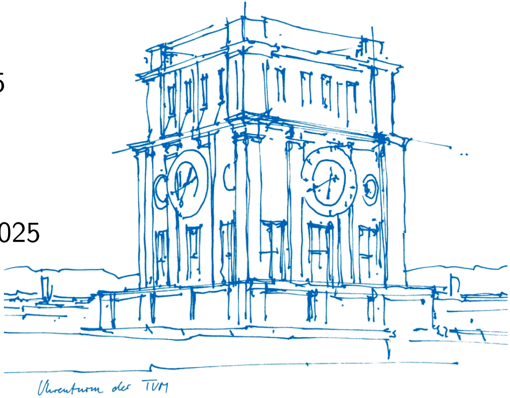
Technical University of Munich
Department of Physics

Establishing the relation between instantons and resonant states (based on arXiv:2507.23125)

DESY Theory
Workshop 2025

Nils Wagner

24th September 2025



Supervisor: Prof. Dr. Björn Garbrecht
Research Group: T70 (TUM)

Overarching goal

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While there exist various methods of computing **decay rates**, most analytical approaches fit into basically **two overarching categories**:

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Wave function techniques based on special solutions of the **Schrödinger equation** [1, 2, etc.]

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Path integral approaches based on the (Euclidean) propagator, hinging on **instanton** calculus [3, 4, etc.]

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Why do both procedures yield identical results?
What is the relation between the two methods?

Traditional instanton method

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$$K_{\mathrm{E}}\left(z_{\mathrm{i}}, z_{\mathrm{f}}\,; T\right) = \left\langle z_{\mathrm{f}} \middle| \exp\left(-\frac{\widehat{H}T}{\hbar}\right) \middle| z_{\mathrm{i}} \right\rangle$$

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The **ground state energy** can be computed from the **late-time behavior** of the **Euclidean propagator**:

$$E_0 = -\hbar \lim_{T\to\infty} \left\{ T^{-1} \log\Big[K_\mathrm{E}\big(z_\mathrm{i}, z_\mathrm{f}\,; T\big)\Big]\right\}.$$

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$$E_0 = -\hbar \lim_{T\to\infty} \left\{ T^{-1} \log \int_{z(0)=z_\mathrm{i}}^{z(T)=z_\mathrm{f}} \mathcal{D}_\mathrm{E}[z] \, \exp\left(-\frac{S_\mathrm{E}[z]}{\hbar} \right) \right\} .$$

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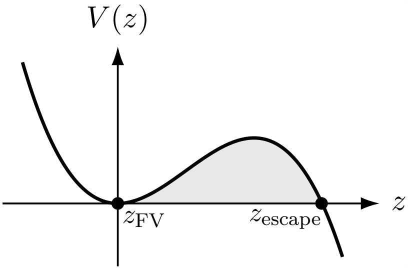

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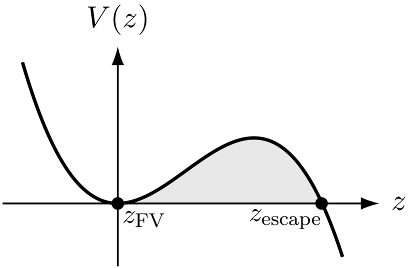

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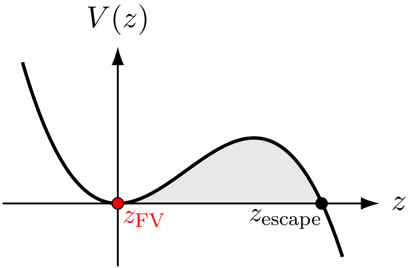

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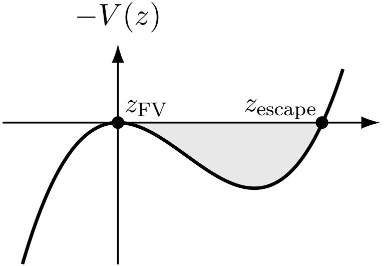

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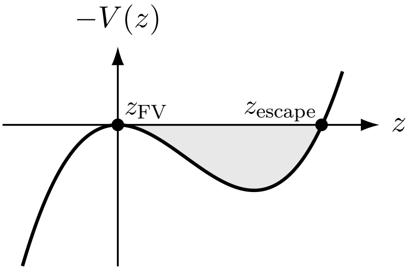

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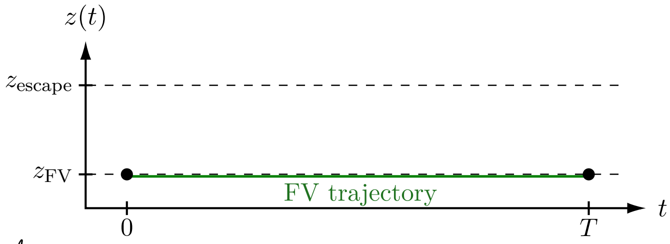

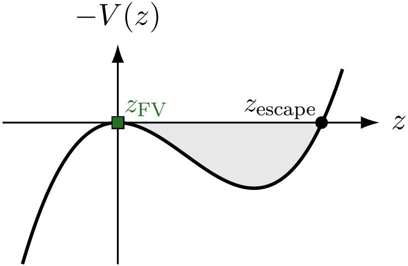

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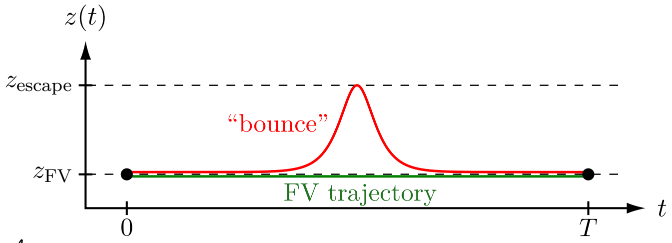

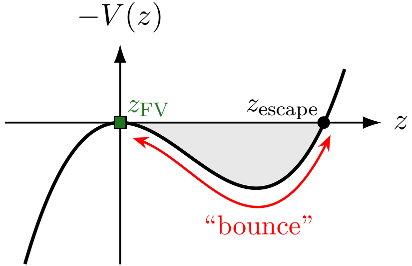

Heuristic potential-deformation argument

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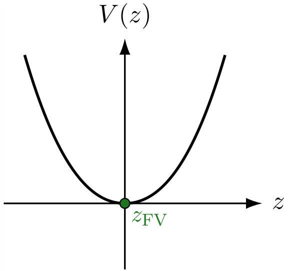

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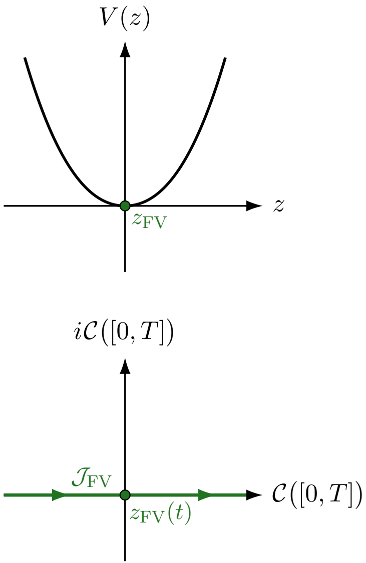

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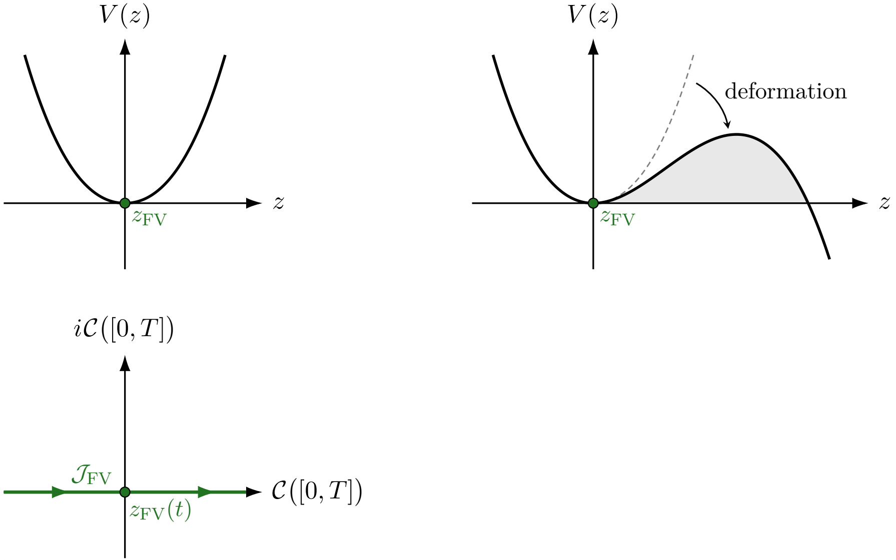

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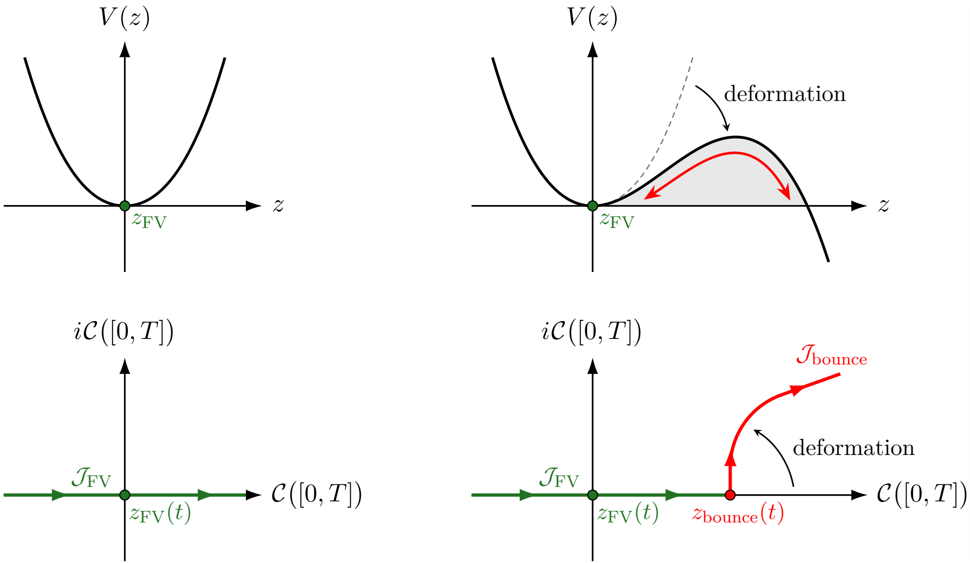

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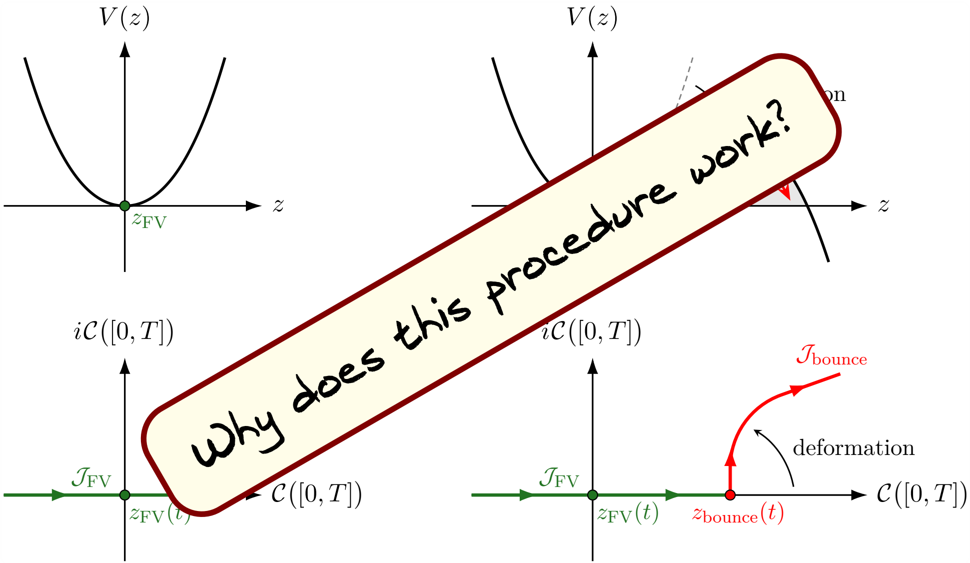

Why does this procedure work?

Nils Wagner

What is the “decay rate”?

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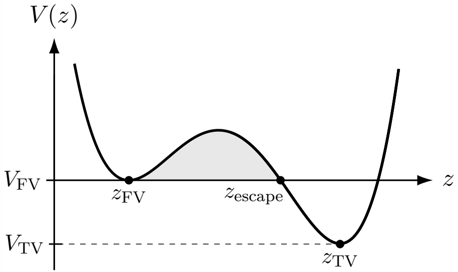

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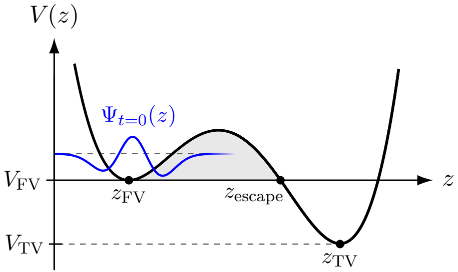

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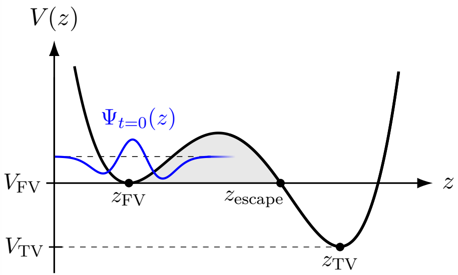

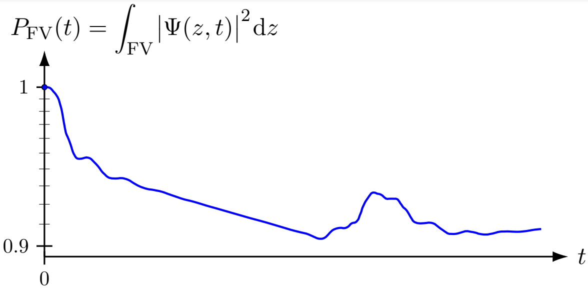

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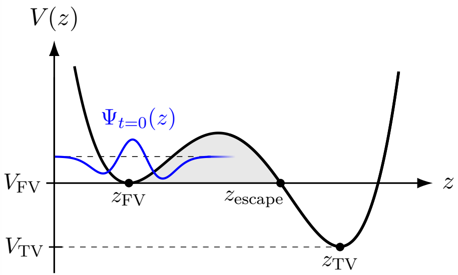
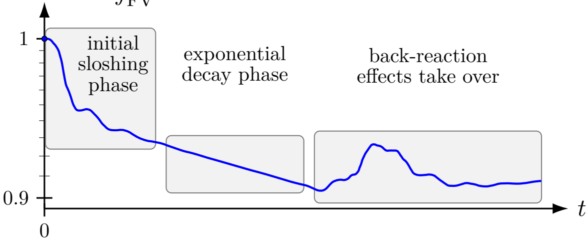


$$P_{\mathrm{FV}}(t) = \int_{\mathrm{FV}} \left|\Psi(z,t)\right|^2 \mathrm{d}z$$



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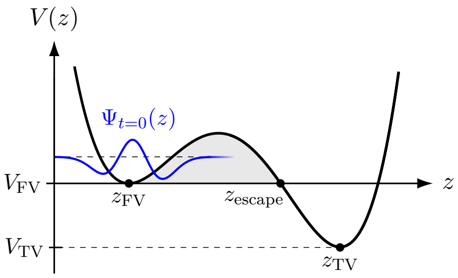

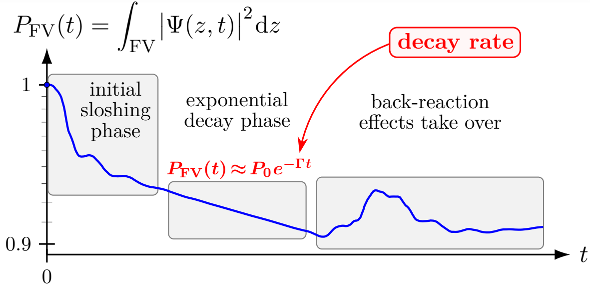

Upshot 1

Only in a special temporal regime does the loss of probability follow a simple **exponential decay law**, amenable to analytical studies [5,6].

What is the "decay rate"?

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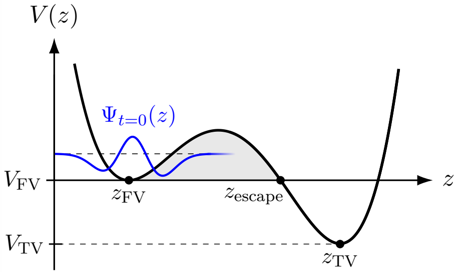

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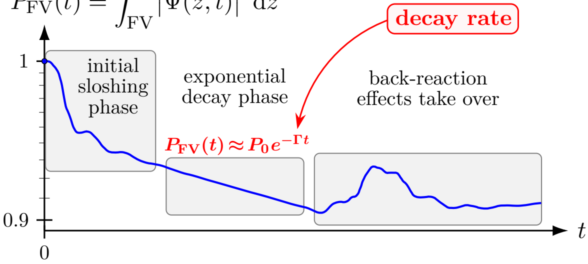

Upshot 1

Only in a special temporal regime does the loss of probability follow a simple **exponential decay law**, amenable to analytical studies [5,6].

Upshot 2

Due to **uniformity** during these intermediate times, we can transform the **time-dependent** problem into a **time-independent** one [1,7].

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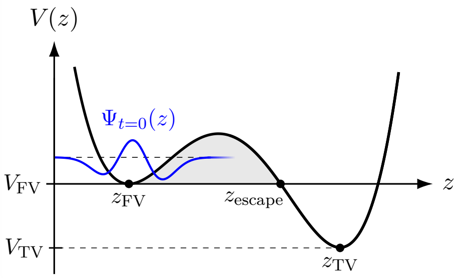
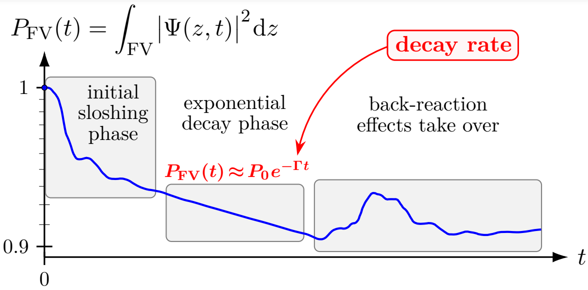
Introduction to resonant states





Key features during the exponential regime:

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Introduction to resonant states

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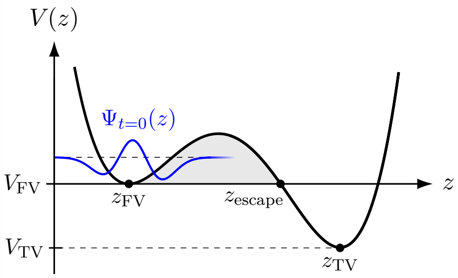

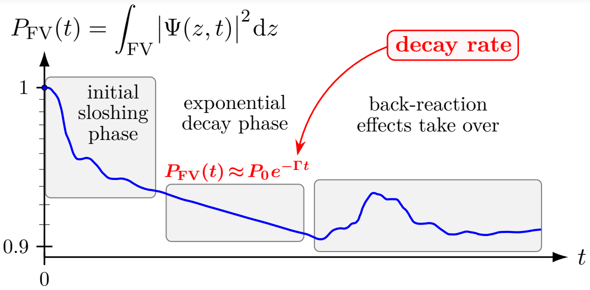

Key features during the exponential regime:

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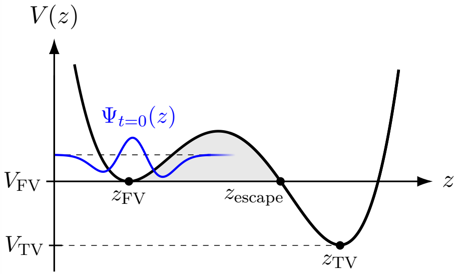

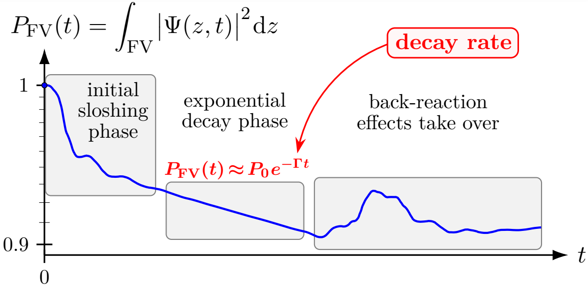

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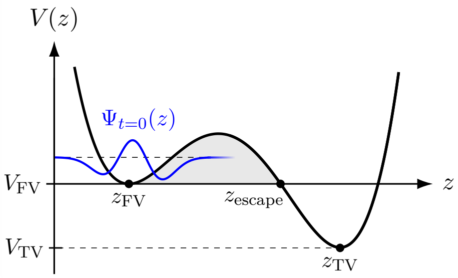

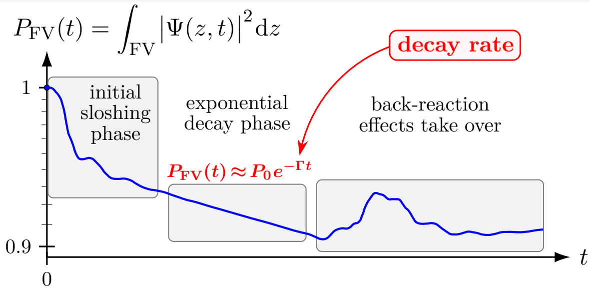

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$\longrightarrow$ equilibrated **steady-state** situation, sustained for a long period of time

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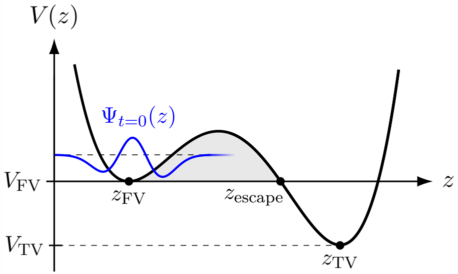

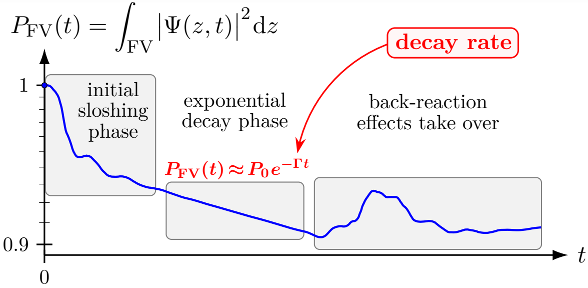

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Solve the **time-independent** Schrödinger equation $\widehat{H}\Psi = E\Psi$

Introduction to resonant states

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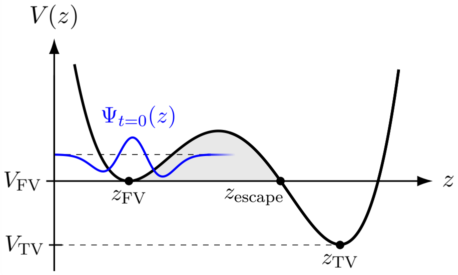

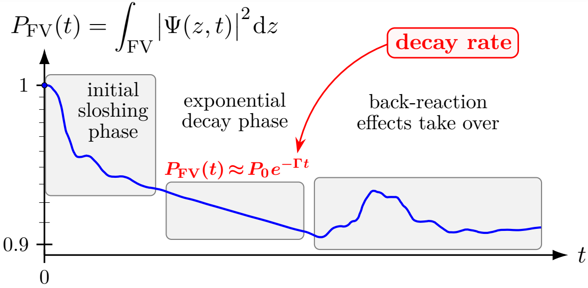

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Solve the **time-independent** Schrödinger equation $\hat{H}\Psi = E\Psi$ demanding **outgoing Gamow–Siegert boundary conditions**:

Introduction to resonant states

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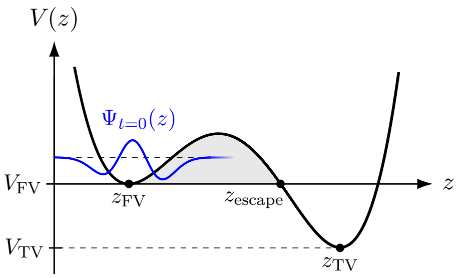

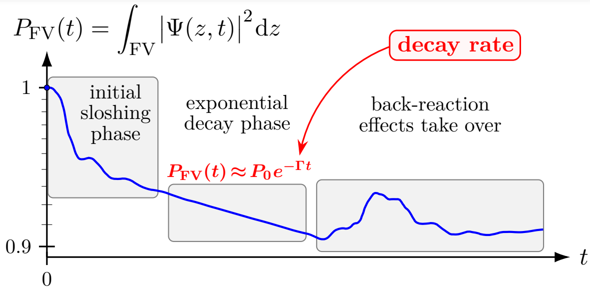

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- **Quasi-stationary** wave function inside the FV region
- Constant, **outward-directed flux**

$\longrightarrow$ equilibrated **steady-state** situation, sustained for a long period of time

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[5] Andreassen, et al. (2017), *PRD 95*(8)
[1] Gamow (1928), *Z. Physik 51*(3)
[6] Peres (1980), *Annals Phys. 129*(1)
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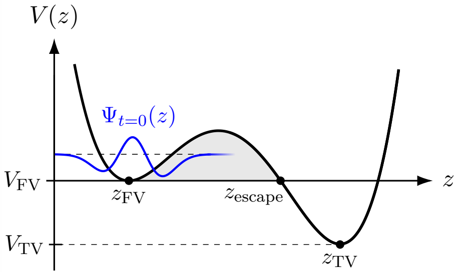

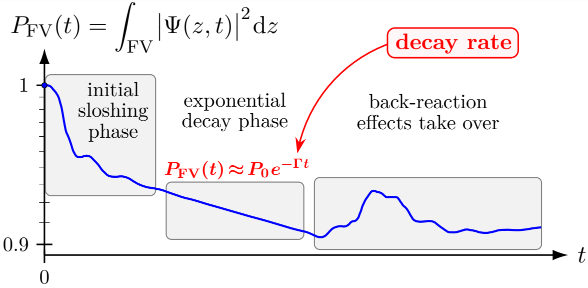

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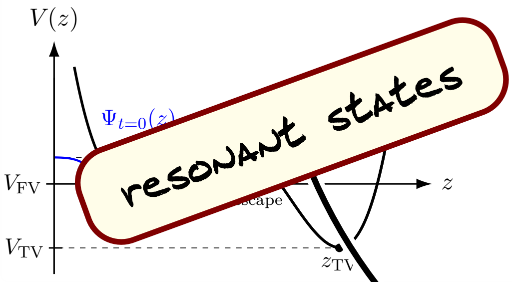

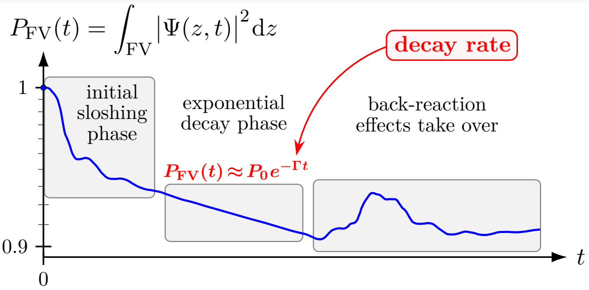

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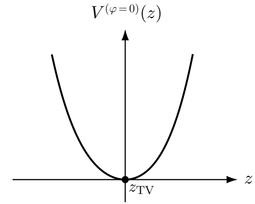

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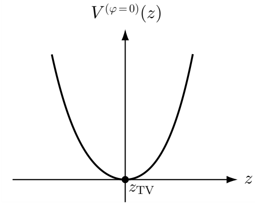

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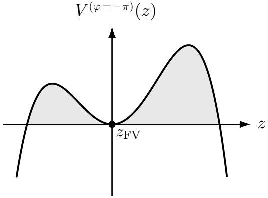

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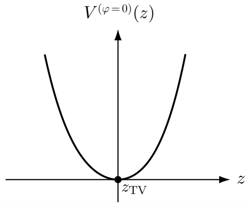

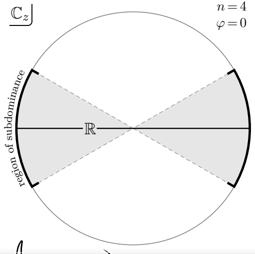

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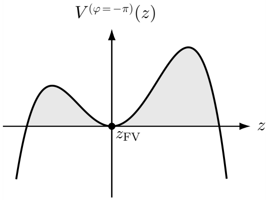

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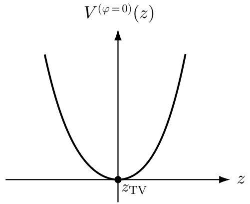

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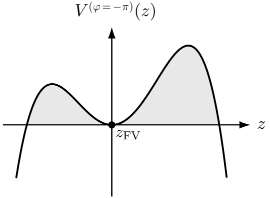

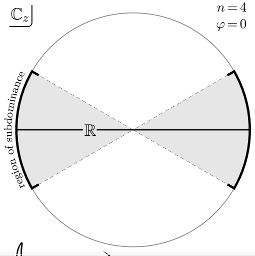

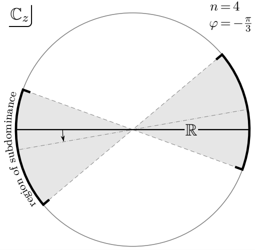

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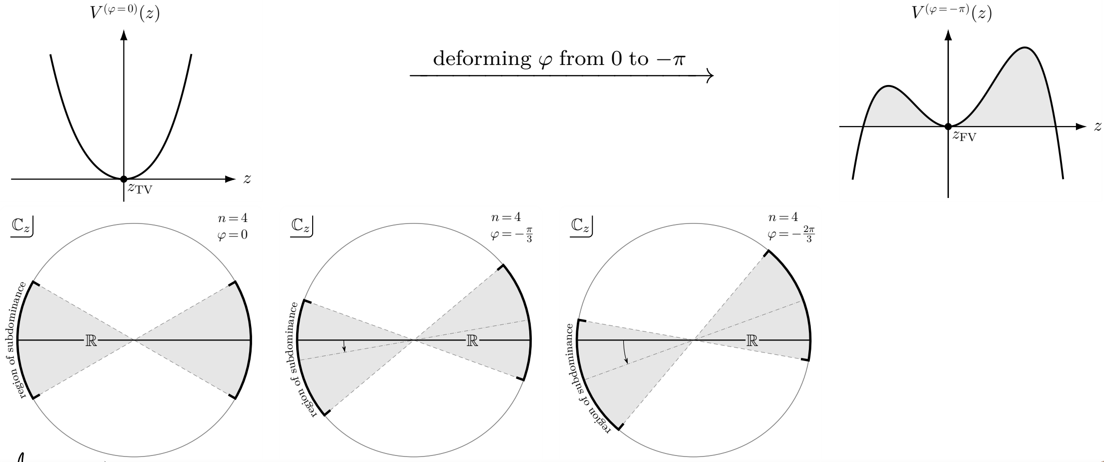

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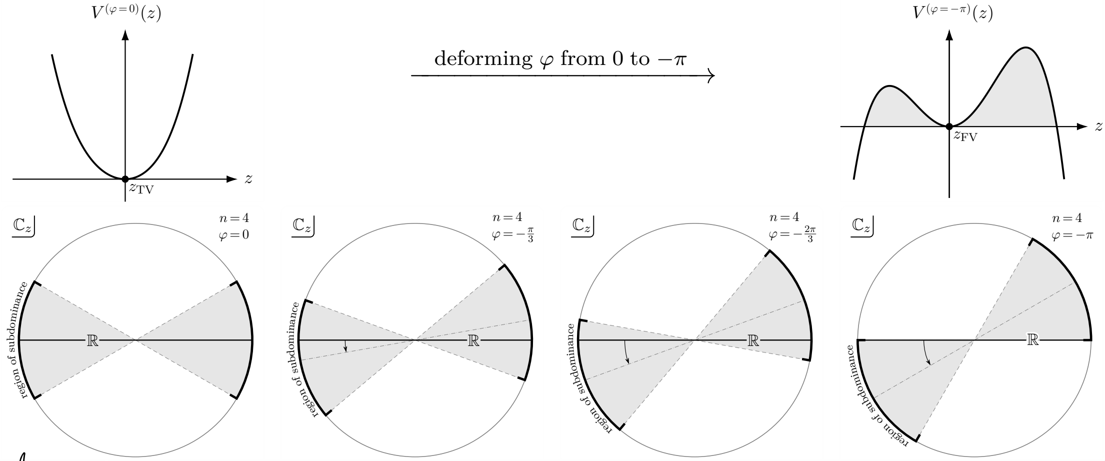

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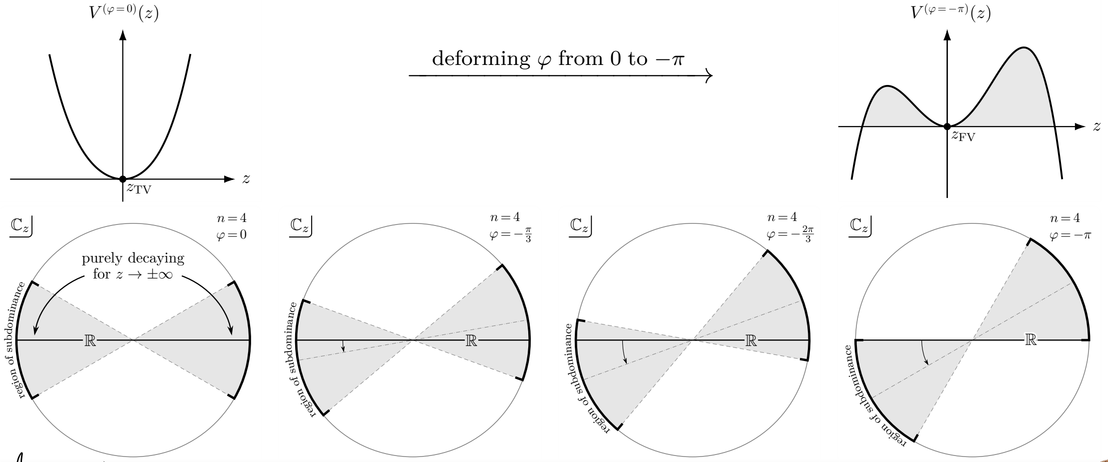

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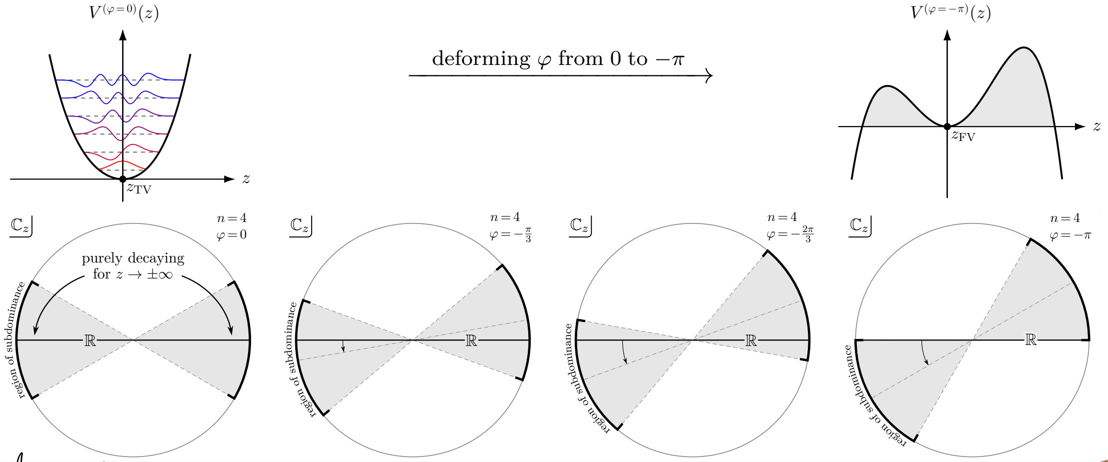

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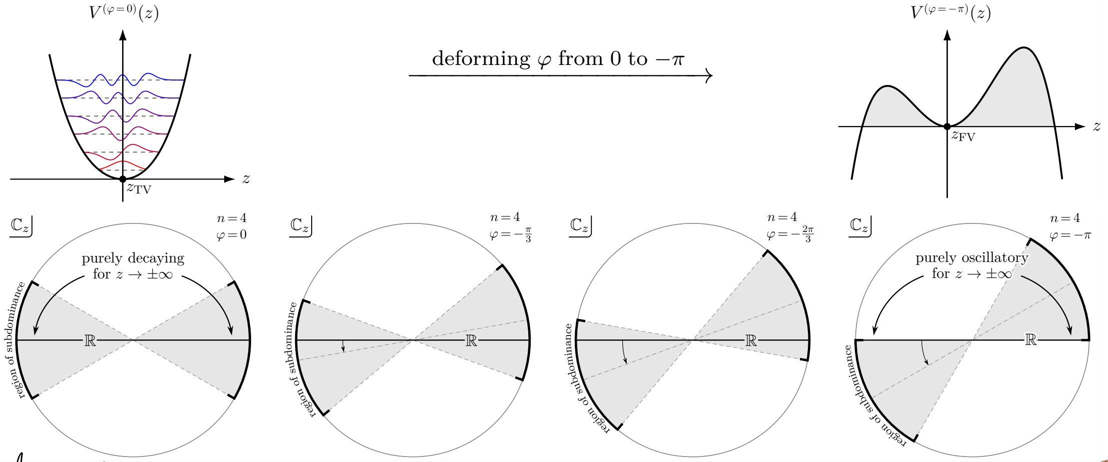

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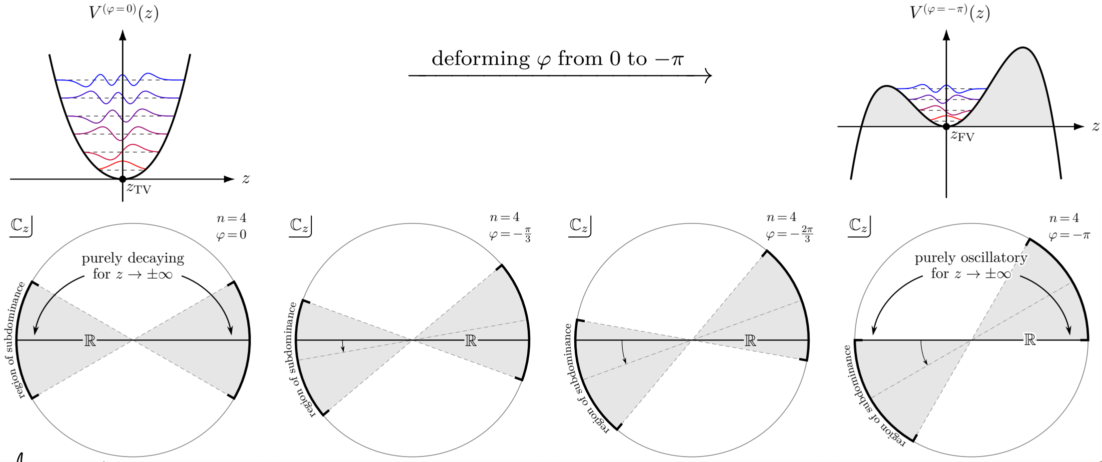

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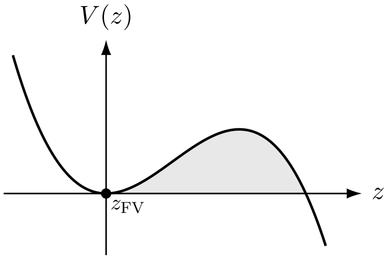

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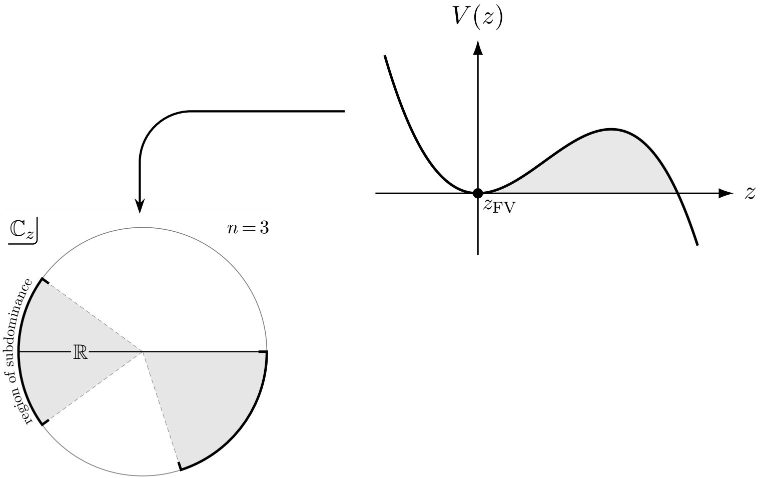

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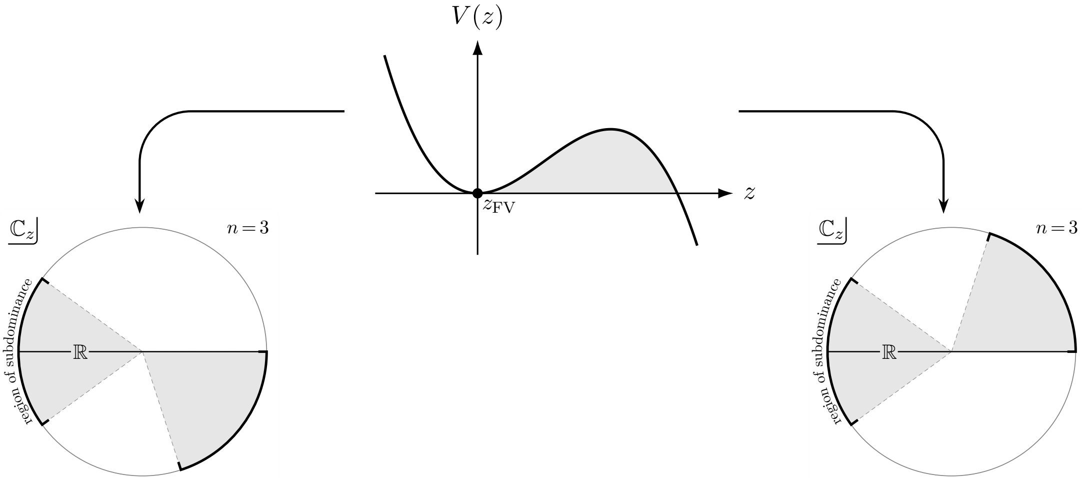

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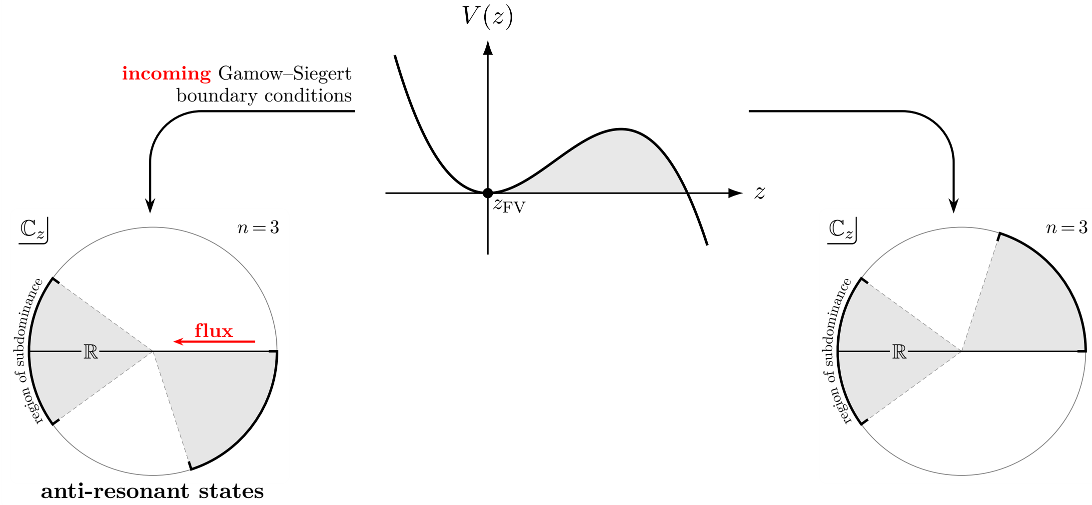

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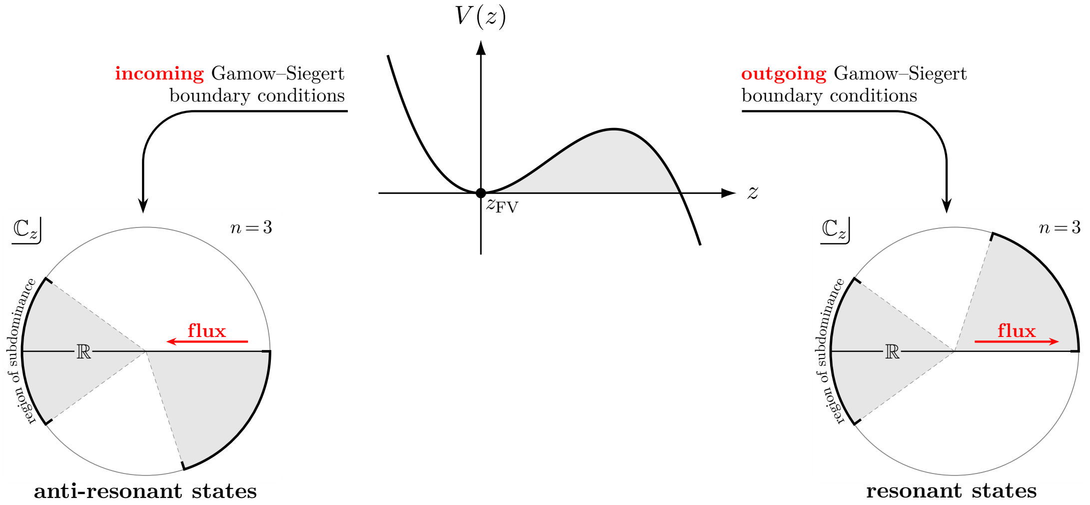

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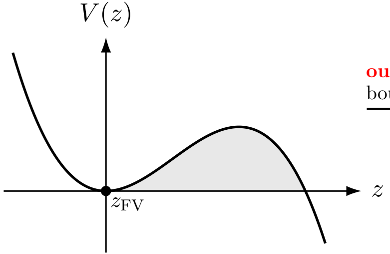

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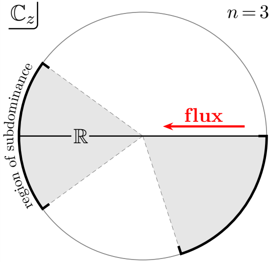

anti-resonant states

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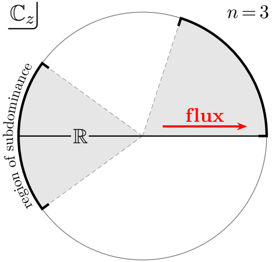


resonant states

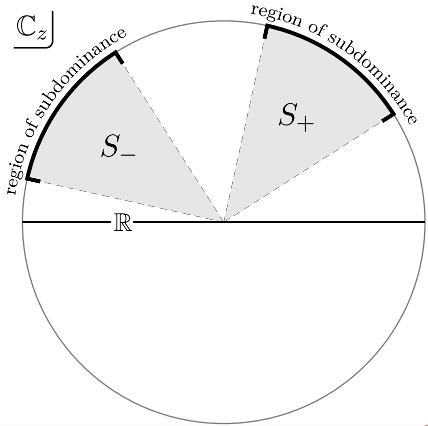
Upshot

Gamow–Siegert BC naturally emerge for real, **unstable potentials**, as the real axis **borders** two **Stokes sectors**.

Retreat to a single real dimension

Let us investigate the **generic eigenvalue problem**

$$\widehat{H}\Psi_\ell(z) = \left\{-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}z^2} + V(z)\right\}\Psi_\ell(z) = E_\ell\,\Psi_\ell(z)\,, \qquad \Psi_\ell(\eta z) \xrightarrow{\eta\to\infty} 0 \;\text{ for } z \in S_\pm\,.$$



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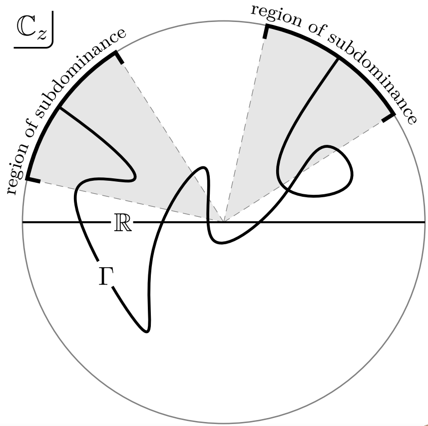

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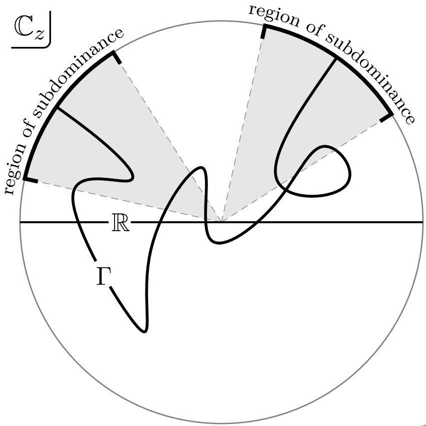

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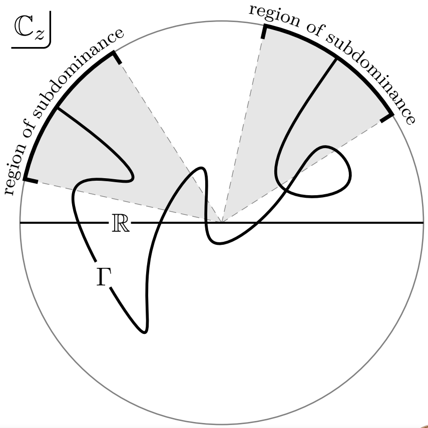

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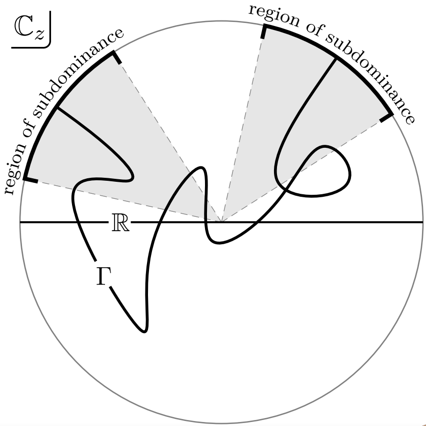

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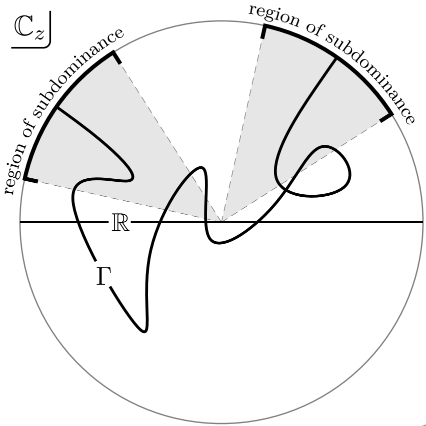

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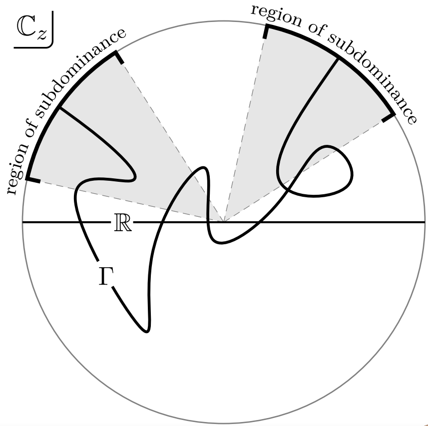

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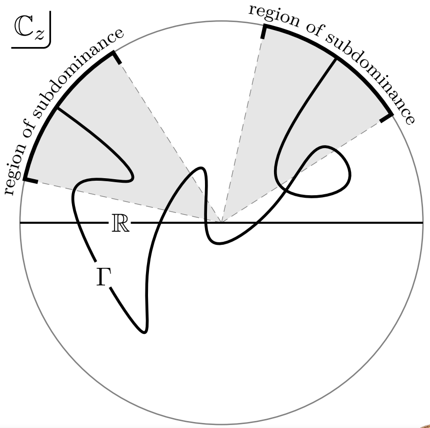

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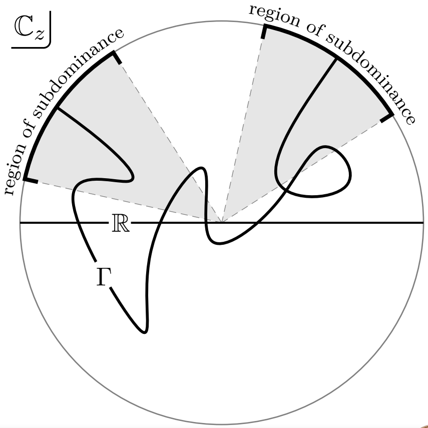

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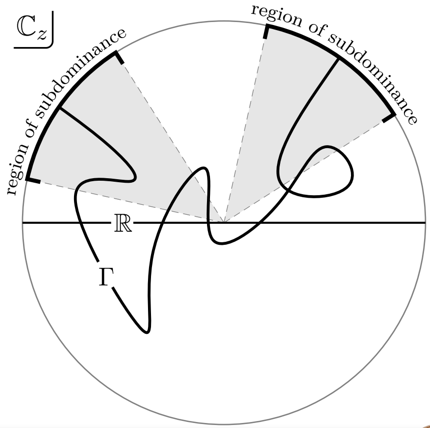

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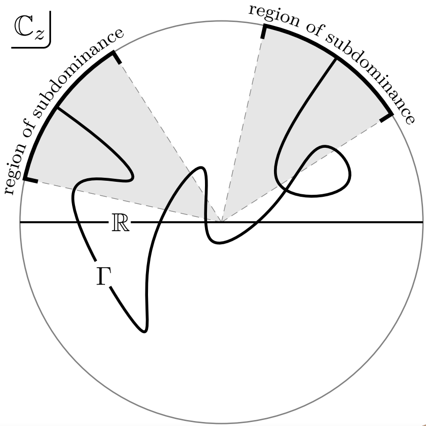

- $\widehat{H}_\gamma\psi_\ell(s) = E_\ell\psi_\ell(s)$ is defined on $\mathbb{R}$
- The **normalizable** eigenfunctions $\psi_\ell(s)$ decay at s-spatial infinity
- Caveat: $\widehat{H}_\gamma$ is **non-Hermitian**

$\longrightarrow$ $\widehat{H}_\gamma\psi_\ell(s) = E_\ell\psi_\ell(s)$ takes a suggestive form

Associated eigenvalue problem on $\mathbb{R}$

Let us investigate the **generic eigenvalue problem**

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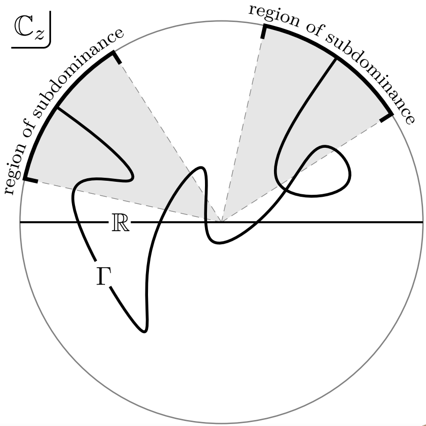

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  $\longrightarrow$ standard QM tools require slight modification

$\longrightarrow$ $\widehat{H}_\gamma\psi_\ell(s) = E_\ell\psi_\ell(s)$ takes a suggestive form

Associated eigenvalue problem on $\mathbb{R}$: "Propagator"

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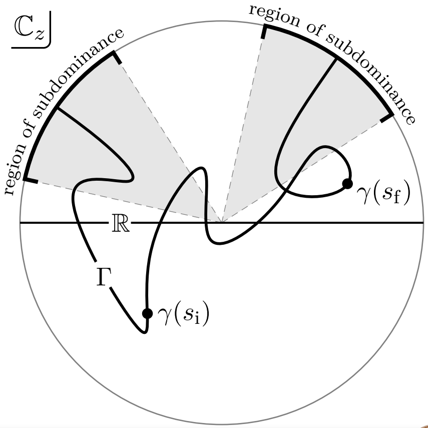
What is the **transition amplitude** of a point particle **propagating** from $\gamma(s_\mathrm{i})$ to $\gamma(s_\mathrm{f})$



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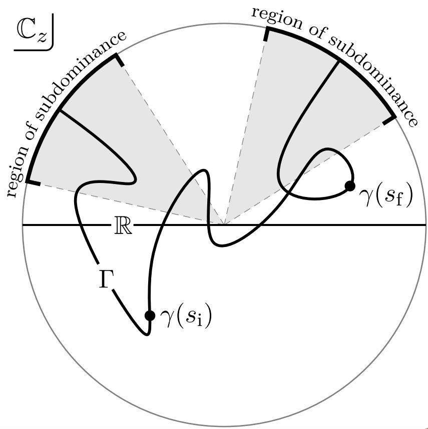
What is the **transition amplitude** of a point particle **propagating** from $\gamma(s_\mathrm{i})$ to $\gamma(s_\mathrm{f})$ in case the motion is fully **constrained** to Γ?



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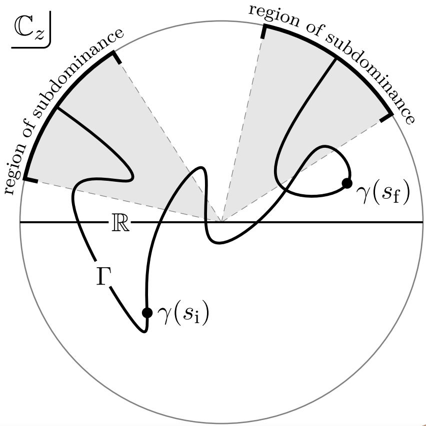


What is the **transition amplitude** of a point particle **propagating** from $\gamma(s_\mathrm{i})$ to $\gamma(s_\mathrm{f})$ in case the motion is fully **constrained** to Γ?

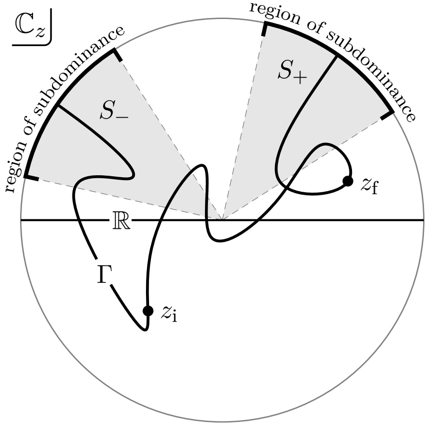
$$K_\mathrm{E}^{(\gamma)}(s_\mathrm{i}, s_\mathrm{f}; T) := \left\langle s_\mathrm{f} \,\middle|\, \exp\left(-\frac{\widehat{H}_\gamma T}{\hbar}\right) \middle|\, s_\mathrm{i}\right\rangle$$

Accessing the spectrum with a functional integral

Given the **generic eigenvalue problem**

$$\widehat{H}\,\Psi_\ell(z) = \left\{-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}z^2} + V(z)\right\}\Psi_\ell(z) = E_\ell\,\Psi_\ell(z)$$

with $\Psi_\ell(\eta z) \xrightarrow{\eta\to\infty} 0$ for $z \in S_\pm$,



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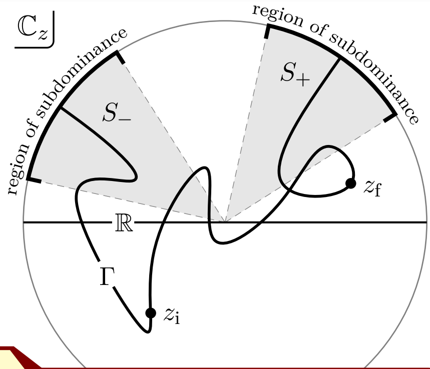
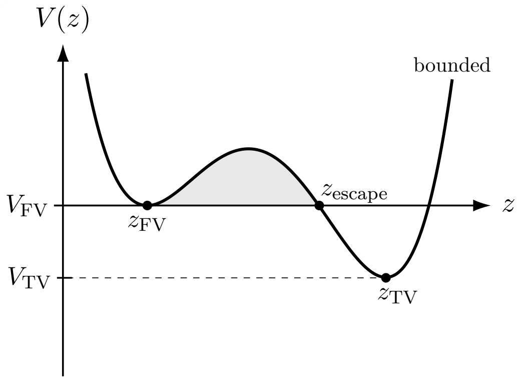
with $\Psi_\ell(\eta z) \xrightarrow{\eta\to\infty} 0$ for $z \in S_\pm$, one finds the relation:



Master formula

$$\int_{\mathcal{C}([0,T],\Gamma)}^{\substack{z(0)=z_\mathrm{i}\\ z(T)=z_\mathrm{f}}} \mathcal{D}_\mathrm{E}[z]\,\exp\left(-\frac{S_\mathrm{E}[z]}{\hbar}\right) = \sum_{\ell=0}^{\infty} \exp\left(-\frac{E_\ell T}{\hbar}\right)\Psi_\ell(z_\mathrm{i})\,\Psi_\ell(z_\mathrm{f})\left\{\int_\Gamma \Psi_\ell(z)^2\,\mathrm{d}z\right\}^{-1}.$$

Extracting the decay rate



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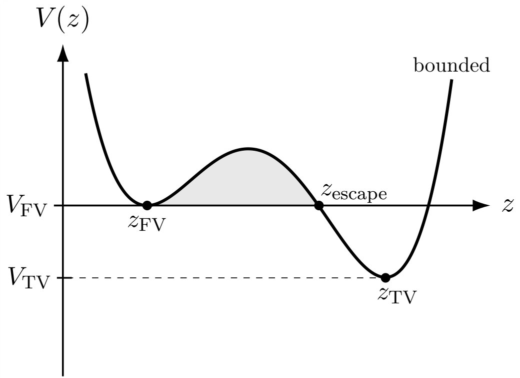

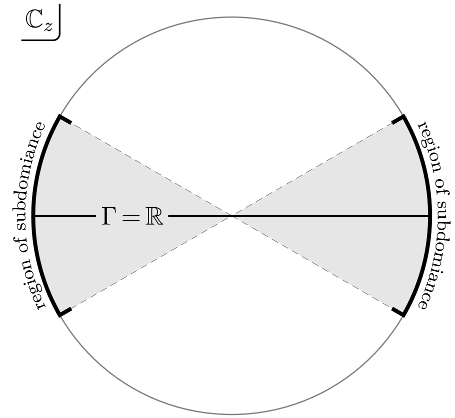

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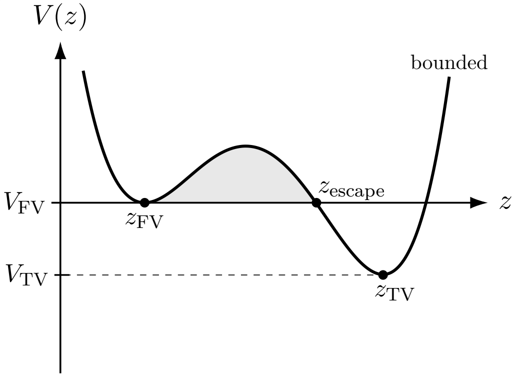
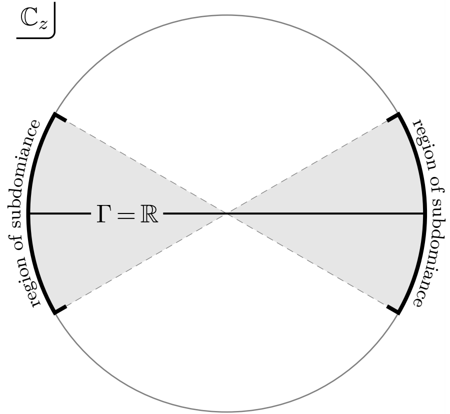
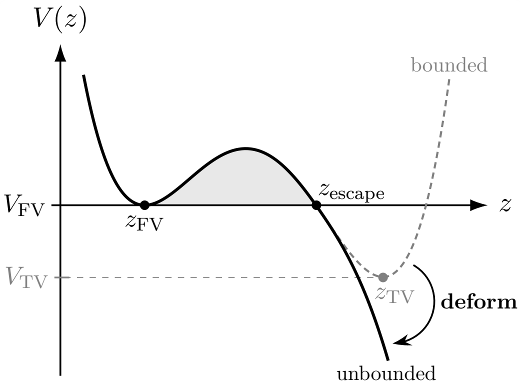
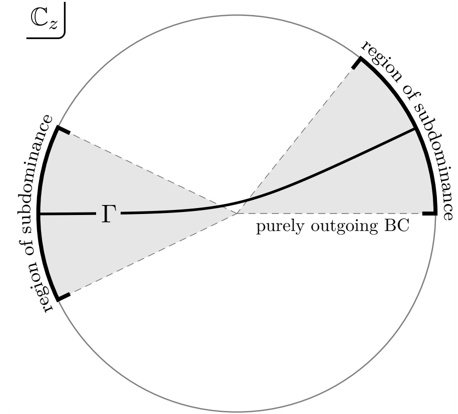




$$E_0^{(\mathrm{global})} = -\hbar \lim_{T\to\infty} \left(\frac{1}{T} \log \int_{\mathcal{C}([0,T])}^{\substack{z(0)=z_{\mathrm{i}} \\ z(T)=z_{\mathrm{f}}}} \mathcal{D}_{\mathrm{E}}[z] \exp\left\{ -\frac{1}{\hbar} \int_0^T \left[\frac{m}{2} \dot{z}(t)^2 + V^{(\mathrm{stable})}\big(z(t)\big) \right] \mathrm{d}t \right\} \right)$$

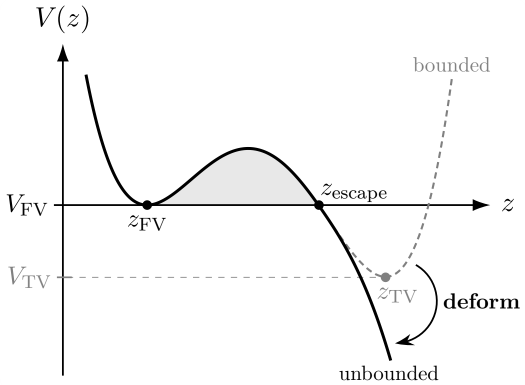
Extracting the decay rate

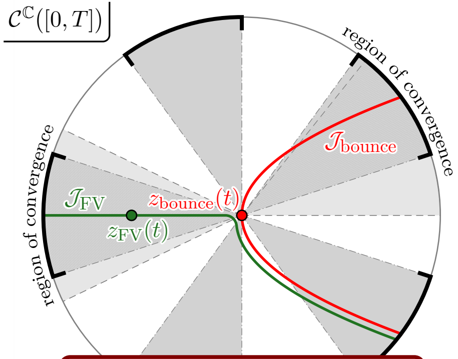




$$E_0^{(\text{resonant})} = -\hbar \lim_{T\to\infty}\left(\frac{1}{T}\log\int_{\mathcal{C}([0,T],\Gamma)}^{\substack{z(0)=z_{\text{i}}\\ z(T)=z_{\text{f}}}} \mathcal{D}_{\text{E}}[z]\,\exp\left\{-\frac{1}{\hbar}\int_0^T\left[\frac{m}{2}\dot{z}(t)^2 + V^{(\text{unstable})}\big(z(t)\big)\right]\text{d}t\right\}\right)$$

Extracting the decay rate





$$\mathcal{C}\big([0,T],\Gamma\big) \equiv \mathcal{J}_{\mathrm{FV}} + \mathcal{J}_{\mathrm{bounce}}$$

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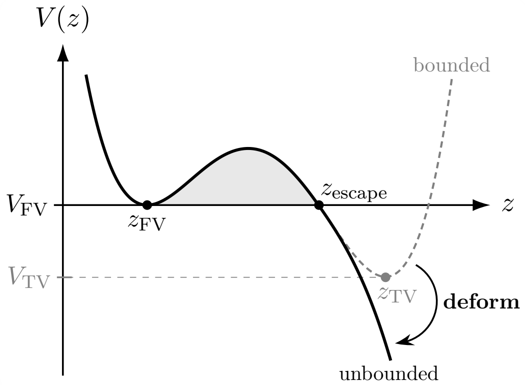
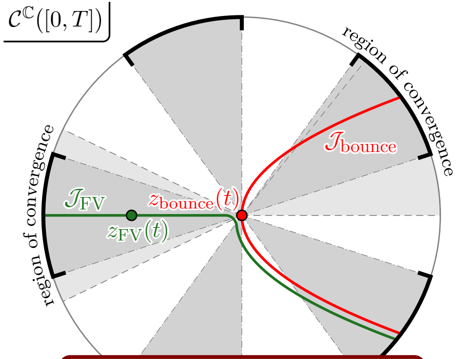
Extracting the decay rate





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Thanks for your attention!