenius-based implementations have existed for Feynman integrals (*DiffExp, SeaSyde, etc.*), none have been purpose-built for hypergeometric systems.

1) Solution in the neighborhood of a singular point.

2) The case of many variables.

3) Analytical continuation.

4) Practical applications, Hypergeometric functions of several variables.

Solution in the neighborhood of a singular point

$$\frac{dJ}{dx} = Mx, \qquad M = \frac{A_0}{x} + R(x)$$

$$U = \sum_{\lambda \in S} x^{\lambda} U^{\lambda} = \sum_{\lambda \in S} x^{\lambda} \sum_{n=0}^{\infty} \sum_{k=0}^{m_\lambda} c_n(\lambda, k) x^n \log^k(x)$$

the set S consists of the non-degenerate eigenvalues of the matrix A_0. If several eigenvalues are degenerate, the set S will include only the smallest of them.

suppose the set of eigenvalues of the matrix A_0 is

$$\{0, -1, -2, 1-\varepsilon, 1-2\varepsilon, -2\varepsilon, 1/2-\varepsilon, 1/2-\varepsilon\}$$

then S = $\{$-2,1-$\varepsilon, -2\varepsilon, 1/2-\varepsilon\}$ and

$$\mathrm{m}_{-2} = 2,\ m_{1-\varepsilon} = 0,\ m_{-2\varepsilon} = 1 \text{ and } \ m_{1/2-\varepsilon} = 1.$$

Solution in the neighborhood of a singular point

$$l_\lambda = \max(s \cap \mathbb{Z})$$

$$\text{where s is the set of solutions to the polynomial equation } \det\left(A_0 - (n+\lambda)\right) = 0$$

$$c_n(\lambda, k) = -\left(A_0 - (n+\lambda)\right)^{-1}\left[\hat{M}_1 c_n(\lambda, k) - \hat{Q}_1 (n+\lambda) c_n(\lambda, k) - (k+1)\hat{Q} c_n(\lambda, k+1)\right],$$

$$n > l_\lambda$$

$$\hat{M}_1 c_n(\lambda, k) - \hat{Q}_1 (n+\lambda) c_n(\lambda, k) - (k+1)\hat{Q} c_n(\lambda, k+1) + \left(A_0 - (n+\lambda)\right) c_n(\lambda, k) = 0,$$

$$0 \leq n \leq l_\lambda$$

$$\text{if } H = \sum_{n=0}^{\infty} h_n x^n, \qquad \text{then} \qquad \hat{H} c_n = \sum_{i=0}^{n} c_i h_{n-i}.$$

Multiple variable case

$$dJ = \omega J$$

$$\omega = M_1 dx_1 + M_2 dx_2 + \cdots + M_i dx_i$$

integrability condition

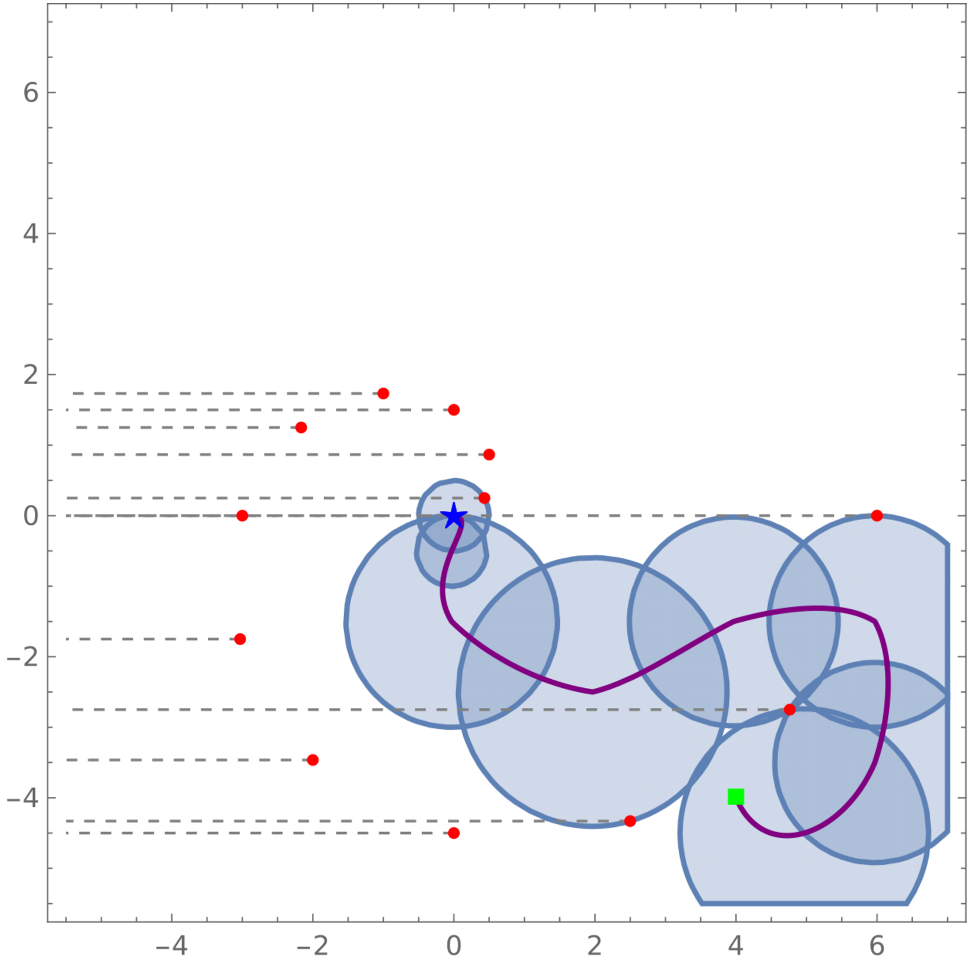
$$d\omega = \omega \wedge \omega.$$

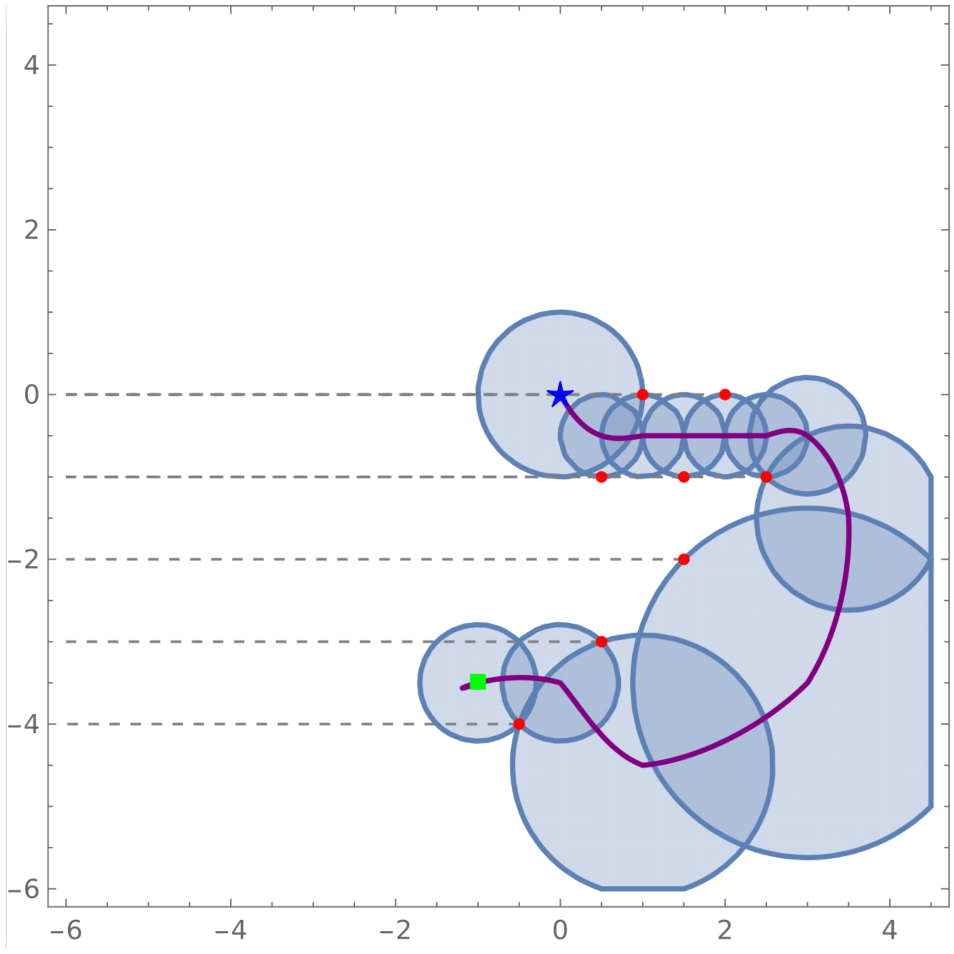
the problem can be reduced to a problem of a lower dimension

$$x_1 = f_1(t), x_2 = f_2(t), \ldots, x_i = f_i(t).$$

$$\frac{dJ}{dt} = M_t J, \qquad M_t = \frac{\partial x_1}{\partial t} M_1 + \frac{\partial x_2}{\partial t} M_2 + \cdots + \frac{\partial x_i}{\partial t} M_i.$$

Analytic continuation





Lauricella functions

$$F_A^{(n)}(\alpha;\beta_1,\dots,\beta_n;\gamma_1,\dots,\gamma_n;x_1,\dots,x_n)= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\dots(\beta_n)_{m_n}}{(\gamma_1)_{m_1}\dots(\gamma_n)_{m_n}m_1!\dots m_n!}x_1^{m_1}\dots x_n^{m_n},$$

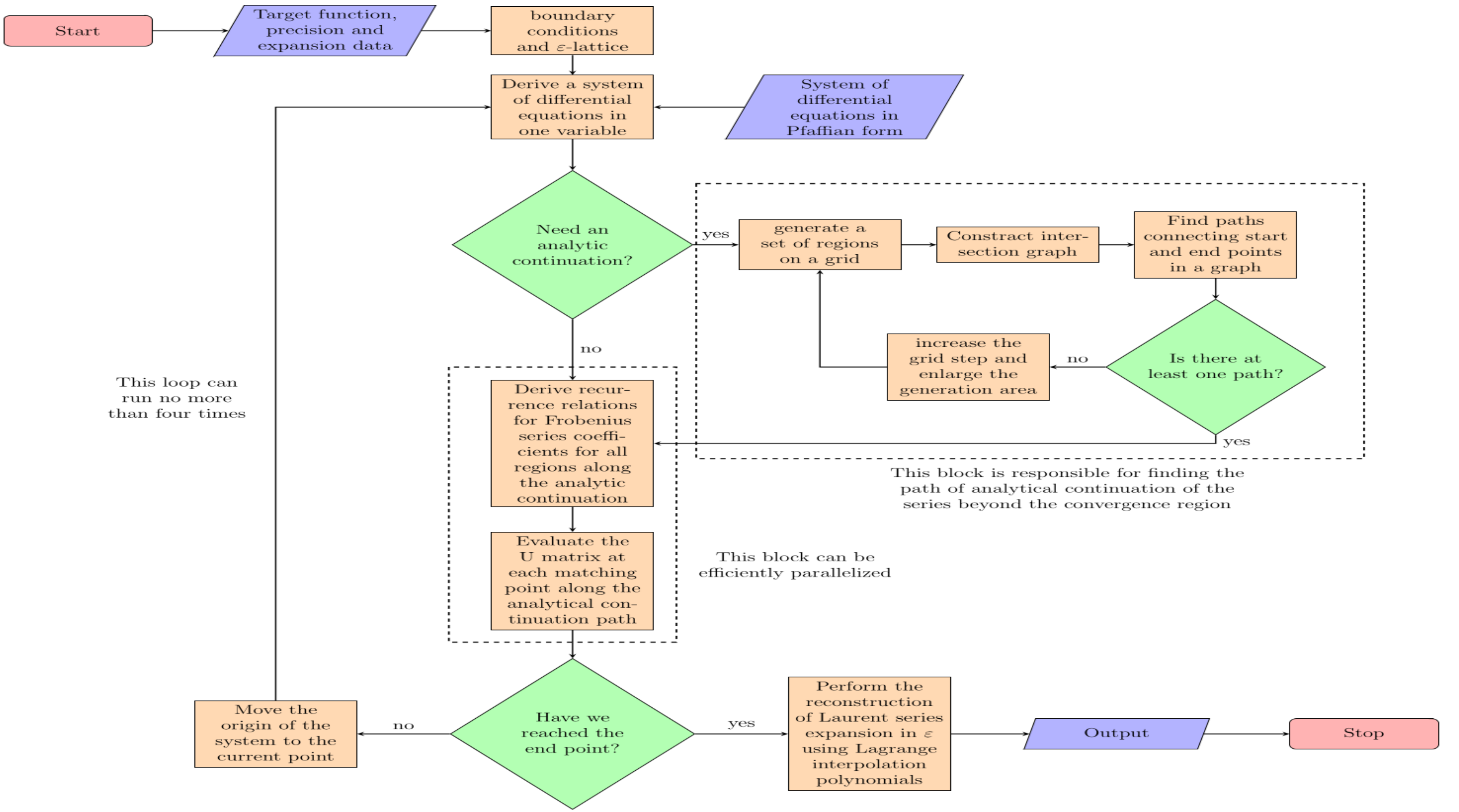
$$F_B^{(n)}(\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_n;\gamma;x_1,\dots,x_n)= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(\alpha_1)_{m_1}\dots(\alpha_n)_{m_n}(\beta_1)_{m_1}\dots(\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}m_1!\dots m_n!}x_1^{m_1}\dots x_n^{m_n},$$

$$F_D^{(n)}(\alpha;\beta_1,\dots,\beta_n;\gamma;x_1,\dots,x_n)= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\cdots+m_n}(\beta_1)_{m_1}\dots(\beta_n)_{m_n}}{(\gamma)_{m_1+\cdots+m_n}m_1!\dots m_n!}x_1^{m_1}\dots x_n^{m_n}.$$

$$\left\{\theta_{x_{j_1}}\dots\theta_{x_{j_k}}F_i \;\middle|\; 0\le k\le n,\ j_1<j_2<\cdots<j_k\right\}, \qquad i=A,B, \qquad \theta_x = x\frac{d}{dx}$$

$$\left\{F_D,\ \theta_{x_j}F_D \;\middle|\; j=1,\dots,n\right\},$$

Implemented in the PrecisionLauricella package using Wolfram Mathematica



Conclusions

New Numerical Method:
Developed a high-precision evaluation method for Lauricella functions.

One-Dimensional Frobenius Series:
Utilized one-dimensional generalized power series for analytic continuation, significantly simplifying and accelerating high-precision computations compared to multidimensional approaches.

Efficiency and Parallelism:
The specialized treatment of ε-dependence enhances computational speed and is well-suited for parallel processing in large-scale calculations.

Practical Implementation:
Implemented in the PrecisionLauricella package using Wolfram Mathematica, and successfully validated against alternative hypergeometric function evaluation tools.

https://bitbucket.org/BezuglovMaxim/precisionlauricella-package/src/main/

Future Directions:
Potential extensions to other classes of hypergeometric functions and direct applications in Feynman integral computations.

Thank you for your attention!