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Non-holomorphic modular forms as iterated integrals

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based on WIP with Oliver Schlotterer and Yi-Xiao Tao

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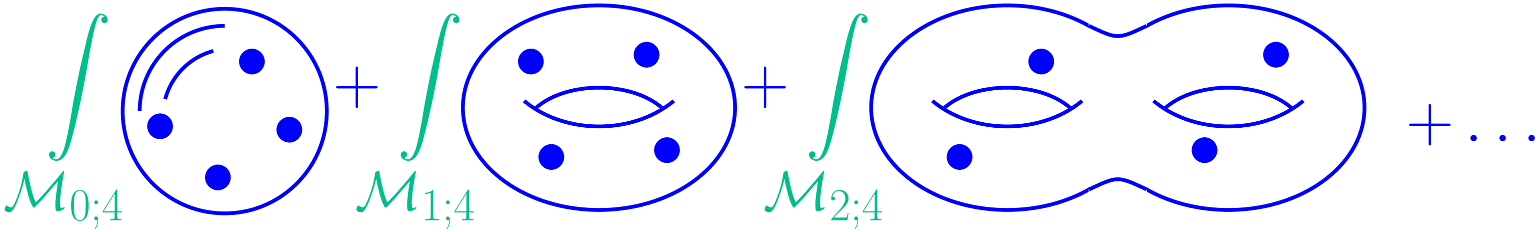
Physics Motivation

- Perturbative expansion of string amplitudes leads to decomposition in integrals over the moduli space of punctured Riemann surfaces:



- At genus zero the integrals evaluate to single valued multiple zeta values.

- At genus one, some integrands are **(elliptic) modular graph forms**:

$$\text{E.g.:} \qquad \mathcal{D}\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](\underbrace{u\tau + v}_{z}, \tau) = \frac{(\mathrm{Im}\tau)^a}{\pi^b} \sum_{(m,n)\neq(0,0)} \frac{e^{2\pi i(nu - mv)}}{(n + m\tau)^a (n + m\bar{\tau})^b}$$

- Understanding analytic structure essential to carry out all integrations.

- **Objective: Express them as iterated integrals!**

Modular forms:

- Functions which transform nicely under rescalings of a torus with periods ψ_1, ψ_2.

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \to \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \qquad \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in SL_2(\mathbb{Z}) \tag{1}$$

- The normalised period $\tau = \frac{\psi_2}{\psi_1}$ and marked points z transform as

$$\tau \to \frac{a\tau+b}{c\tau+d}, \quad z \to \frac{z}{c\tau+d}, \qquad \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in SL_2(\mathbb{Z}) \tag{2}$$

- Eisenstein series G_k modular forms of weight k:

$$\mathrm{G}_k(\tau) = \sum_{(m,n)\neq(0,0)} \frac{1}{(n+m\tau)^k}, \quad \mathrm{G}_k(\tau) \to \mathrm{G}_k\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \mathrm{G}_k(\tau)$$

Modular forms:

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- We can deform G_k to depend on $z = u\tau + v$.

$$\mathrm{G}_k(\tau) = \sum_{(m,n)\neq(0,0)} \frac{1}{(n+m\tau)^k}\,, \quad f^{(k)}(u\tau+v,\tau) = (-)^{k-1} \sum_{(m,n)\neq(0,0)} \frac{e^{2\pi i(mv-nu)}}{(n+m\tau)^k}$$

- functions $f^{(k)}$ are still is a modular form:

$$f^{(k)}(u\tau+v,\tau) \to f^{(k)}\left(\frac{u\tau+v}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f^{(k)}(u\tau+v,\tau)\,.$$

- **Question:** Are integrals of these modular forms still modular forms?

NO! $$\int_\tau^{i\infty} \mathrm{d}\tau_1 f^{(2)}(u\tau_1+v,\tau_1) \to \int_\tau^{i\infty} \mathrm{d}\tau_1 f^{(2)}(u\tau_1+v,\tau_1) + \int_{a/c}^{i\infty} \mathrm{d}\tau_1 f^{(2)}(u\tau_1+v,\tau_1)$$

Modular forms:

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- However, linear combinations are modular again!

$$\beta^{\mathrm{eqv}}\left[\begin{smallmatrix}0\\2\\z\end{smallmatrix}\right] := \int_\tau^{i\infty} \mathrm{d}\tau_1 f^{(2)}(u\tau_1+v,\tau_1) + \int_{\bar\tau}^{-i\infty} \mathrm{d}\bar\tau_1 \overline{f^{(2)}(u\tau_1+v,\tau_1)}$$
$$\to \beta^{\mathrm{eqv}}\left[\begin{smallmatrix}0\\2\\z\end{smallmatrix}\right]$$

- **Nice feature:** decouple holomorphic and anti-holomorphic in eMGFs!

E.g.: $$\mathcal{D}[\begin{smallmatrix}1\\1\end{smallmatrix}](\underbrace{u\tau+v}_{z},\tau) = \frac{(\tau-\bar\tau)}{2\pi i}\sum_{(m,n)\neq(0,0)} \frac{e^{2\pi i(nu-mv)}}{(n+m\tau)(n+m\bar\tau)} = -\beta^{\mathrm{eqv}}\left[\begin{smallmatrix}0\\2\\z\end{smallmatrix}\right]$$

- **Our results:** Whole class of such modular forms as iterated integrals:

$$\beta^{\mathrm{eqv}}\left[\begin{smallmatrix}j_1 & \dots & j_r\\ k_1 & \dots & k_r\\ z & \dots & z\end{smallmatrix};\tau\right] \to \left(\prod_{i=1}^{r}(c\bar\tau+d)^{k_i-2-2j_i}\right)\beta^{\mathrm{eqv}}\left[\begin{smallmatrix}j_1 & \dots & j_r\\ k_1 & \dots & k_r\\ z & \dots & z\end{smallmatrix};\tau\right]$$

elliptic Modular Graph forms (eMGFs):

- At depth one only need iterated integrals and complex conjugate:

$$\beta^{\mathrm{eqv}}\left[\begin{smallmatrix} j \\ k \\ z \end{smallmatrix}\right] = \int_{\tau}^{i\infty} \mathrm{d}\tau_1 (\dots) \tau_1^j f^{(k)}(u\tau_1 + v, \tau_1) + \int_{\bar\tau}^{-i\infty} \mathrm{d}\bar\tau_1 (\dots) \bar\tau_1^j \overline{f^{(k)}(u\tau_1 + v, \tau_1)}$$

- At depth two encounter **multiple zeta values** and **Bernoulli polynomials** as coefficients:

$$\beta^{\mathrm{eqv}}\left[\begin{smallmatrix} 0 & 2 \\ 2 & 4 \\ z & z \end{smallmatrix}\right] = \int_{\tau}^{i\infty} \mathrm{d}\tau_1 (\dots) \tau_1^2 f^{(4)}(u\tau_1 + v, \tau_1) \int_{\tau_1}^{i\infty} \mathrm{d}\tau_2 (\dots) f^{(2)}(u\tau_2 + v, \tau_2) + \int_{-i\infty}^{\bar\tau} \mathrm{d}\bar\tau_1 (\dots) \overline{f^{(2)}(u\tau_1 + v, \tau_1)} \int_{-i\infty}^{\bar\tau_1} \mathrm{d}\bar\tau_2 (\dots) \bar\tau_2^2 \overline{f^{(4)}(u\tau_2 + v, \tau_2)} + \frac{2}{3}\zeta_3 \left(-\frac{B_2(u)}{2}(-2\pi i \bar\tau) + \int_{-i\infty}^{\bar\tau} \mathrm{d}\bar\tau_1 (\dots) \overline{f^{(2)}(u\tau_1 + v, \tau_1)} \right) + \dots$$

Generating series for eMGFs

- **Main result:** A generating series for eMGFs:

$$\mathbb{I}^{\text{eqv}} = (\mathbb{M}_{\text{Z}}^{\text{sv}})^{-1}\, \widetilde{\mathbb{I}_{-}}\, \mathbb{M}_{\Sigma}^{\text{sv}}\, \mathbb{I}_{+}$$

- Can be expanded in free Lie algebra generators $b_k^{(j)}$:

$$U\mathbb{I}^{\text{eqv}}U^{-1} = 1 + \sum_{k=2}^{\infty}\sum_{j=0}^{k-2}\frac{(-1)^j(k-1)}{j!}\beta^{\text{eqv}}\left[\begin{smallmatrix} j \\ k \\ z \end{smallmatrix};\tau\right] b_k^{(j)}$$
$$+\sum_{k_1,k_2\geq 2}\sum_{j_1,j_2\geq 0}\frac{(-1)^{j_1+j_2}(k_1-1)(k_2-1)}{j_1!j_2!}\beta^{\text{eqv}}\left[\begin{smallmatrix} j_1 & j_2 \\ k_1 & k_2 \\ z & z \end{smallmatrix};\tau\right] b_{k_2}^{(j_2)}b_{k_1}^{(j_1)} + \dots$$

- with $\mathbb{I}_{+} \to M_{+}^{\text{bad}}\,\mathbb{I}_{+}\,M_{+}^{\text{good}}$ and $\widetilde{\mathbb{I}_{-}} \to M_{-}^{\text{good}}\,\widetilde{\mathbb{I}_{-}}\,M_{-}^{\text{bad}}$ such that

$$\mathbb{I}^{\text{eqv}} \to M_{-}^{\text{good}}\,(\mathbb{M}_{\text{Z}}^{\text{sv}})^{-1}\,\widetilde{\mathbb{I}_{-}}\,\underbrace{M_{-}^{\text{bad}}\,\mathbb{M}_{\Sigma}^{\text{sv}}\,M_{+}^{\text{bad}}}_{\mathbb{M}_{\Sigma}^{\text{sv}}}\,\mathbb{I}_{+}\,M_{+}^{\text{good}}$$

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- $\mathbb{I}_{+}$ is the generating series for the iterated integrals in $\tau^{j} f^{(k)}$ is solution to a differential system $\partial_{\tau}\mathbb{I}_{+} = -\mathbb{I}_{+}\mathcal{K}_{\tau}$ which is solved through

$$\mathbb{I}_{+} = \mathcal{P}\exp\left(\int_{\tau}^{i\infty} \mathrm{d}\tau_1 \mathcal{K}_{\tau_1}\right) = \mathbb{1} + \int_{\tau}^{i\infty} \mathrm{d}\tau_1 \mathcal{K}_{\tau_1} + \int_{\tau}^{i\infty} \mathrm{d}\tau_1 \mathcal{K}_{\tau_1} \int_{\tau_1}^{i\infty} \mathrm{d}\tau_2 \mathcal{K}_{\tau_2} + \ldots$$

with the connection reading:

$$\mathcal{K}_{\tau} = \sum_{k=2}^{\infty} \sum_{j=0}^{k-2} \frac{k-1}{j!} \tau^{j} \left(\mathrm{G}_k(\tau)\epsilon_k^{(j)} - f^{(k)}(u\tau + v, \tau) b_k^{(j)} \right).$$

- The Lie algebra generators $\epsilon_k^{(j)}$ have relations with $b_k^{(j)}$, (e.g. $[\epsilon_4^{(0)}, b_2^{(0)}] = [b_2^{(0)}, b_4^{(0)}]$).

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- $\mathbb{I}_{-}$ is the generating series for the iterated integrals $\bar{\tau}^{j}\,\overline{f^{(k)}}$ and can be constructed in a similar fashion to $\mathbb{I}_{+}$ with the with the appropriate complex conjugations. Note that the tilde simply signals to invert the order of the Lie algebra generators $\epsilon_k^{(j)}$ and $b_k^{(j)}$ (like a transpose).

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- $\mathbb{M}_{\Sigma}^{\mathrm{sv}}$ (with $\mathbb{M}_{\mathrm{Z}}^{\mathrm{sv}}$ being a subset of this series), accounts for the single valued multiple zeta values and the Bernoulli polynomials in u.

$$\mathbb{M}_{\Sigma}^{\mathrm{sv}} = 1 + 2 \sum_{i_1 \in 2\mathbb{N}+1} \Sigma_{i_1}(u) \zeta_{i_1} + 2 \sum_{i_1, i_2 \in 2\mathbb{N}+1} \Sigma_{i_1}(u) \Sigma_{i_2}(u) \zeta_{i_1} \zeta_{i_2} + \ldots \, ,$$

  where the coefficients $\Sigma_n(u)$ are non-trivial infinite series in the generators $\epsilon_k^{(j)}$ and $b_k^{(k)}$.

$$\Sigma_3(u) = -\frac{1}{2} \epsilon_4^{(2)} + B_1(u)[b_2^{(0)}, b_3^{(1)}] + \frac{1}{2} B_2(u)[b_3^{(0)}, b_3^{(1)}] + \frac{1}{4} B_2(u)[b_2^{(0)}, b_4^{(1)}] + \ldots$$

- Differential equation and modular properties fully fix Σ_k to given order.

Conclusion and Outlook

- Have constructed generating series for eMGFs expressible as iterated integrals over modular forms $f^{(k)}$ and G_k but will also include single valued multiple zeta values and Bernoulli polynomials.

- Additionally connection for $\mathbb{I}_+$ can be shown to generate elliptic multiple polylogarithms (eMPLs) **[Brown, Levin, 1110.6917]** Implies that $\mathbb{I}^{\mathrm{eqv}}$ depends on eMPLs and their c.c. (as for single valued MPLs).

- Towards one-loop analog of (**open string**)^2 = (**closed string**)

Thank you for your attention !