UNIVERSITÄT BONN

Bethe Center for
Theoretical Physics

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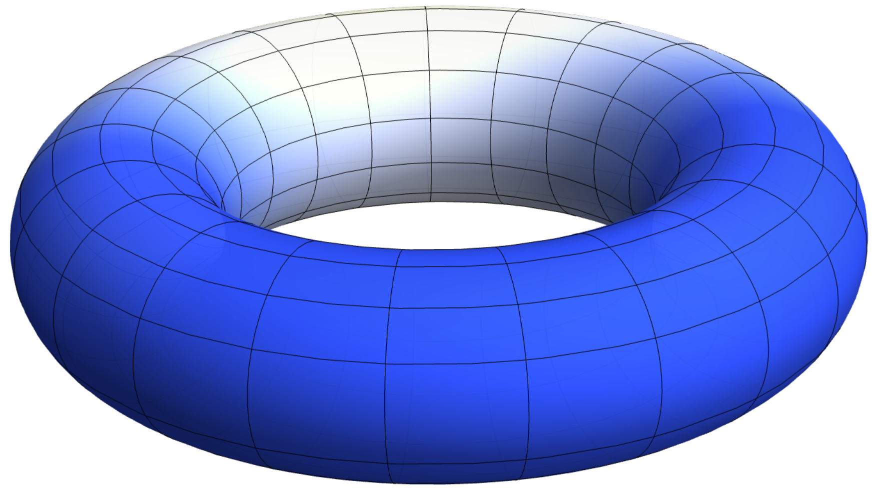
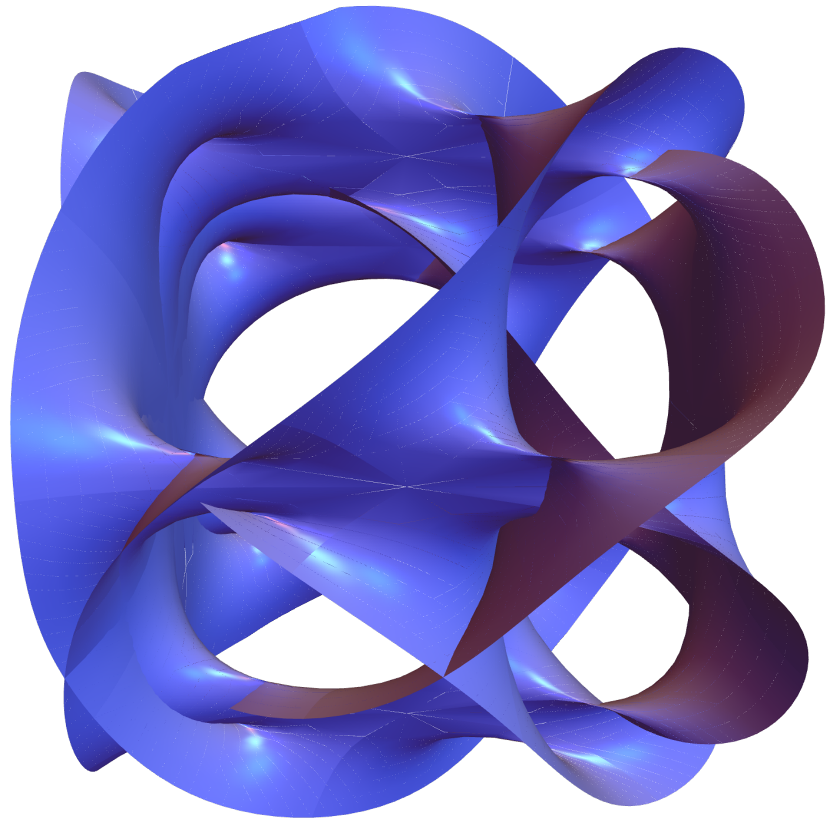
Canonical Feynman integrals beyond polylogarithms

with **Claude Duhr, Christoph Nega, Benjamin Sauer, Lorenzo Tancredi, Fabian J. Wagner**



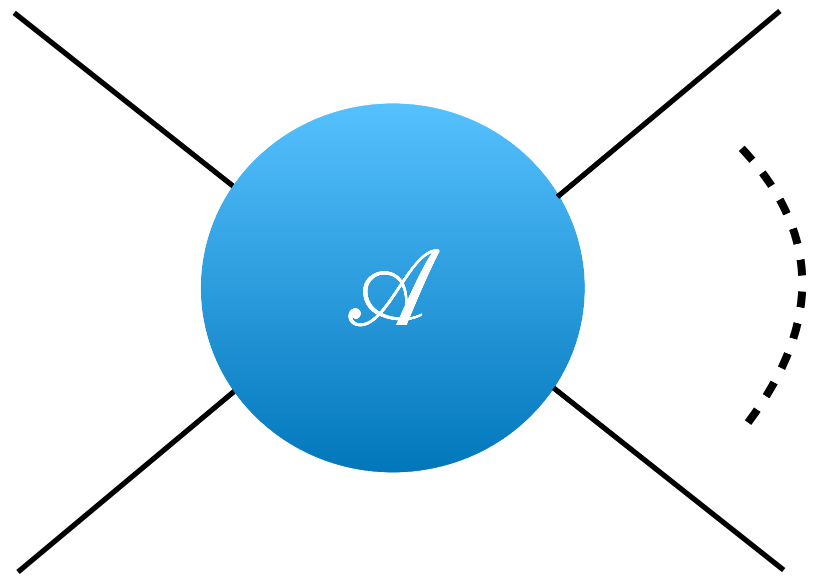


Based on: *JHEP* 06 (2025) 128, arXiv 2503.20655

Sara Maggio | Desy Theory Workshop, Desy Hamburg | 24/09/2025

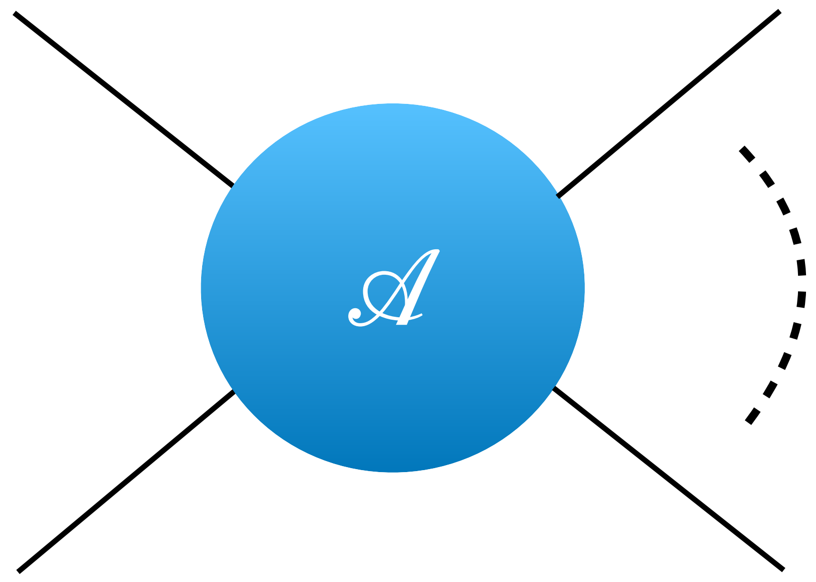
Motivation

Motivation



Motivation

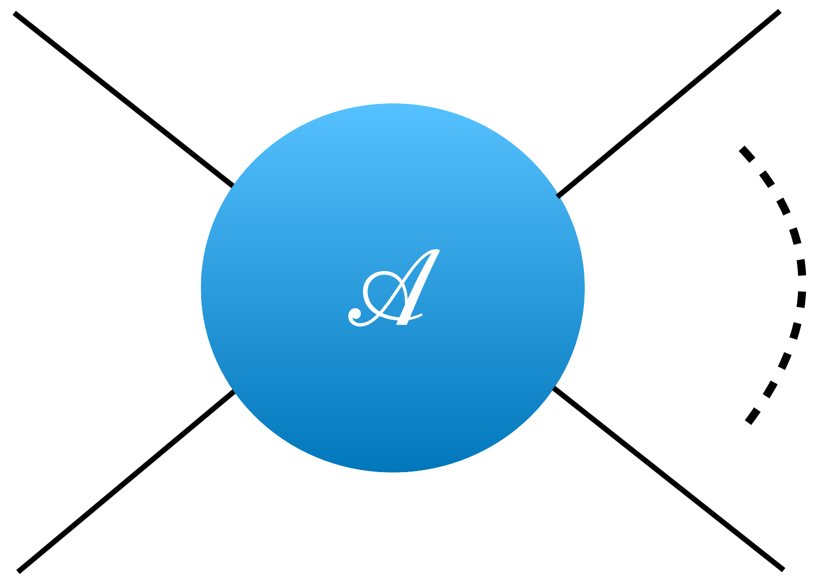
Singularities



Motivation

Singularities

poles



Motivation

Singularities

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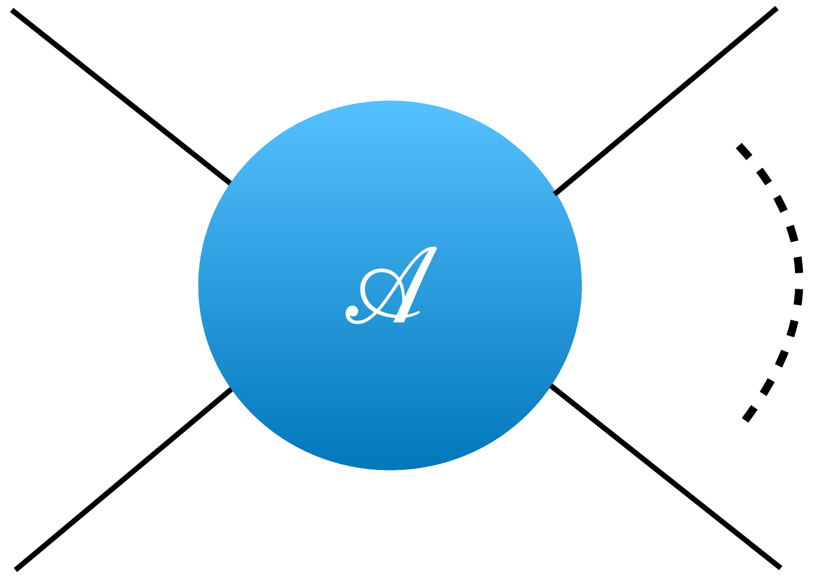
single-particle
states go on-shell



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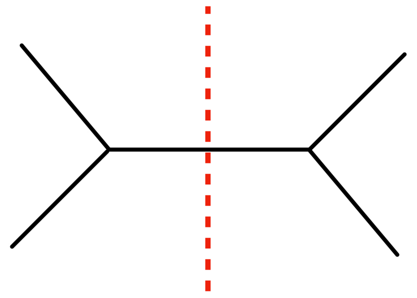
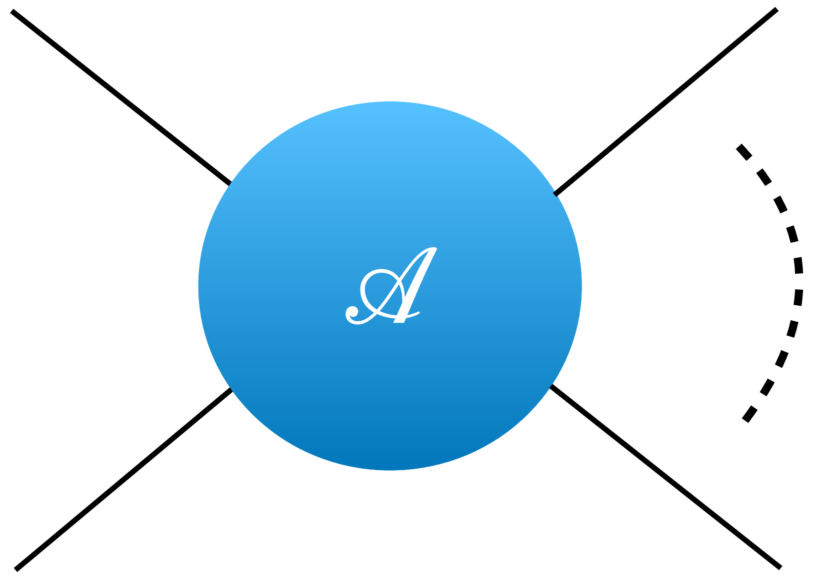


$$\sim \frac{1}{s}$$



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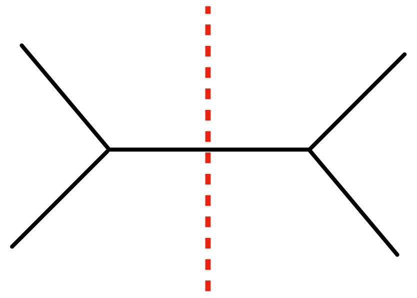
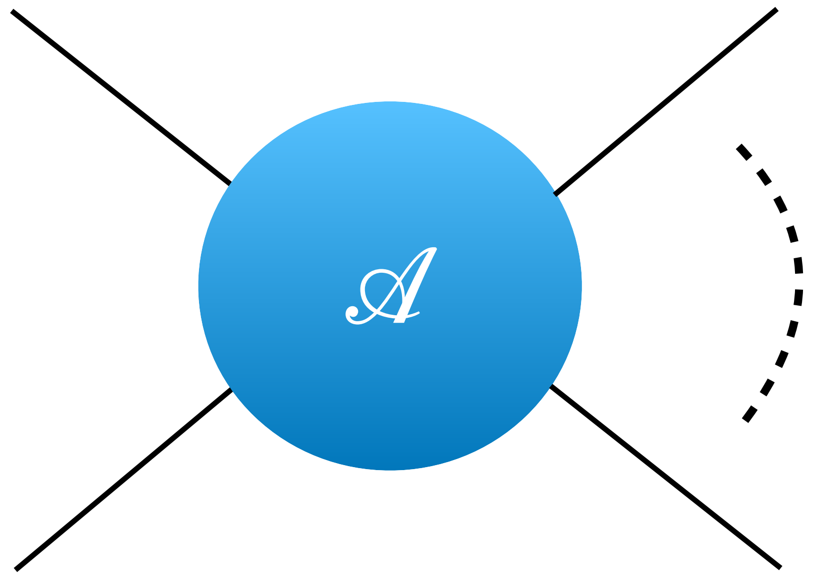


$\sim \frac{1}{s}$

"cutting" the propagator ~
the particle goes on shell



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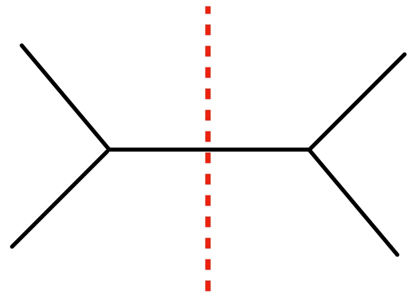

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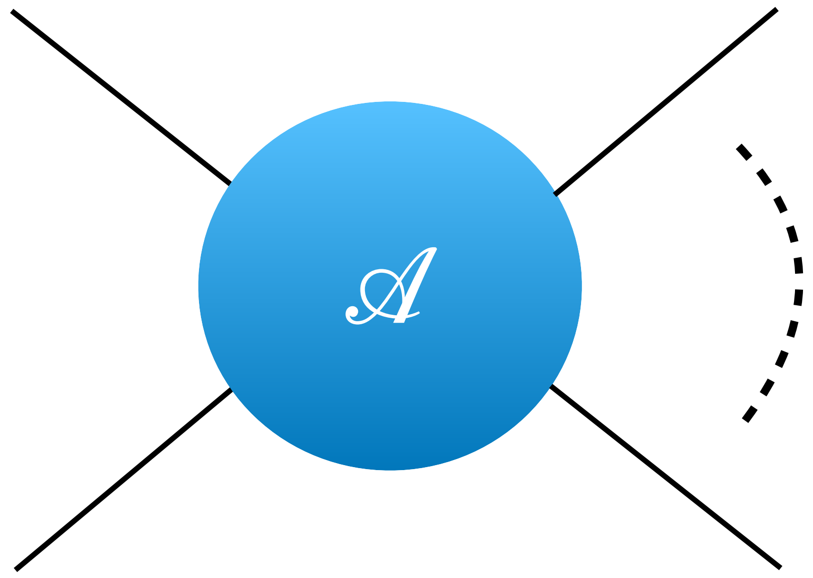

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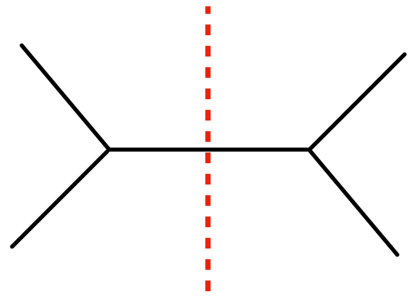

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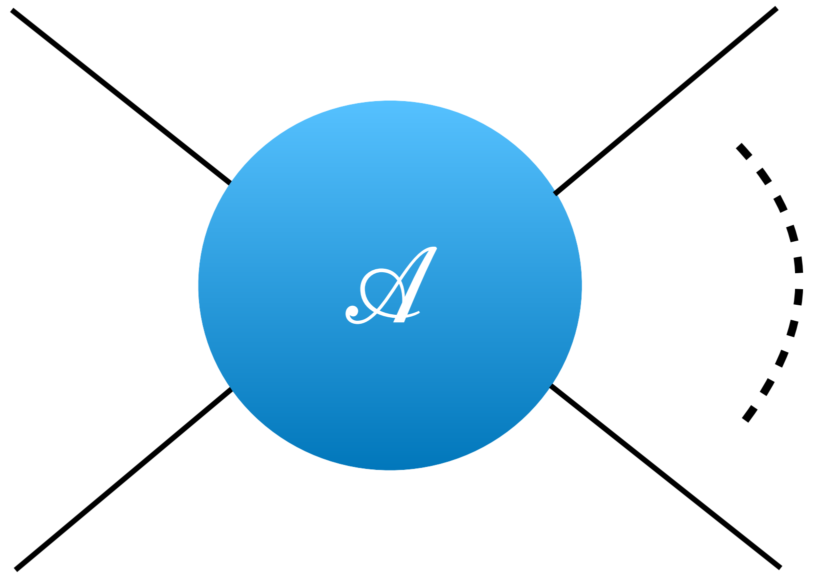

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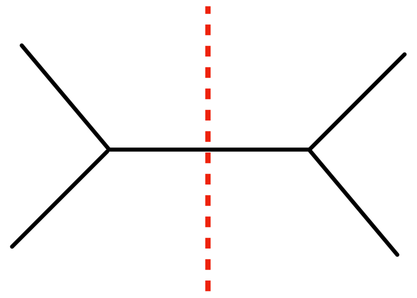

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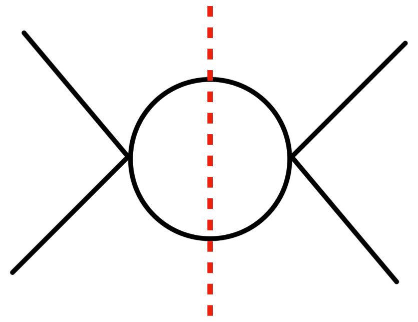
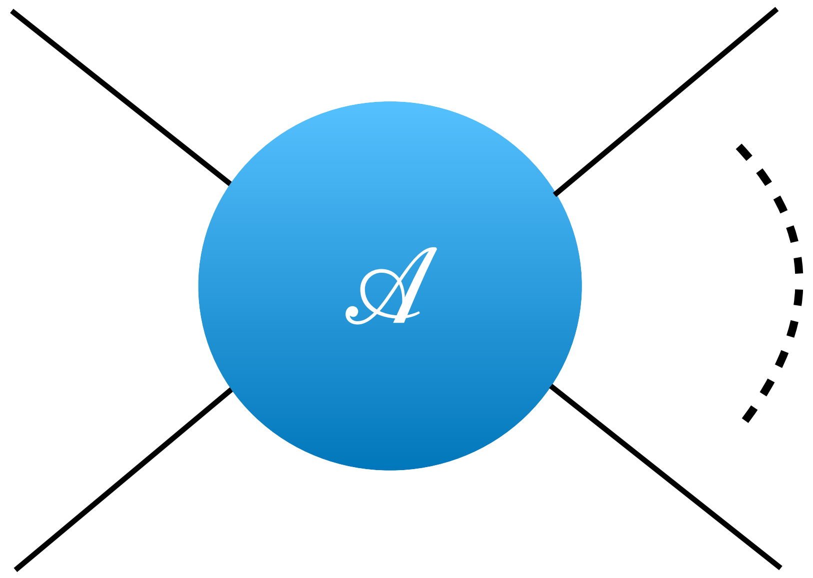


$$\sim \sqrt{s}\,,\log(s)$$



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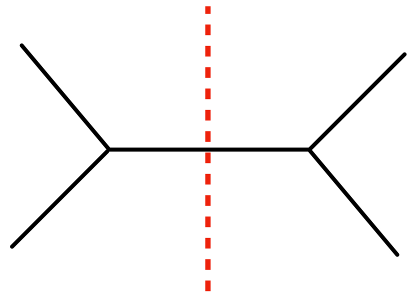

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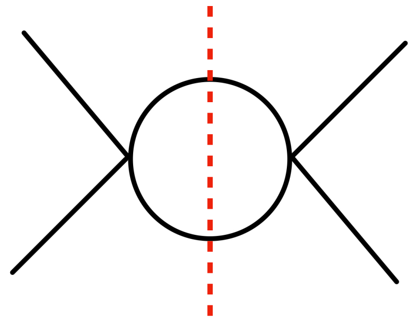

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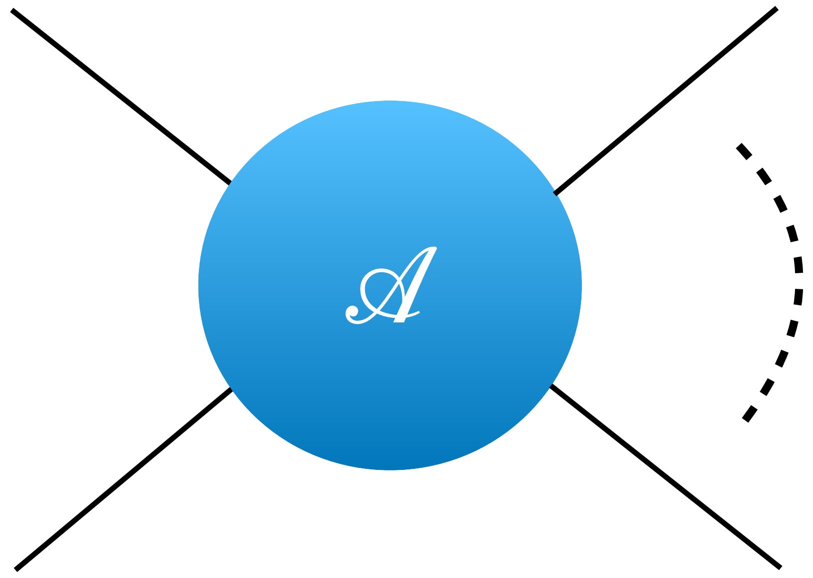

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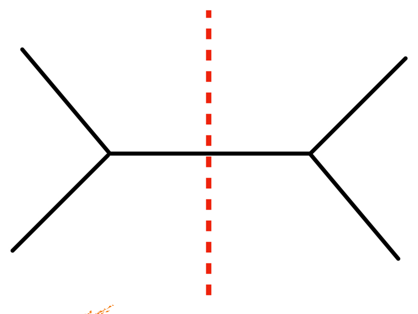

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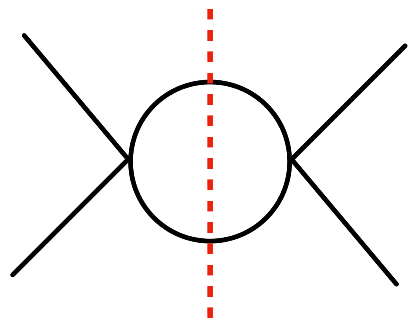

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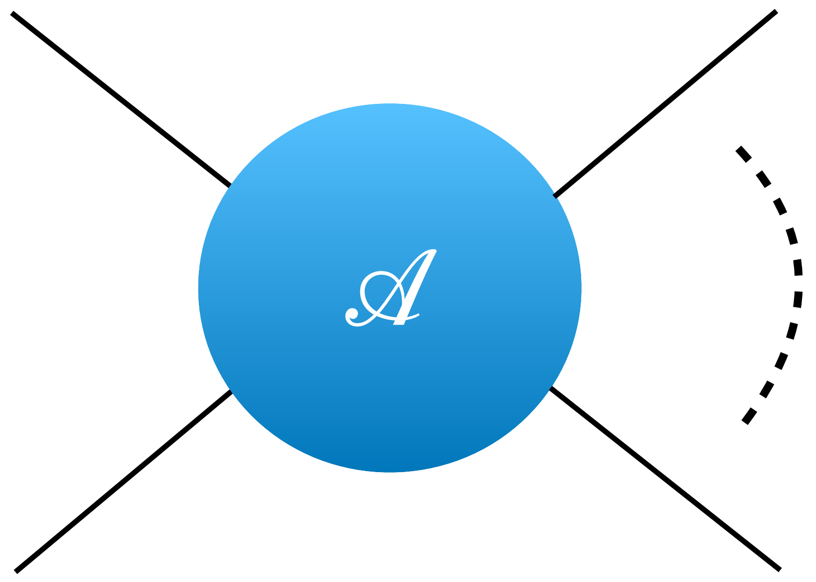

Computation

$\mathscr{A} \sim$

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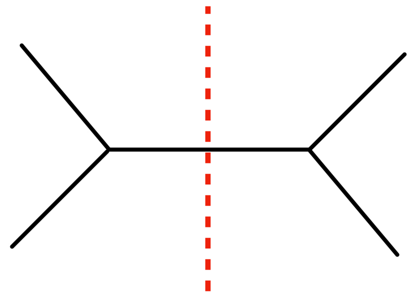

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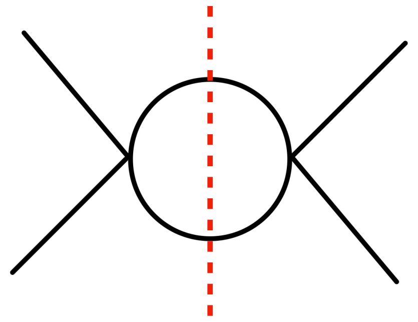

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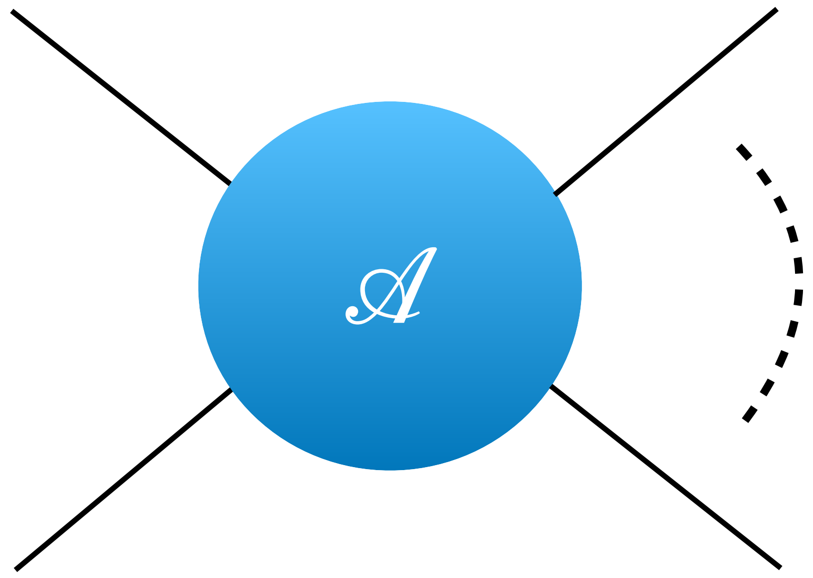

Computation

$$\mathscr{A} \sim \qquad T^{\mu\nu}$$

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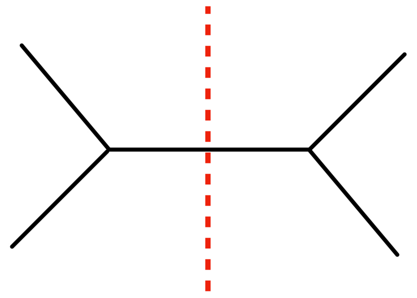

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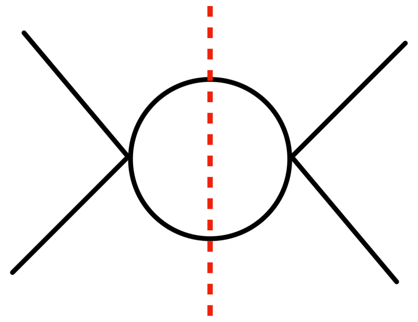

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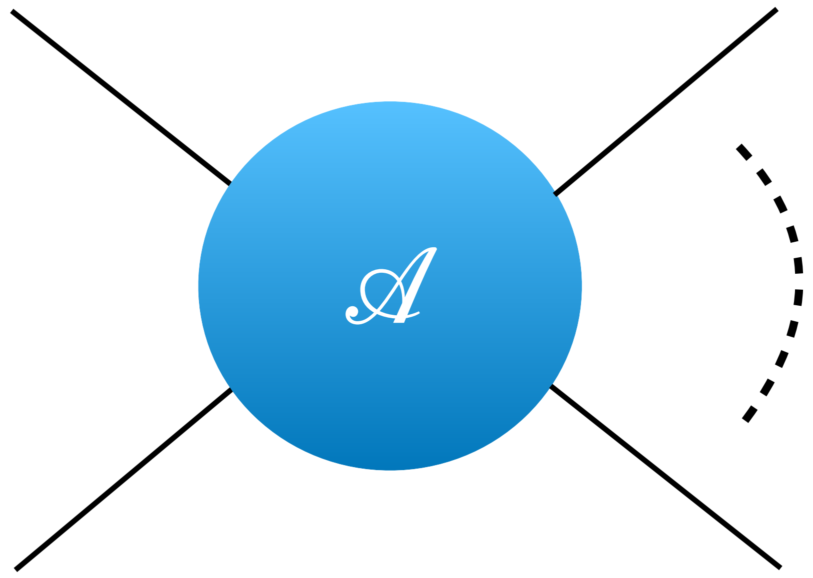

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tensor
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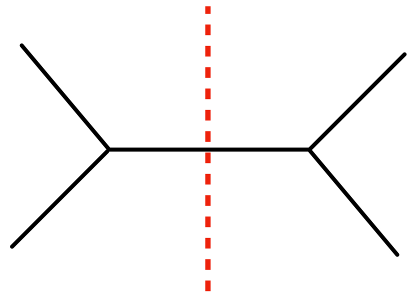

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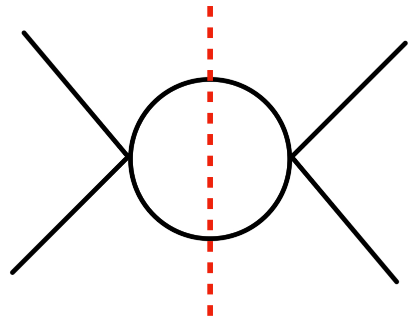

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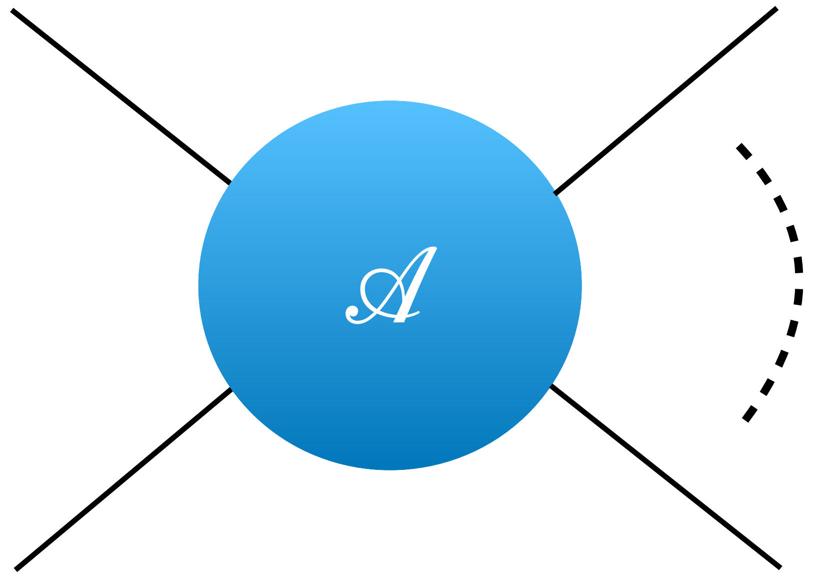

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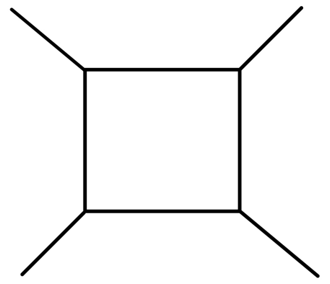

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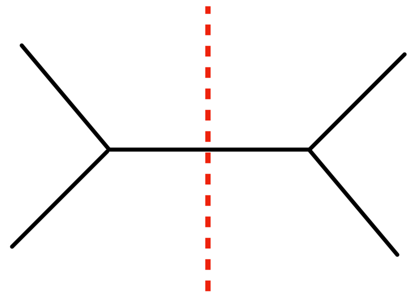

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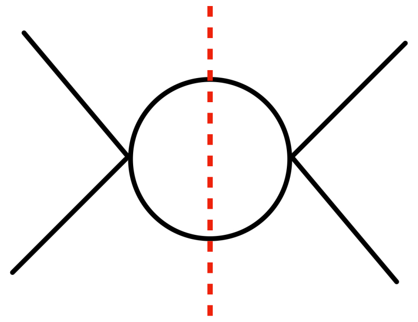

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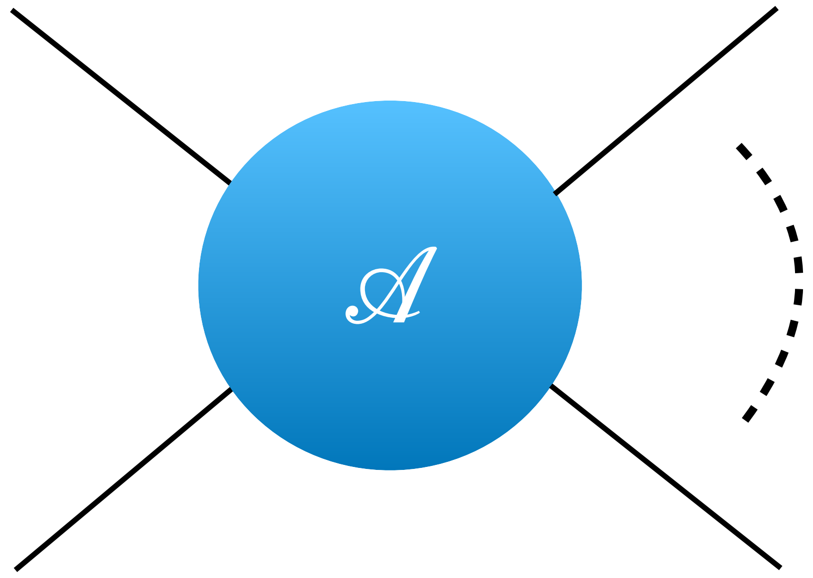

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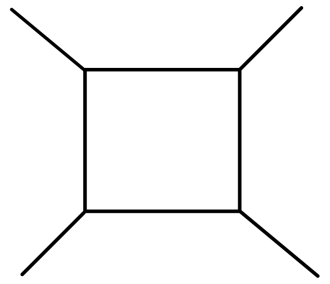

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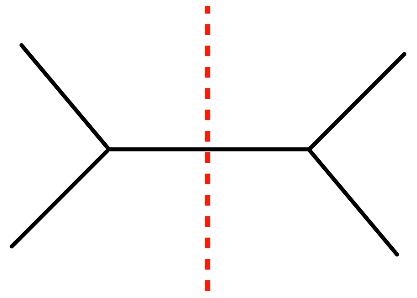

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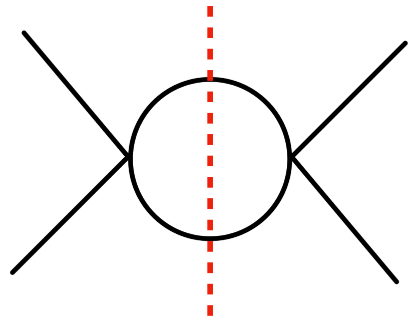

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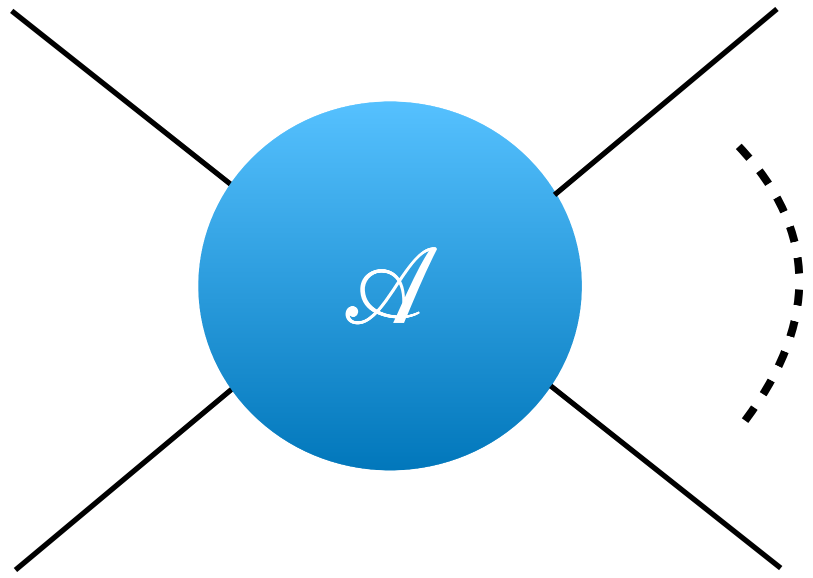

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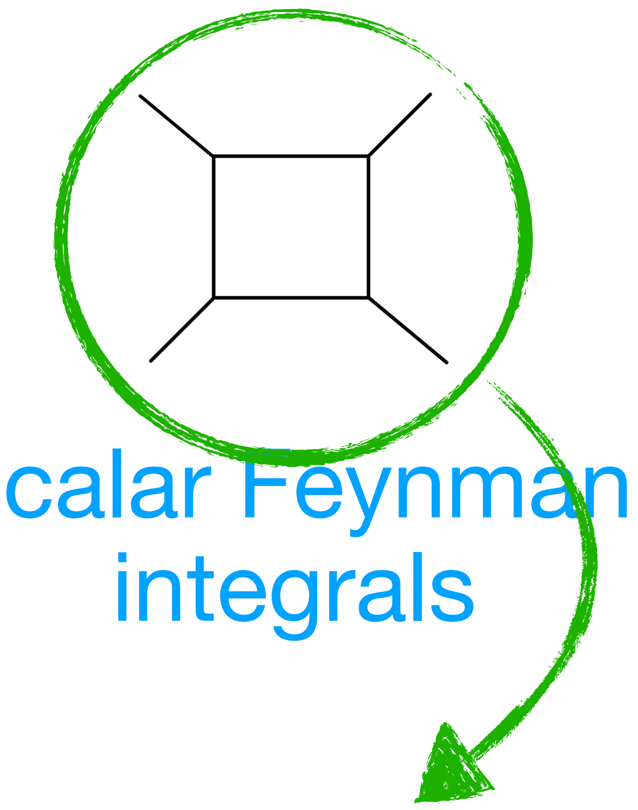

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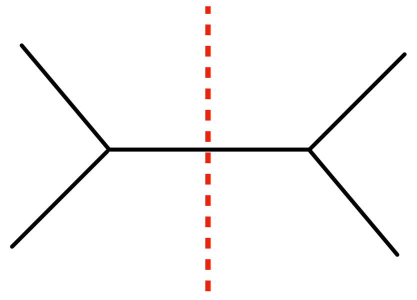

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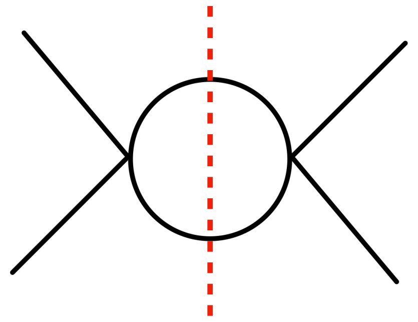

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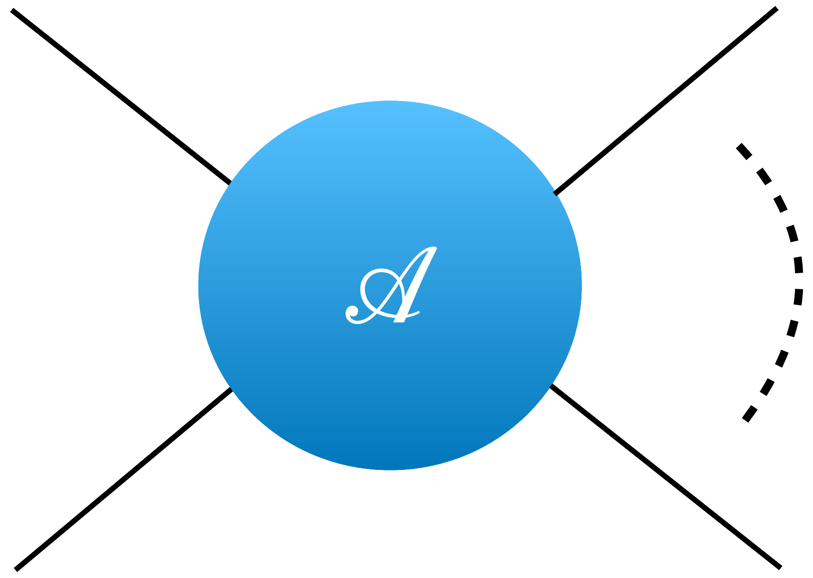

one-loop

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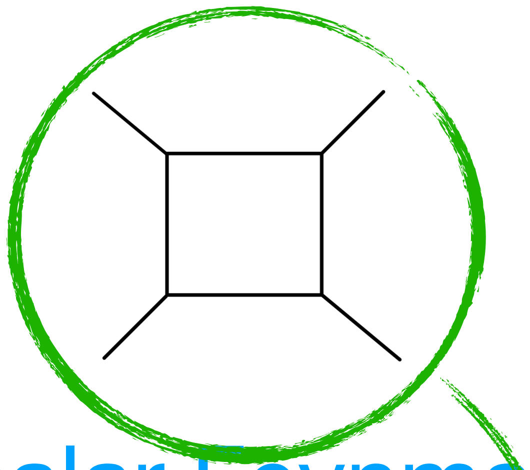

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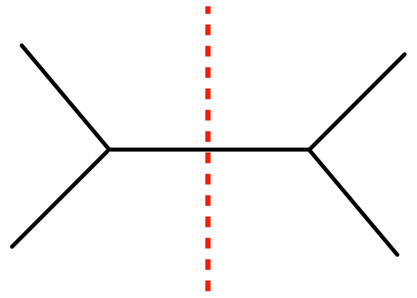

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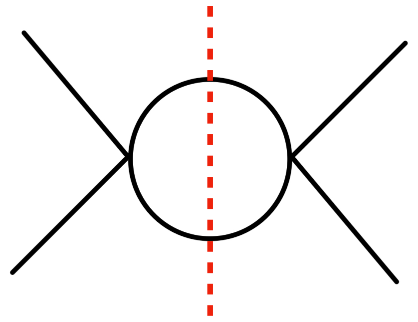

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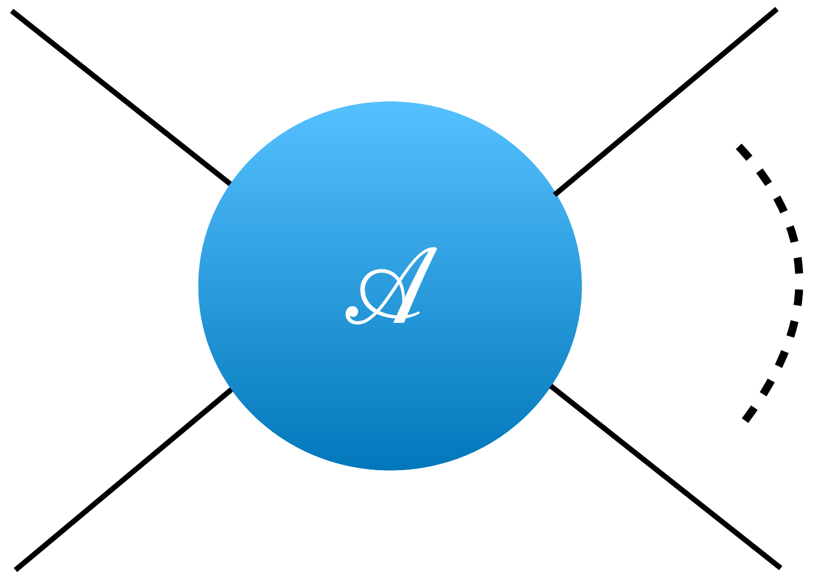

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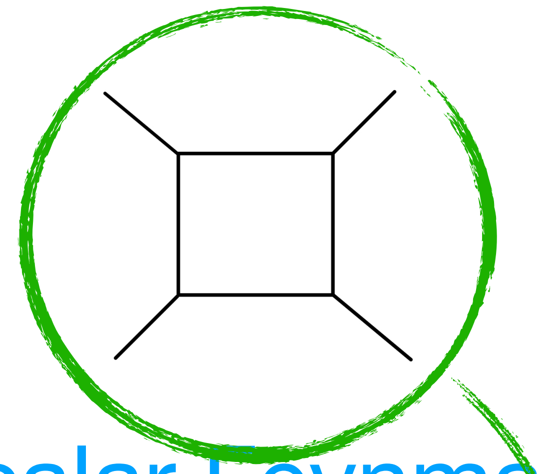

tensor
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one-loop

$$\sum_i R_i(s_{ij}) \int_\gamma \mathrm{d}\log f_n \wedge \ldots \wedge \mathrm{d}\log f_1$$

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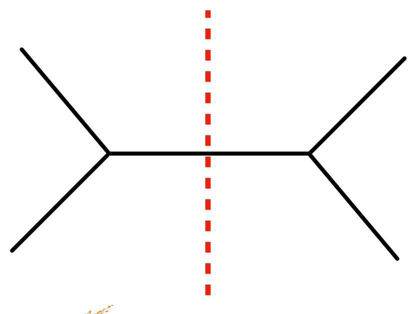

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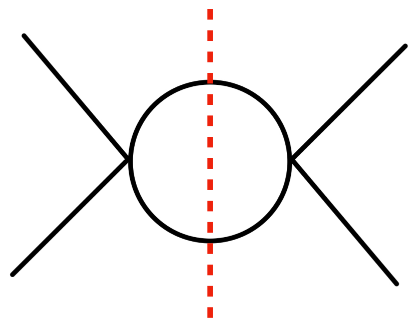

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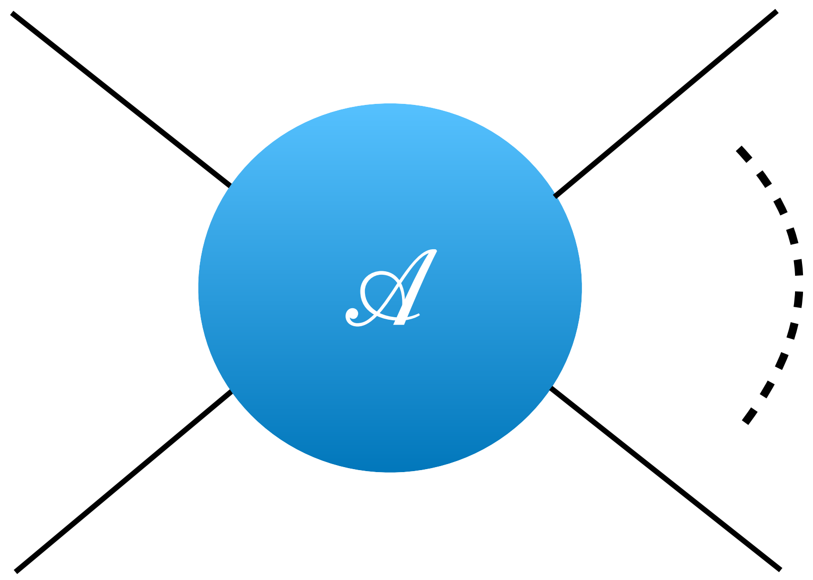

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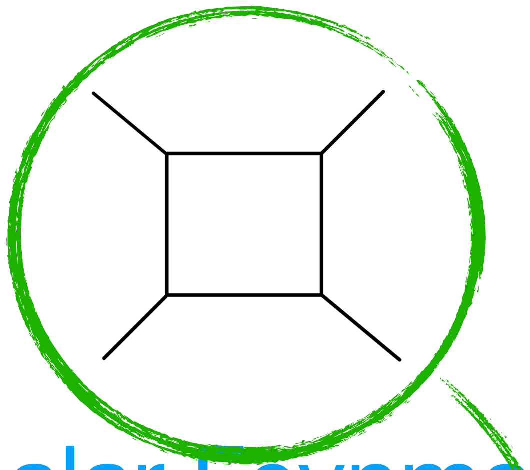

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algebraic
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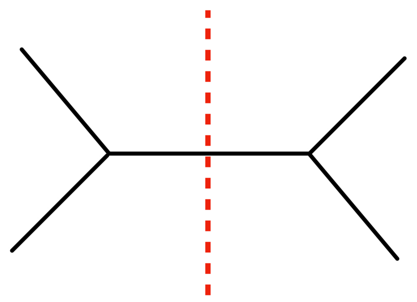

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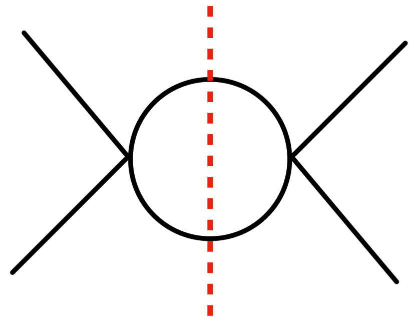

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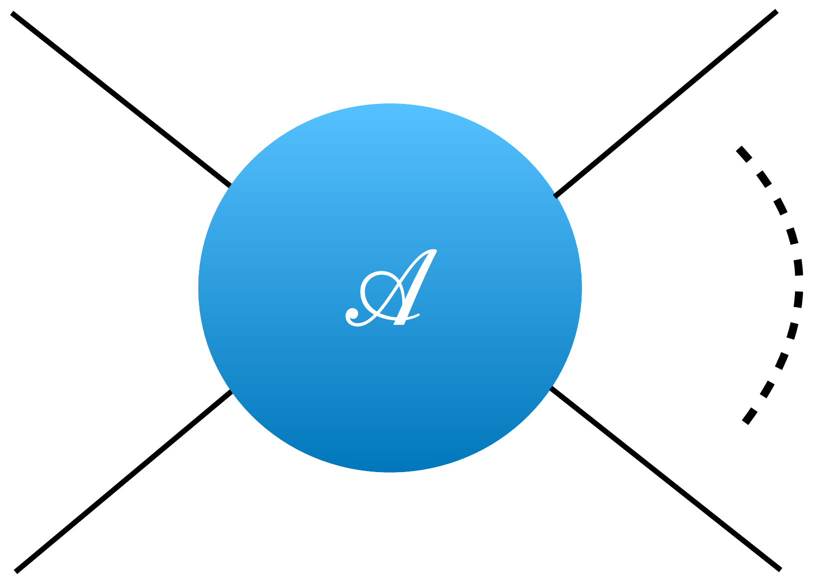

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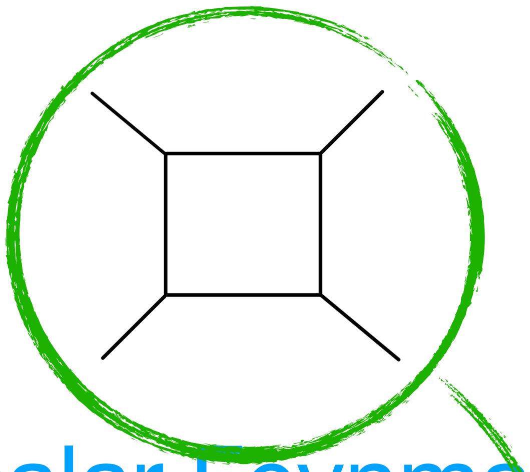

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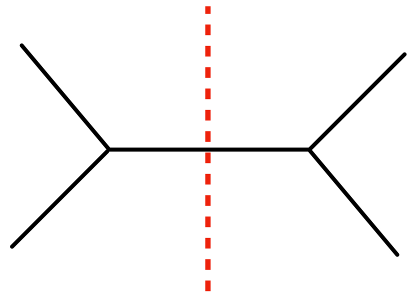

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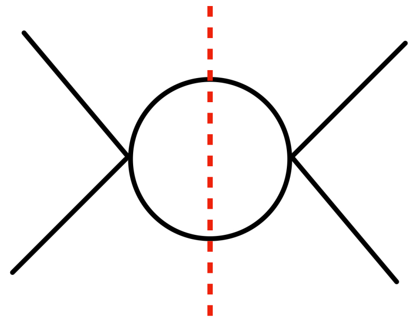

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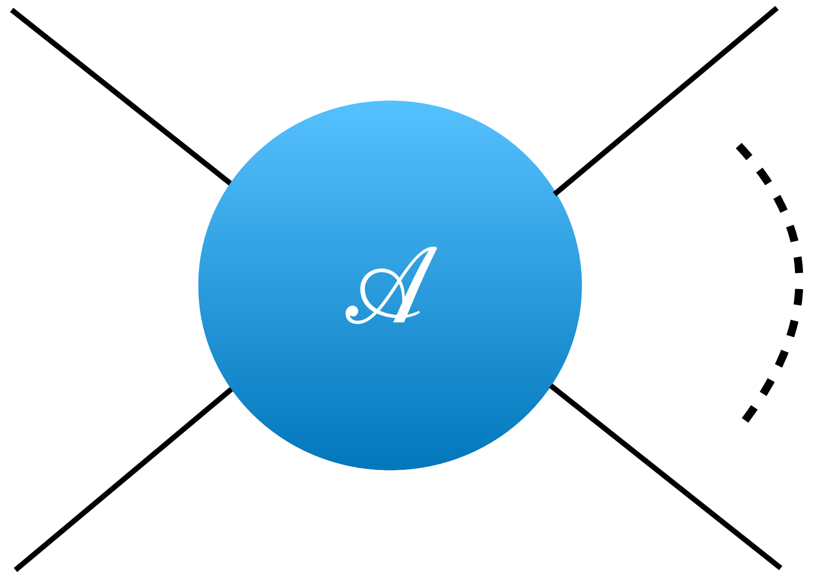
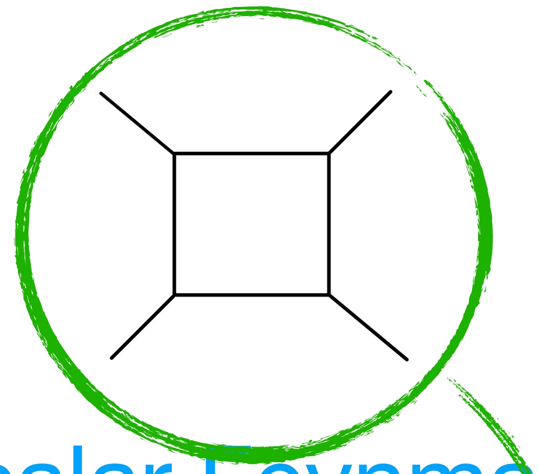


Computation

$\mathscr{A} \sim$ $T^{\mu\nu}$



tensor
structure

scalar Feynman
integrals

one-loop

$$\sum_i R_i(s_{ij}) \int_\gamma \mathrm{d}\log f_n \wedge \ldots \wedge \mathrm{d}\log f_1$$

algebraic
functions

special functions

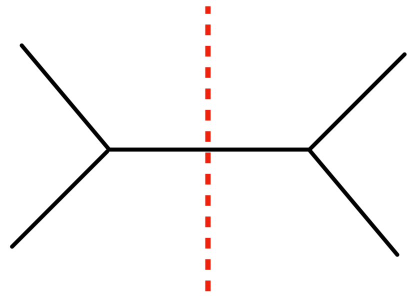
poles

Motivation

Singularities

poles

single-particle
states go on-shell

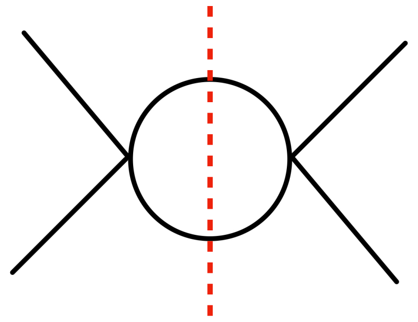


$$\sim \frac{1}{s}$$

“cutting” the propagator ~
the particle goes on shell

branch-cuts

multi-particle
states go on-shell



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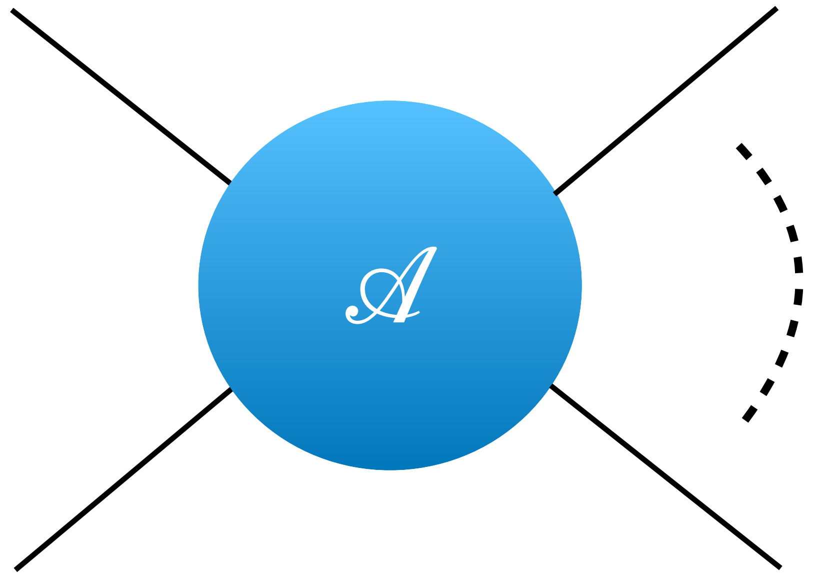

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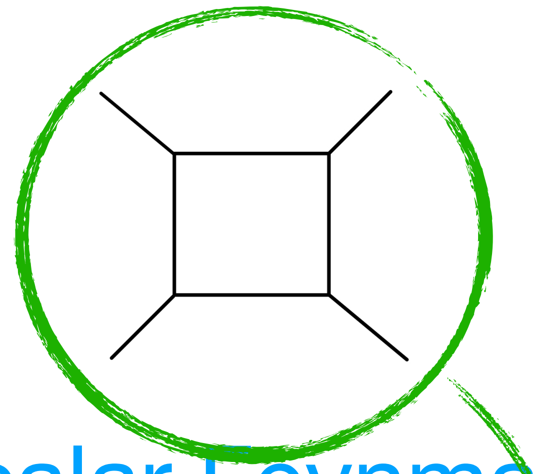

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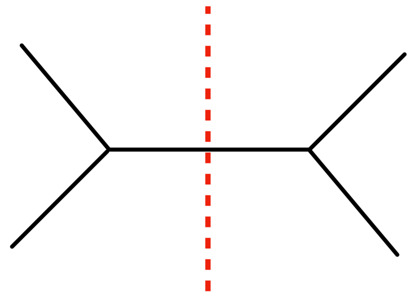

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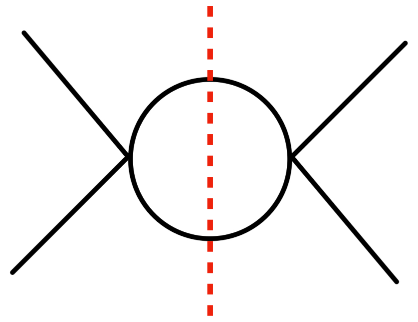

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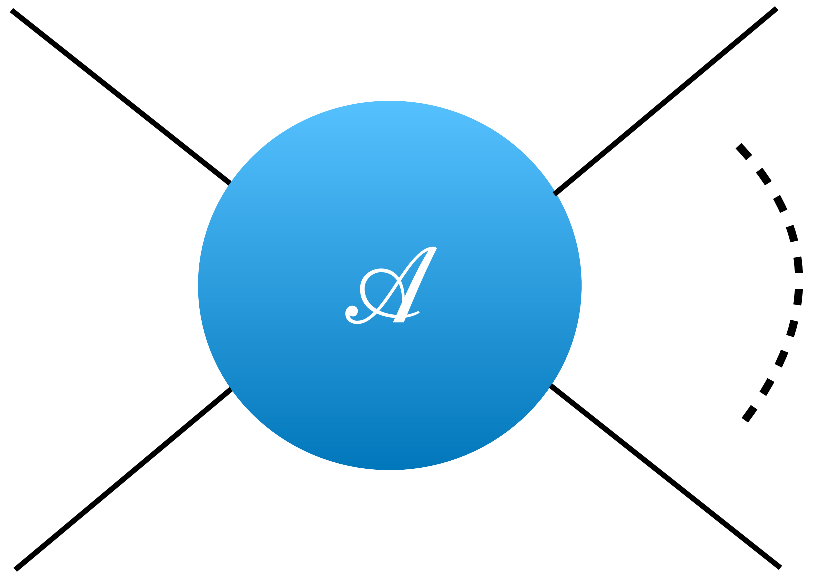

Computation

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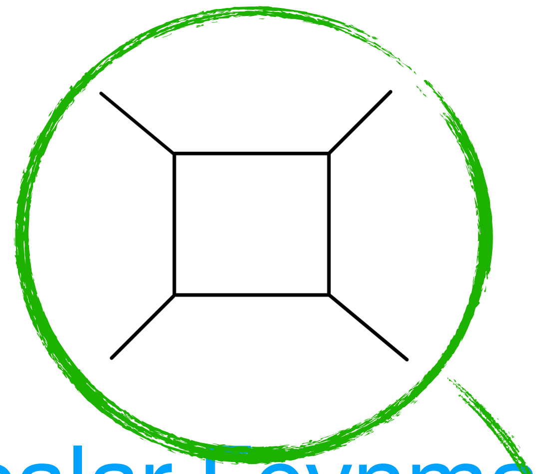

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transcendental:
from the integration

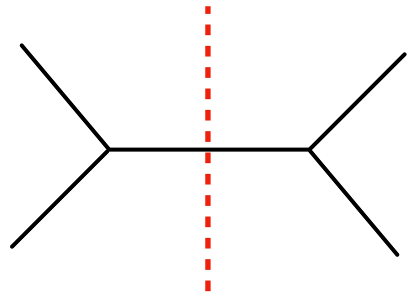
evident with *pure* integrals
[Arkani et al., ’10]

Motivation

Singularities

poles

single-particle states go on-shell



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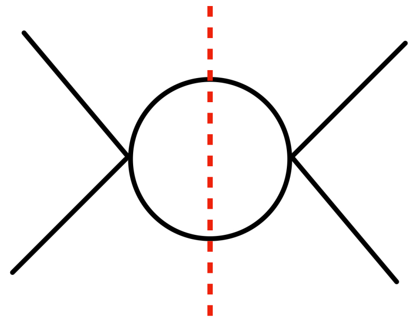

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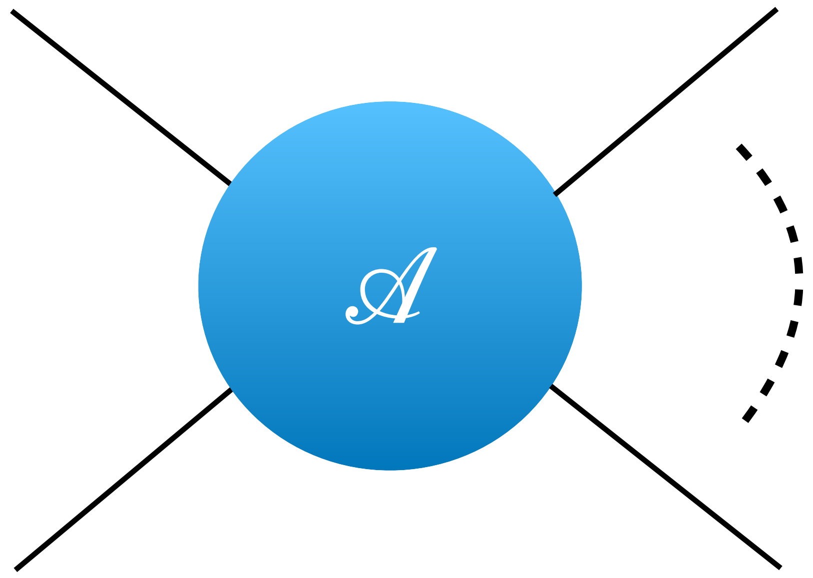
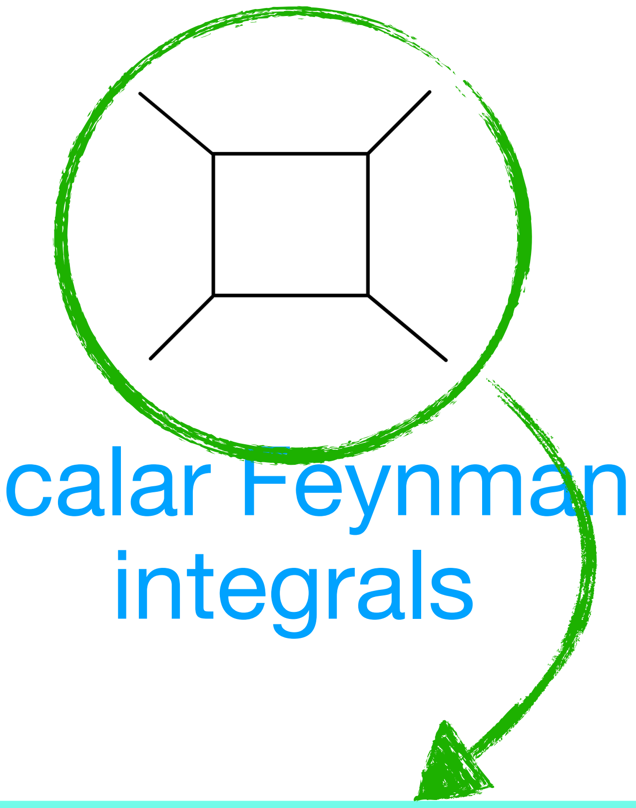


What stays beyond?

Computation

$$\mathscr{A} \sim T^{\mu\nu}$$

tensor structure



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First of all… when are there only (poly)logs?

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Evaluating (generalised) discontinuities → algebraic functions

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The *integrand* lives on the Riemann sphere with some marked points

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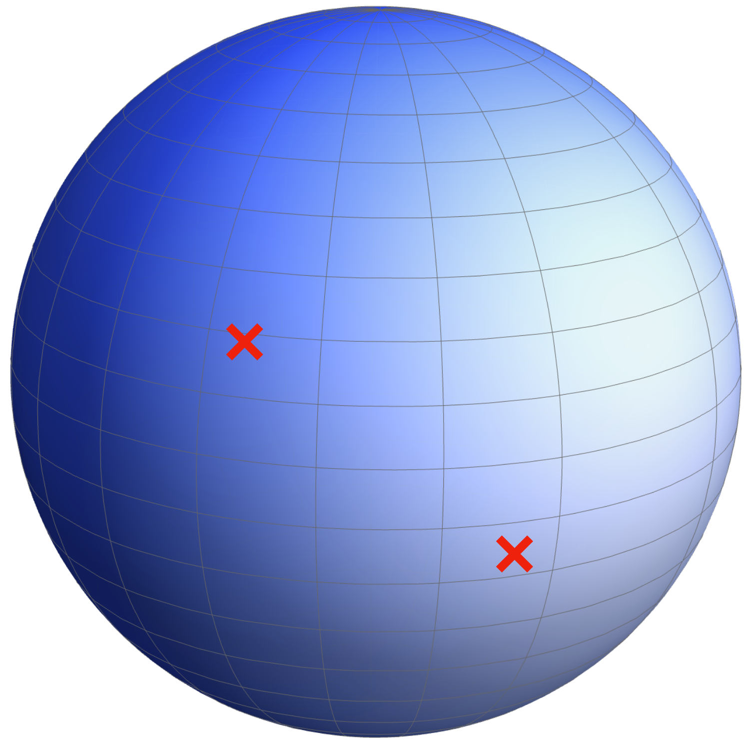

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Evaluating (generalised) discontinuities ➡ algebraic functions

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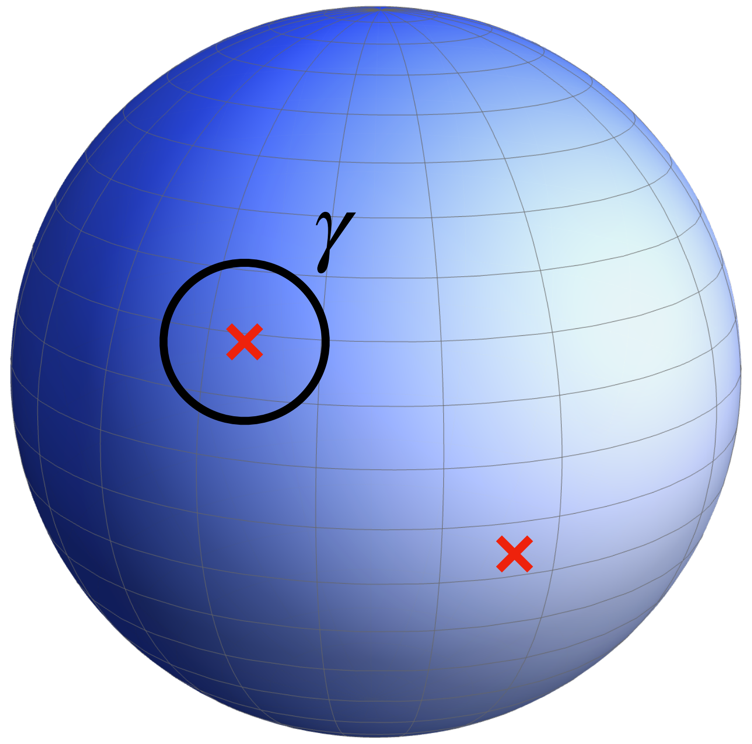

Independent forms: $\int \frac{\mathrm{d}t}{t-a}$

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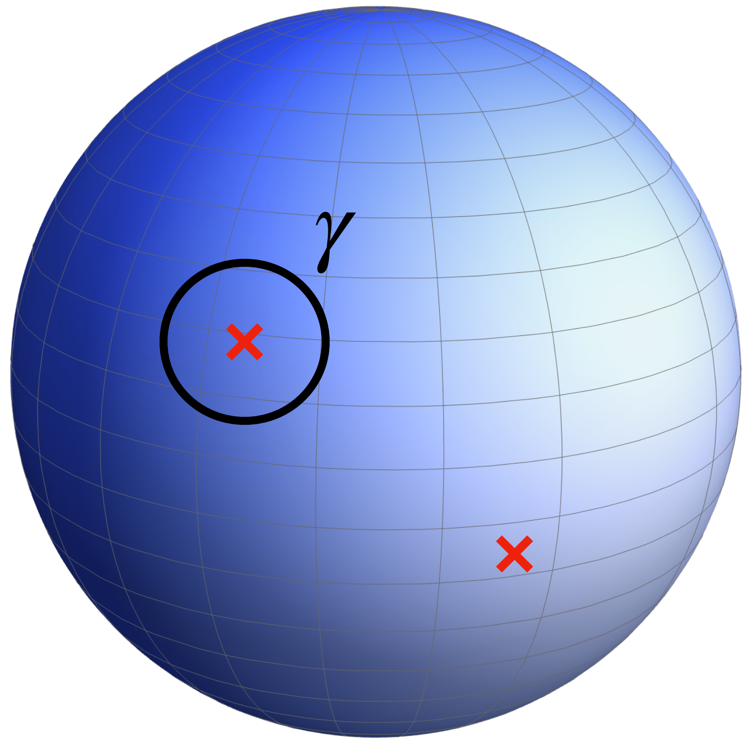

Independent forms: $\int \frac{\mathrm{d}t}{t-a}$ ➡ *multiple polylogarithms*

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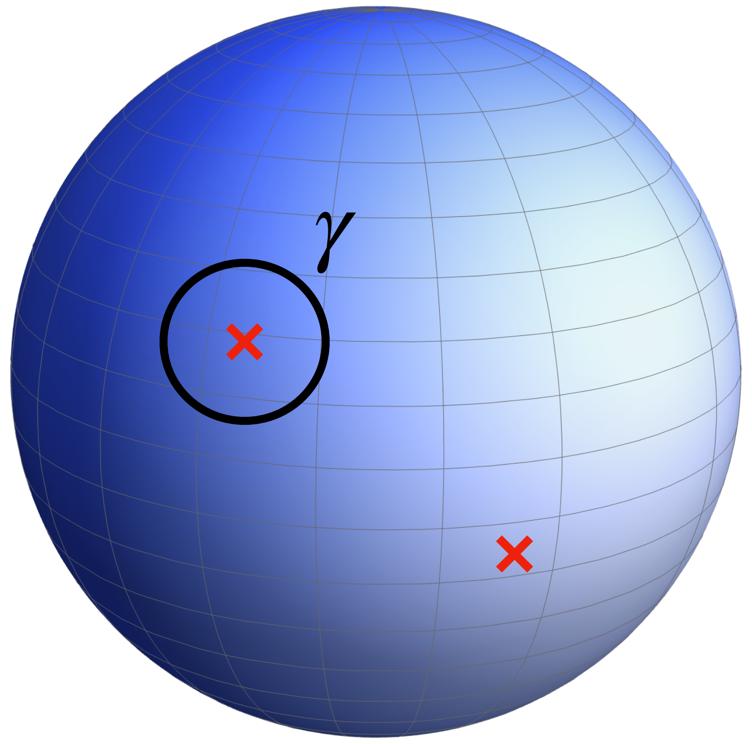

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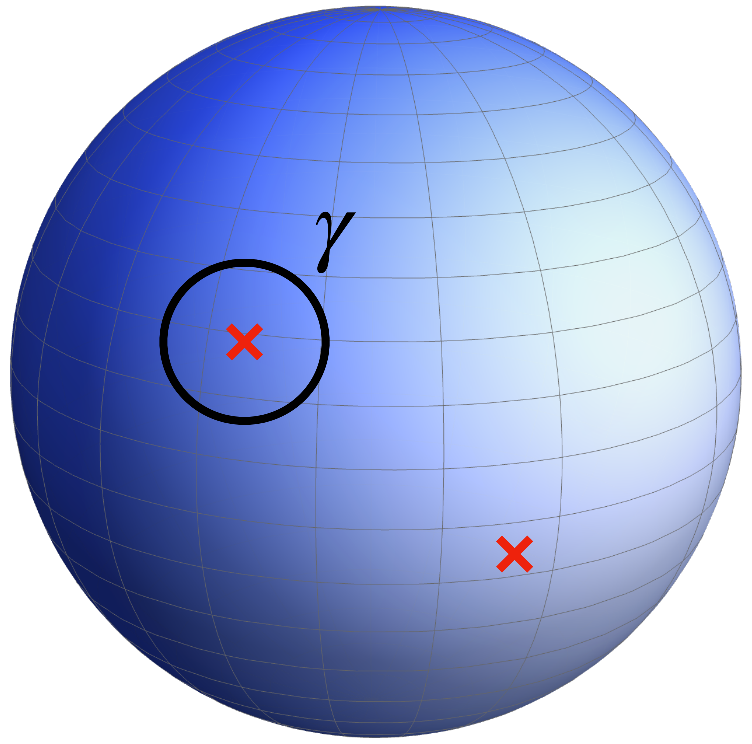

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dim-reg parameter

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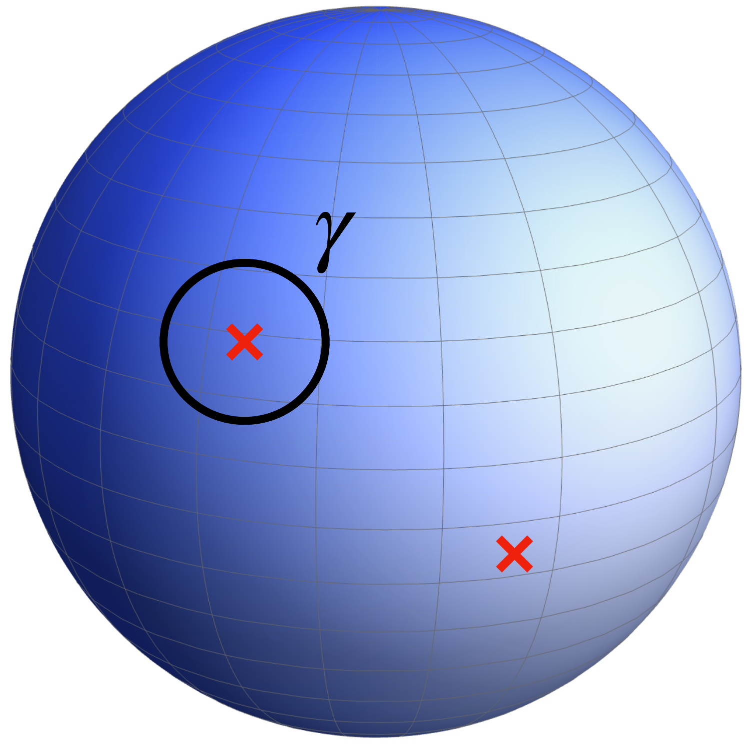

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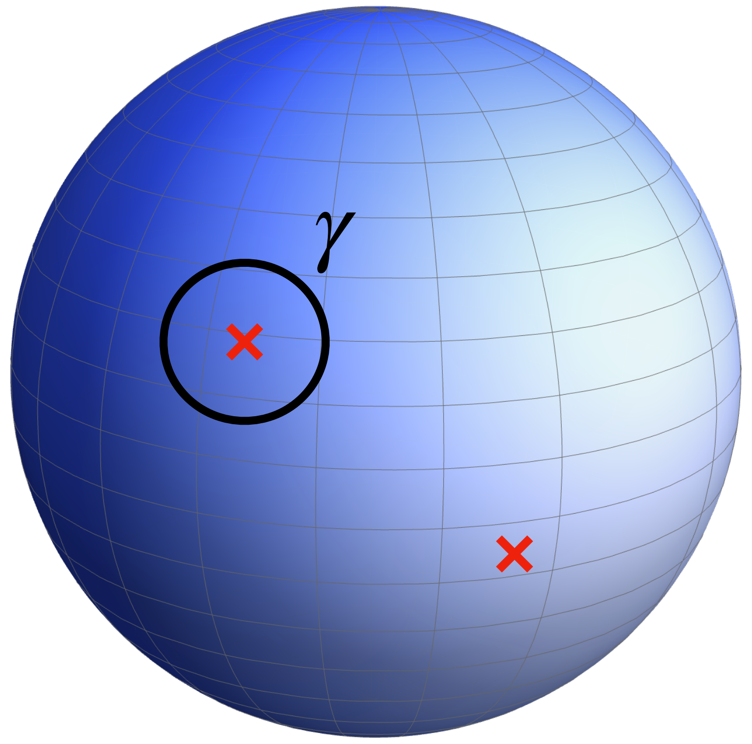

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[Henn, '13]

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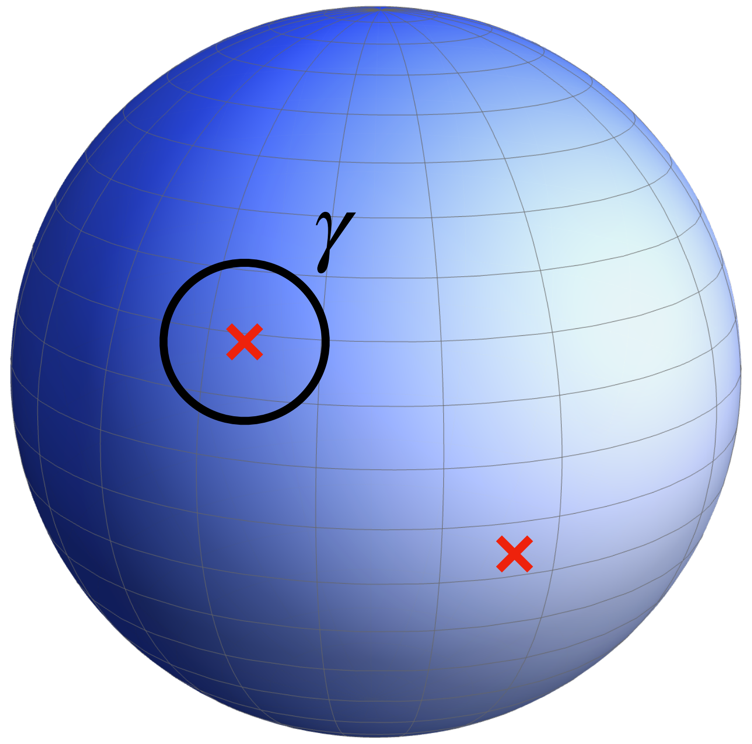

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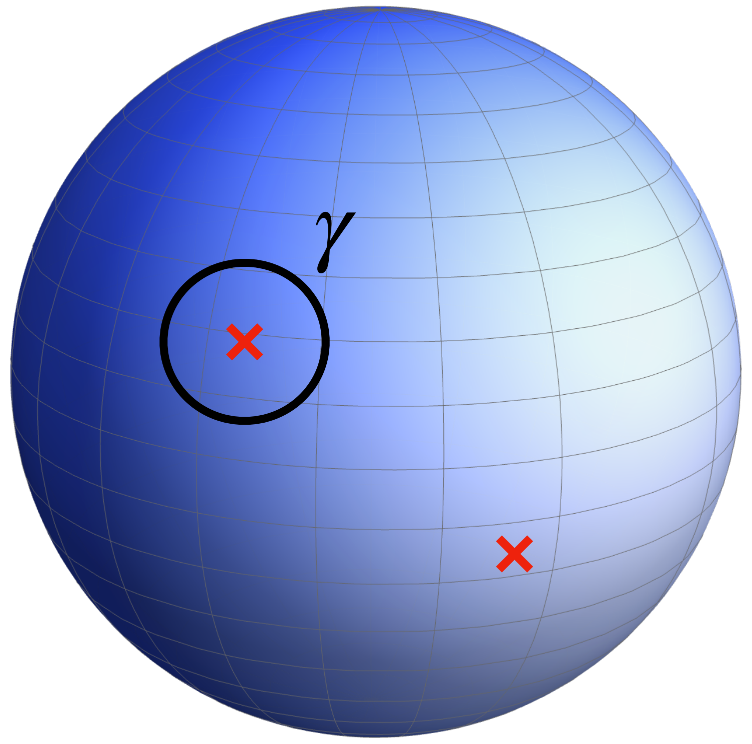

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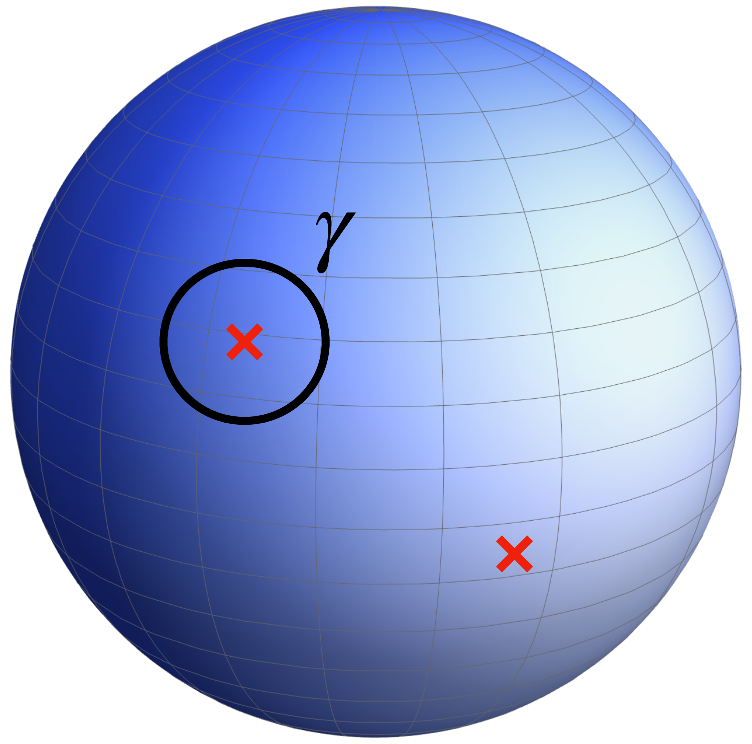

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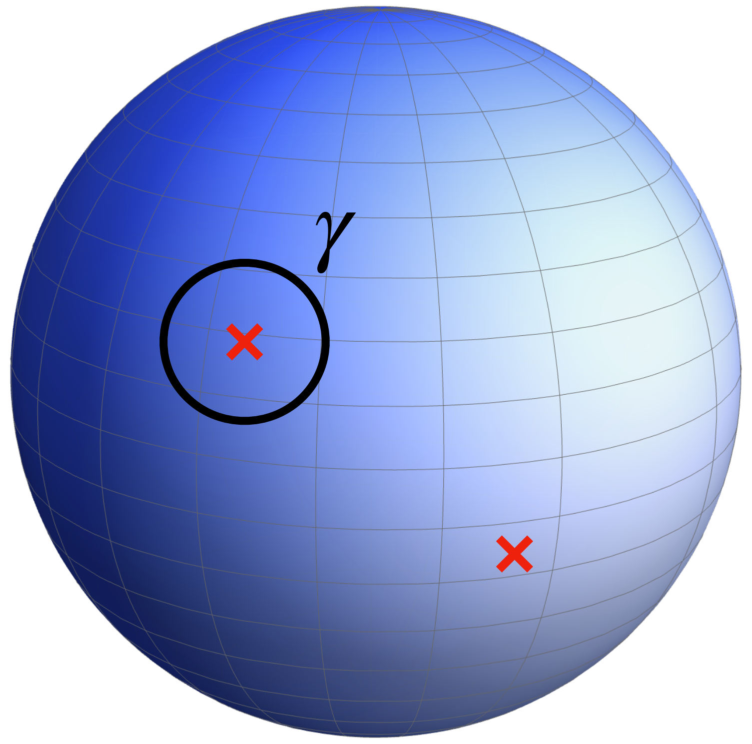

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pure

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Evaluating (generalised) discontinuities → algebraic functions

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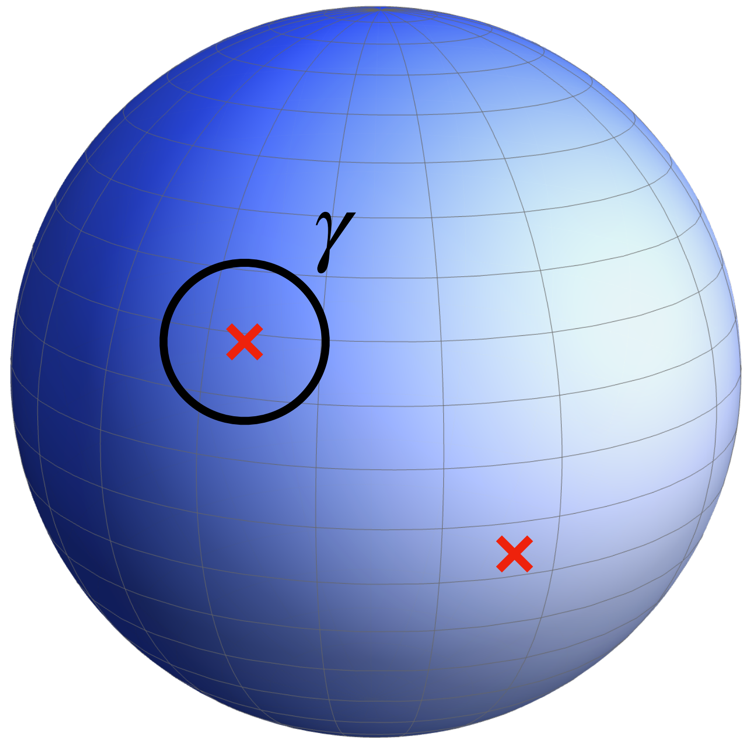

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pure
~only logs

uniform transcendental weight
~ same transcendentality

And now beyond

And now beyond

Evaluating (generalised) discontinuities → trascendental functions

And now beyond

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simplest generalisation: **elliptic curves**

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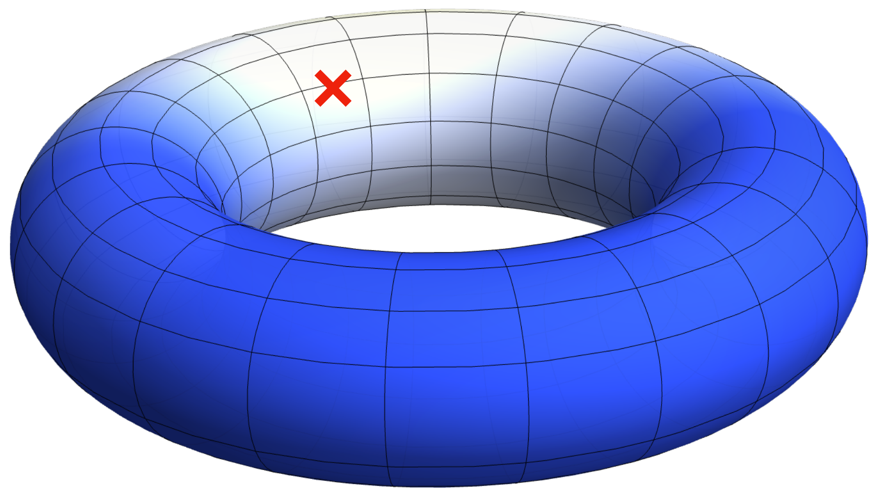

And now beyond

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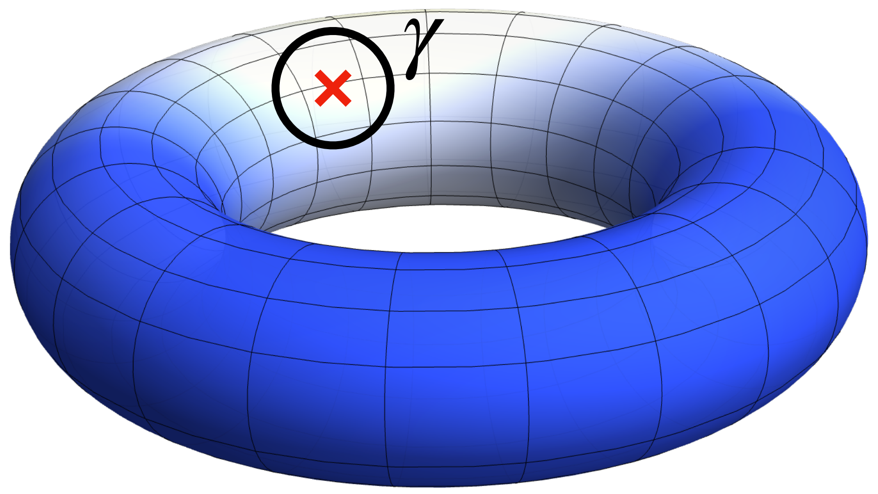


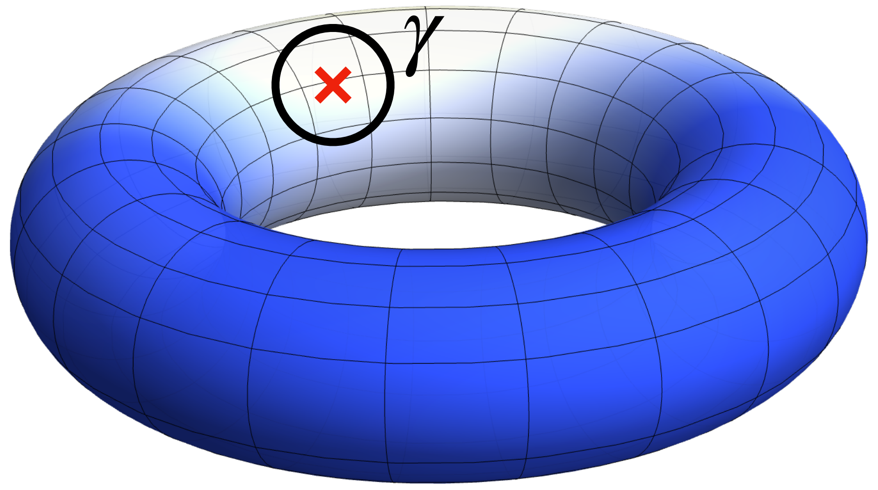
still forms that produce log singularities: $\int_\gamma \frac{\mathrm{d}x}{(x-a)\,y}$

And now beyond

Evaluating (generalised) discontinuities ➡ trascendental functions

simplest generalisation: **elliptic curves**

$\boxed{y = \sqrt{P_3(x)}}$



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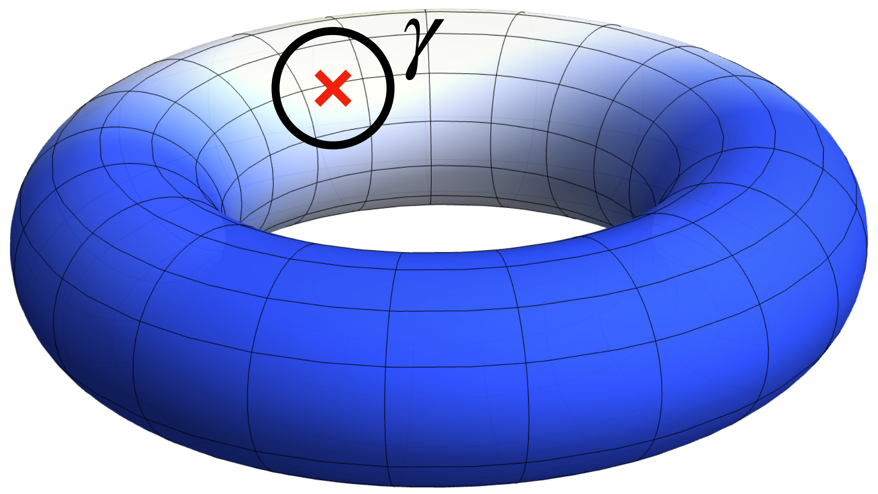

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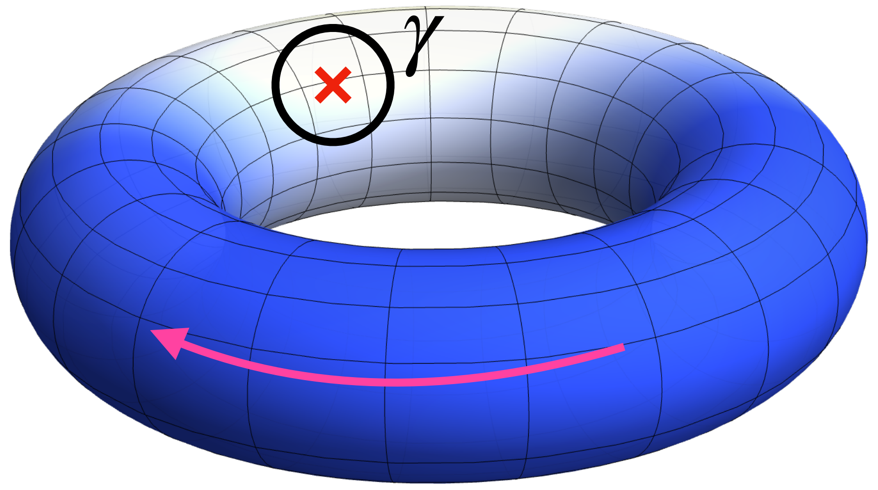

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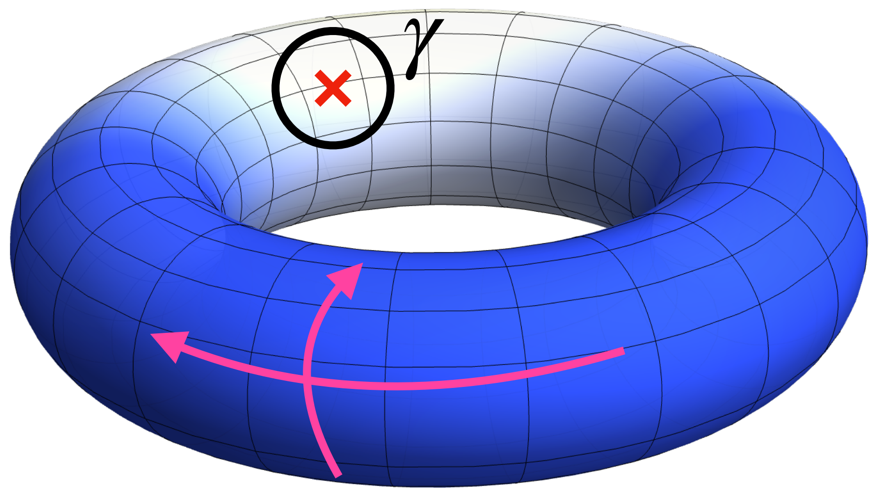

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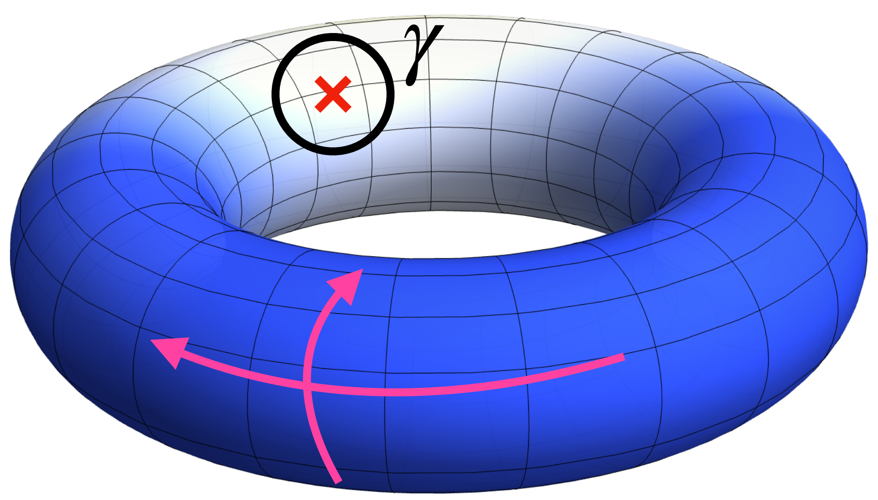

still forms that produce log singularities: $\int_\gamma \frac{\mathrm{d}x}{(x-a)\,y}$ simple poles

and others that don't! $\int \frac{\mathrm{d}x}{y}$,

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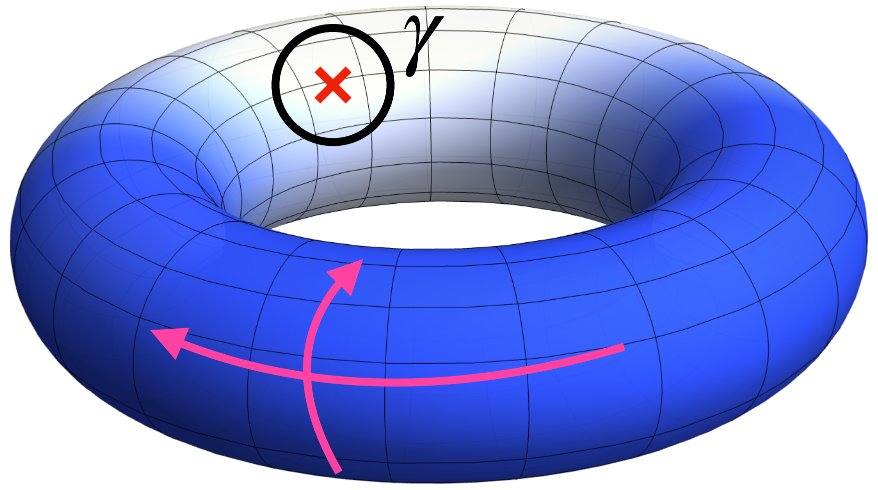

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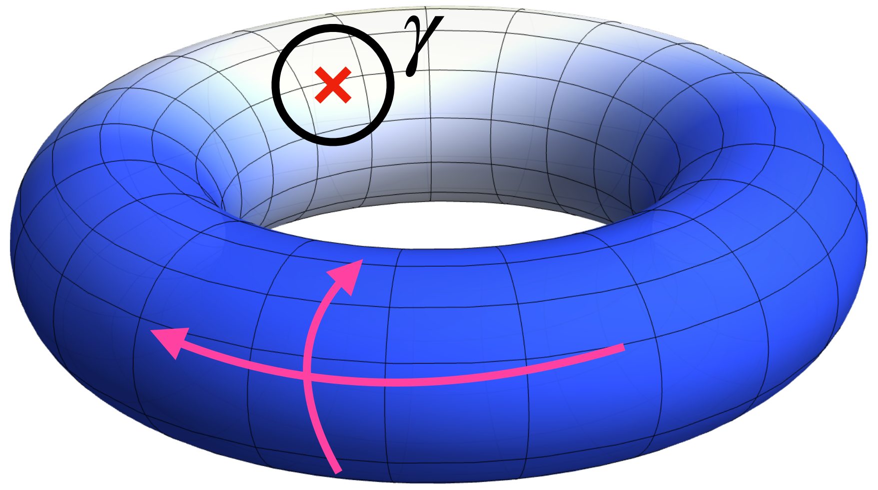

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$\int \frac{\mathrm{d}x\, x}{y}$

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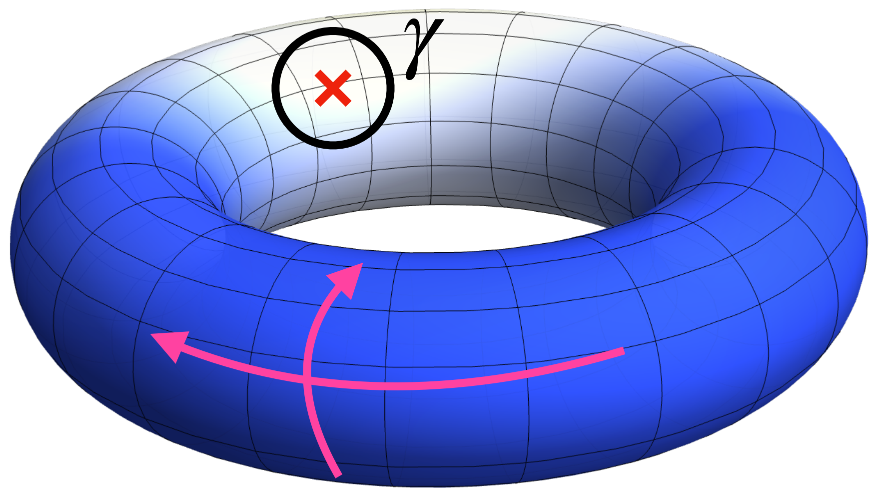

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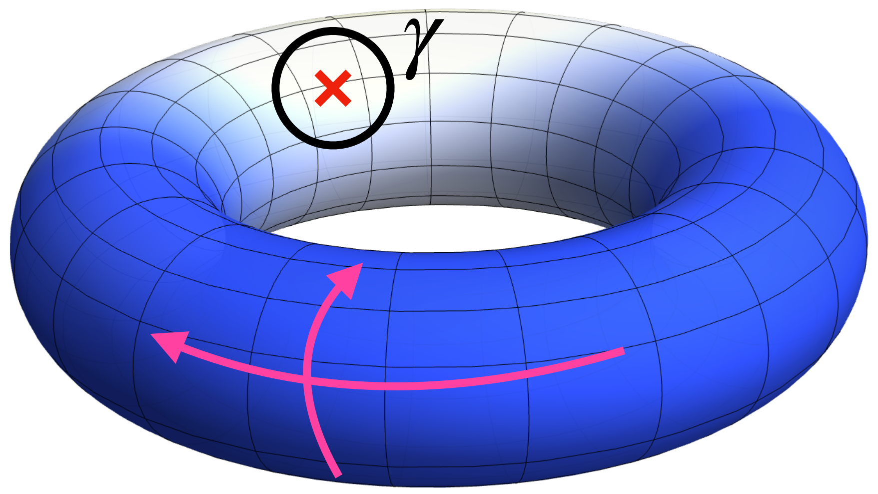

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Can we keep similar properties as before?

And now beyond

Evaluating (generalised) discontinuities → trascendental functions

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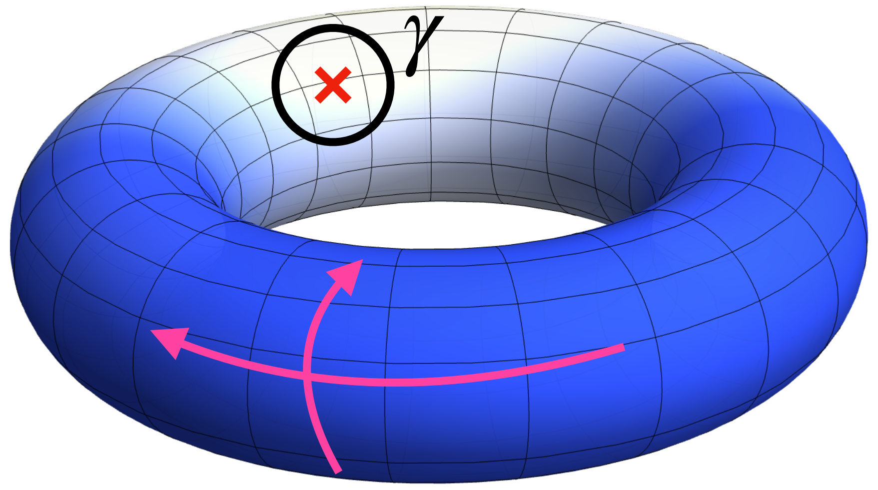

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Can we keep similar properties as before? → yes!

And now beyond

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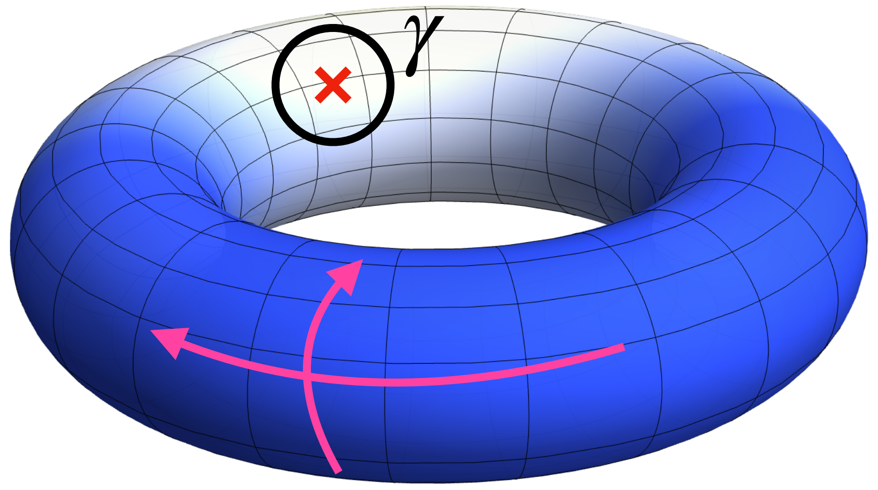

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And now beyond

Evaluating (generalised) discontinuities → trascendental functions

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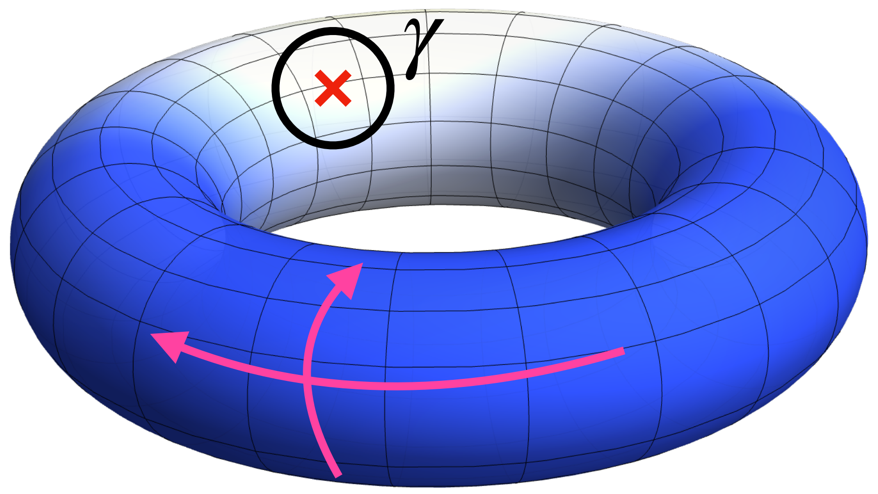

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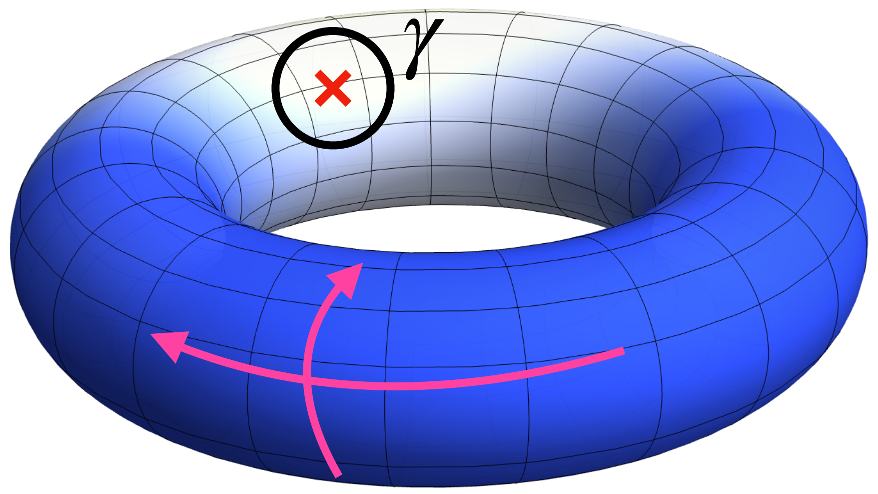

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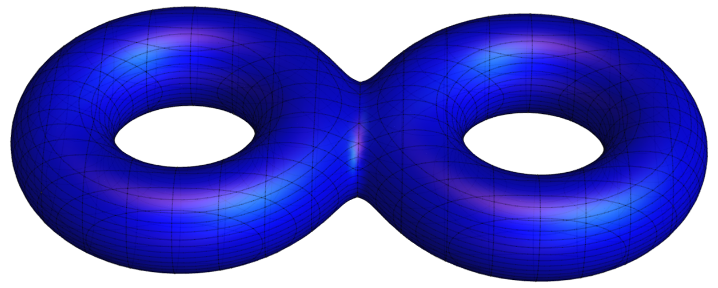

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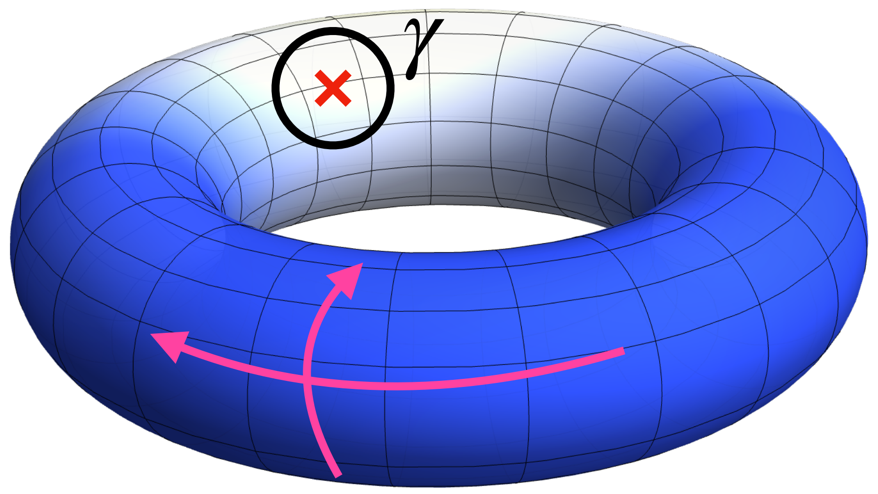

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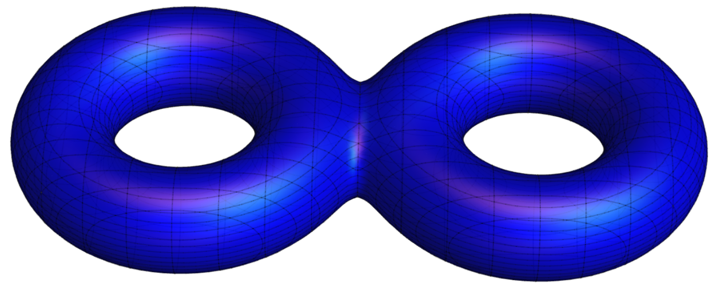
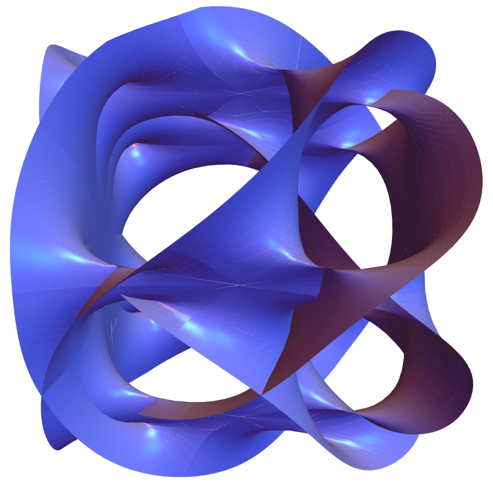
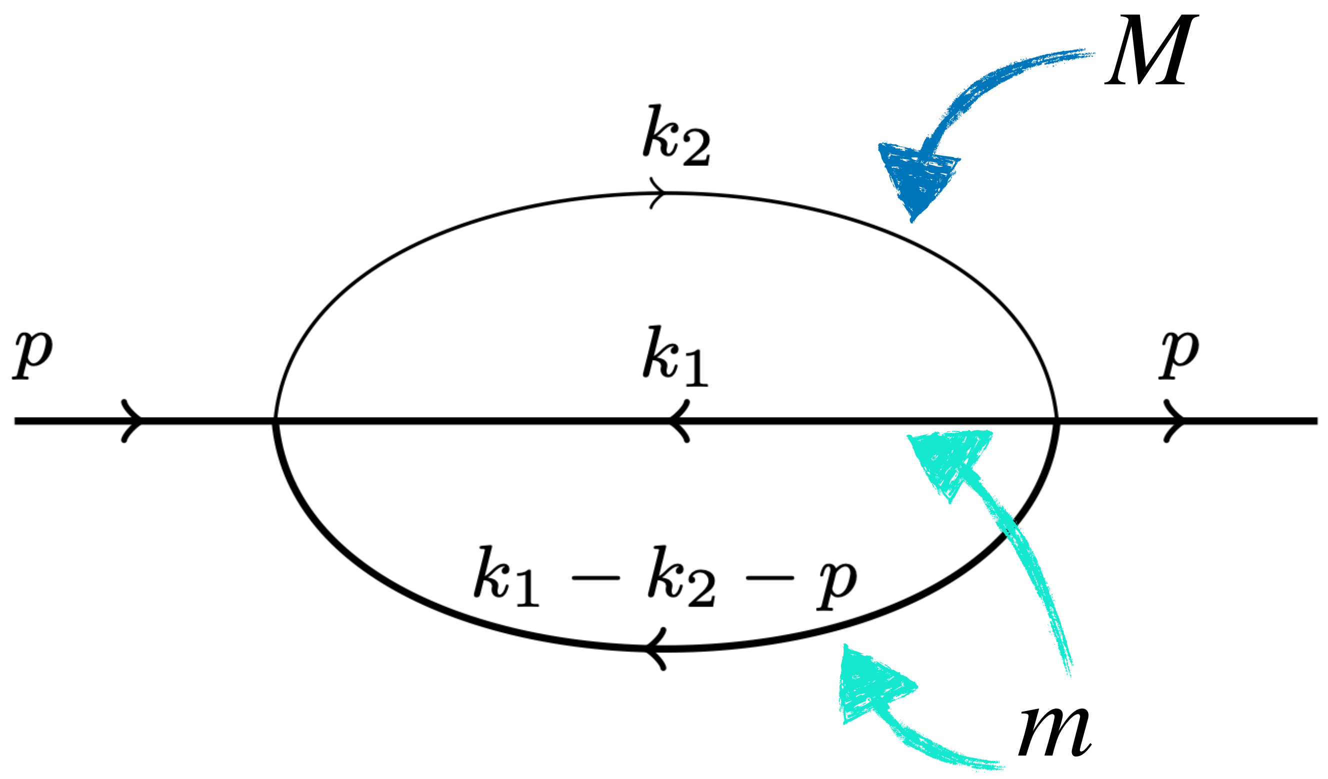




Running example: two-loop sunrise



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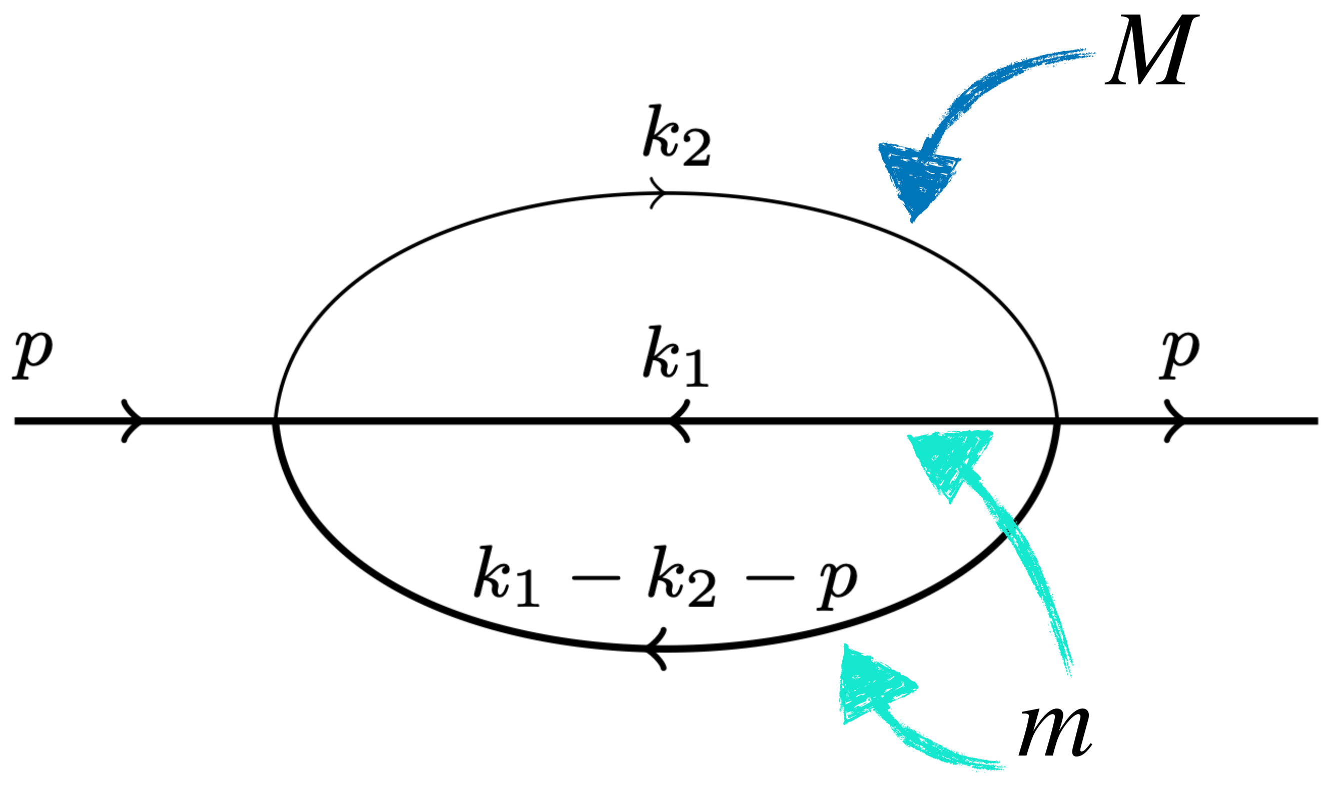

Two cases:

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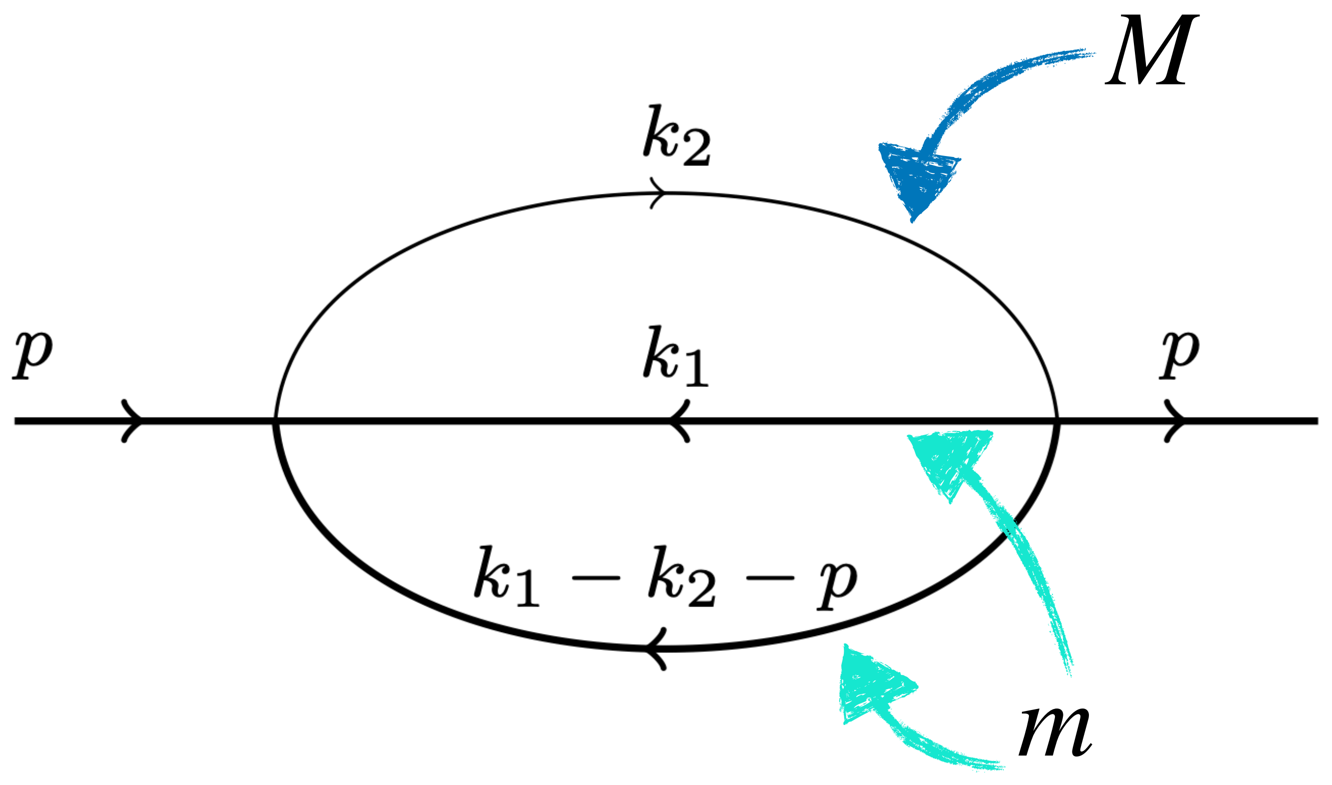

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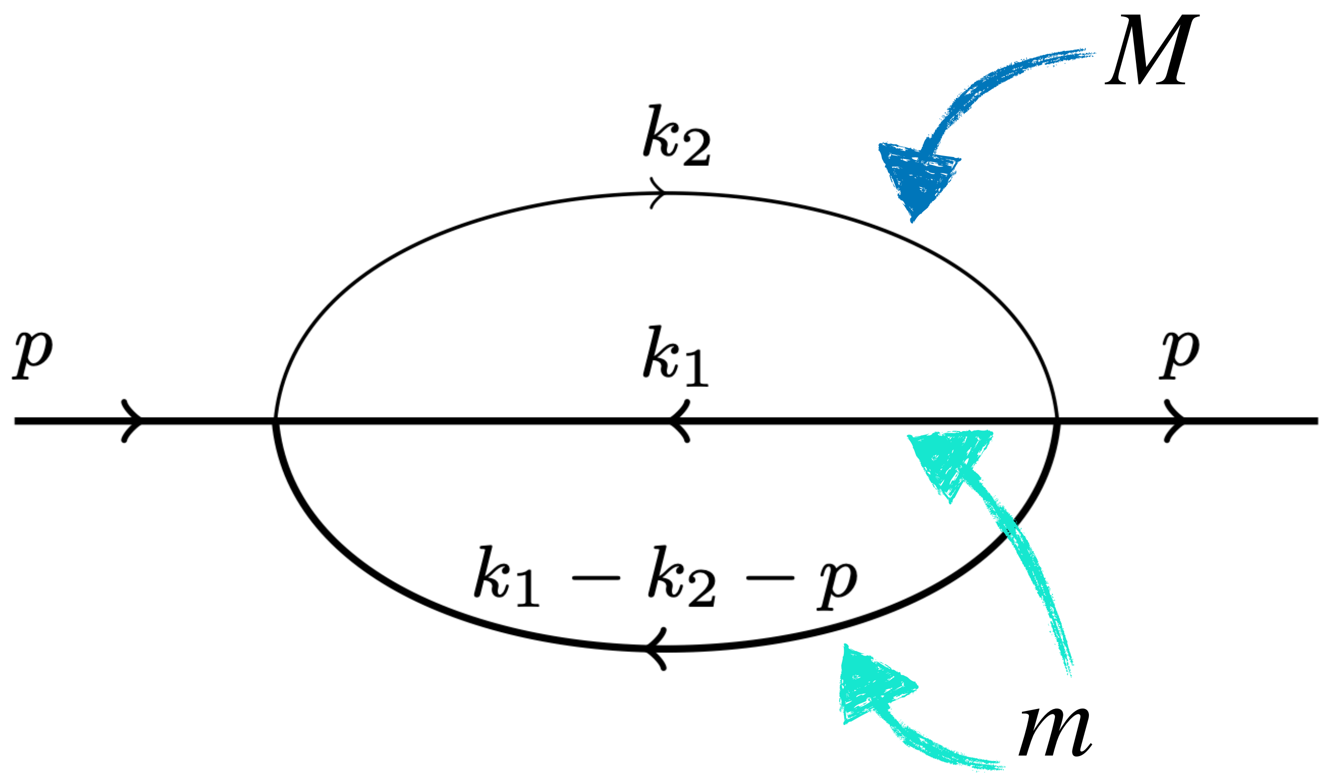

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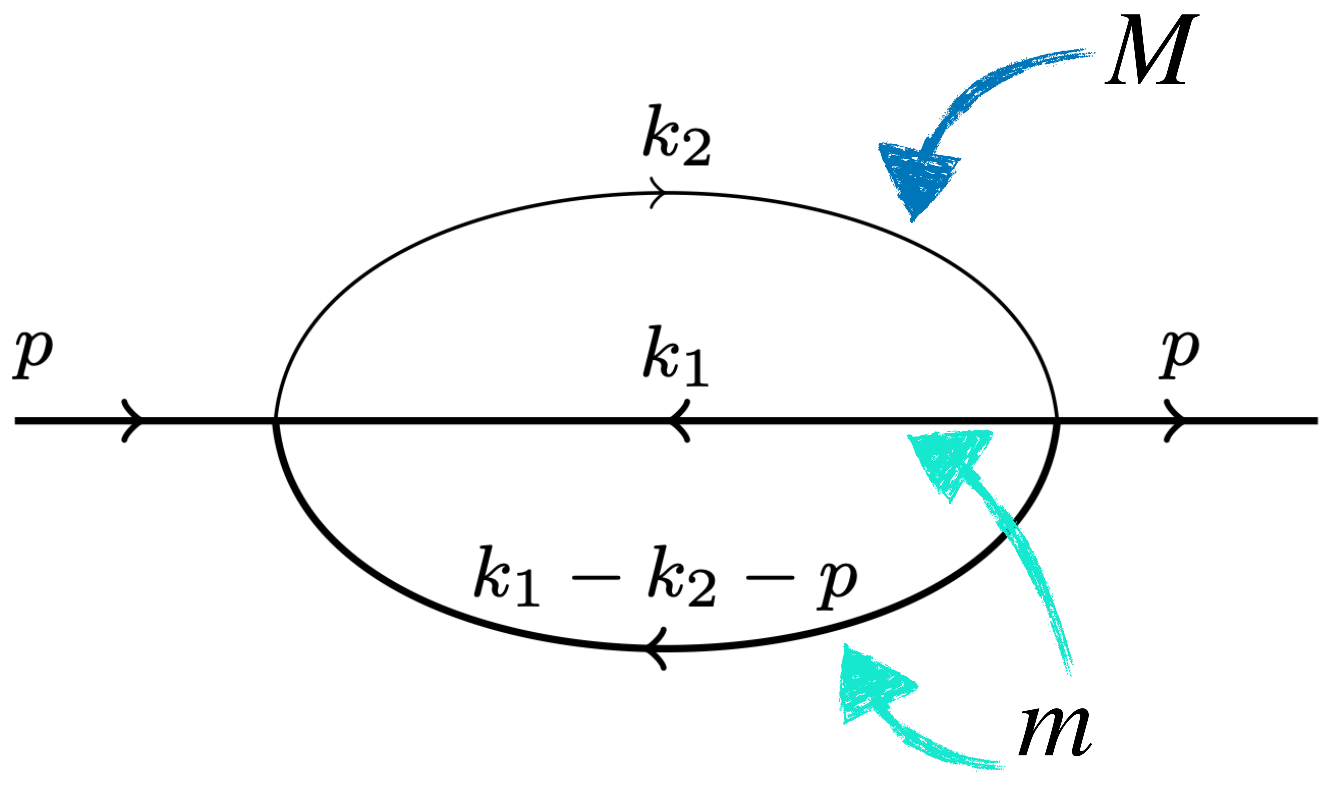

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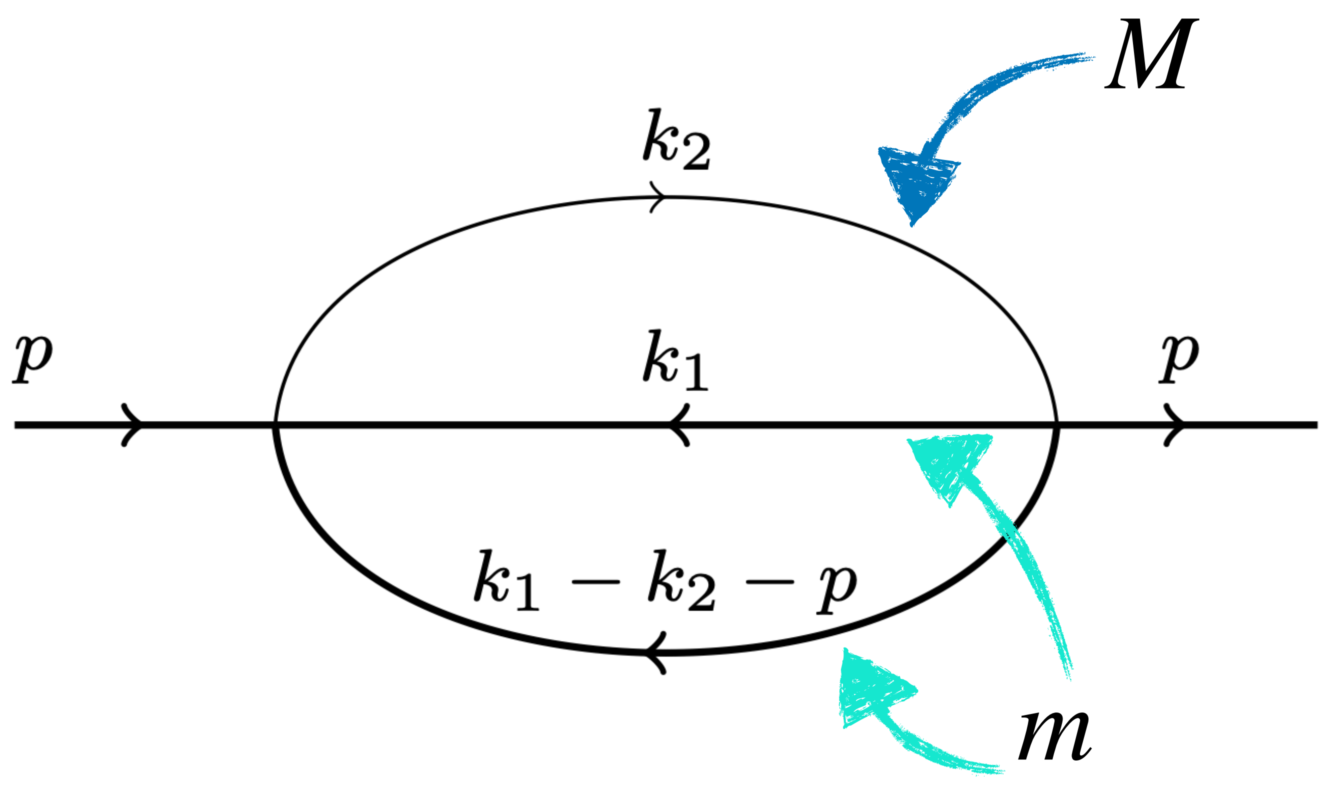

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Integrand analysis

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$\sqrt{P_4(z_5)}$: elliptic curve!

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simple pole in z_5

Case 1: $M = 0$

$$I_{1,1,1,0,\nu_5}\big|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5 z_5^{-\nu_5}}{\underbrace{z_5}_{\text{simple pole in } z_5}\sqrt{\underbrace{(2z_5+s)(4m^2-s-2z_5)}_{\sqrt{P_2(z_5)}}}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

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simple pole in z_5

$\sqrt{P_2(z_5)}$

- $\nu_5 = 0$

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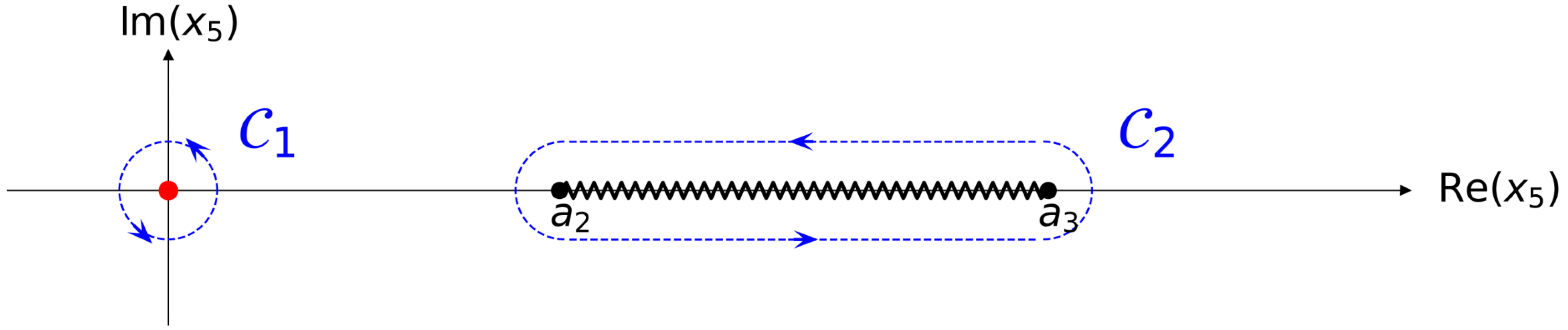

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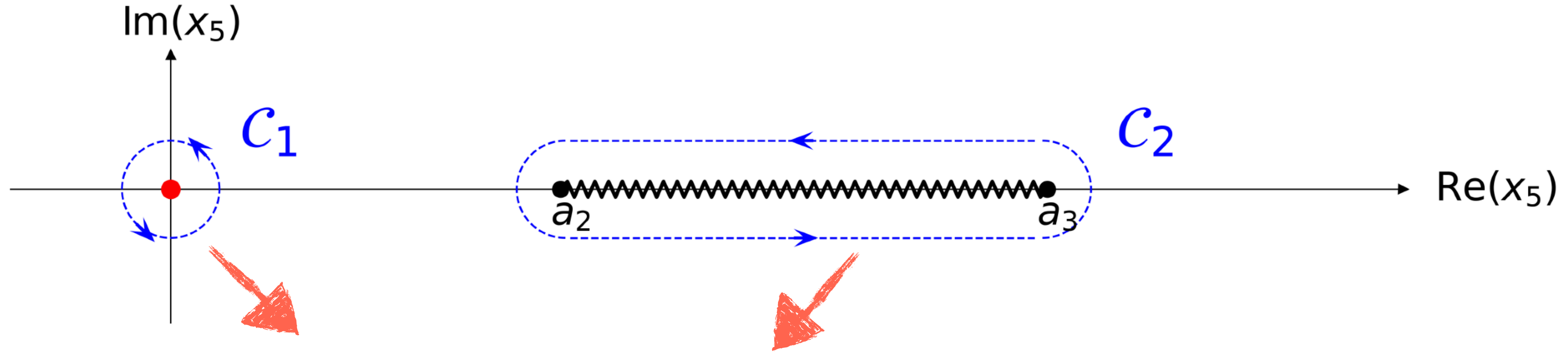

one independent cycle!

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simple pole in z_5

$\sqrt{P_2(z_5)}$

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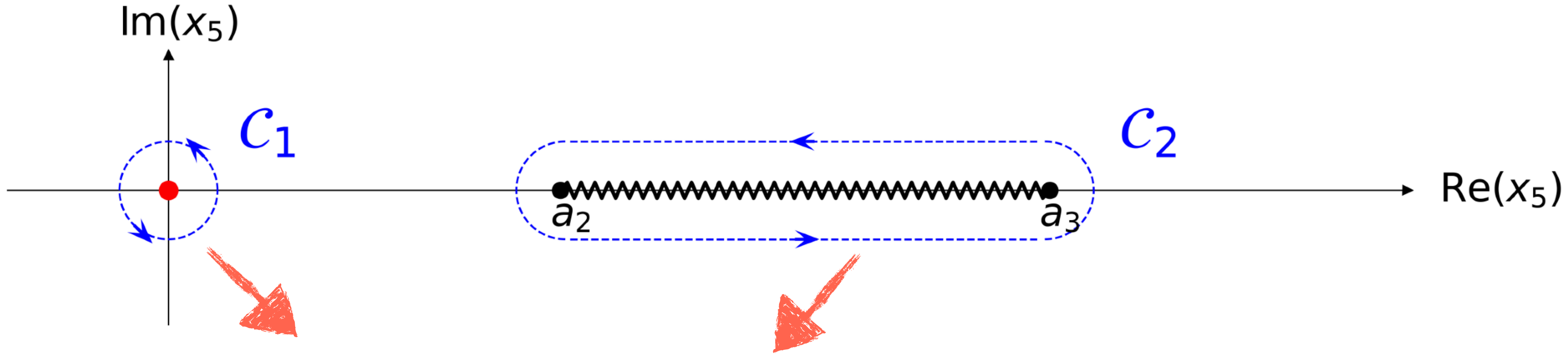

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$$\int_{\mathcal{C}_1} \frac{\mathrm{d}z_5}{z_5\sqrt{(2z_5+s)(4m^2-s-2z_5)}} \propto \int_{\mathcal{C}_2} \frac{\mathrm{d}z_5}{z_5\sqrt{(2z_5+s)(4m^2-s-2z_5)}} \propto \frac{1}{\sqrt{s(s-4m^2)}}$$

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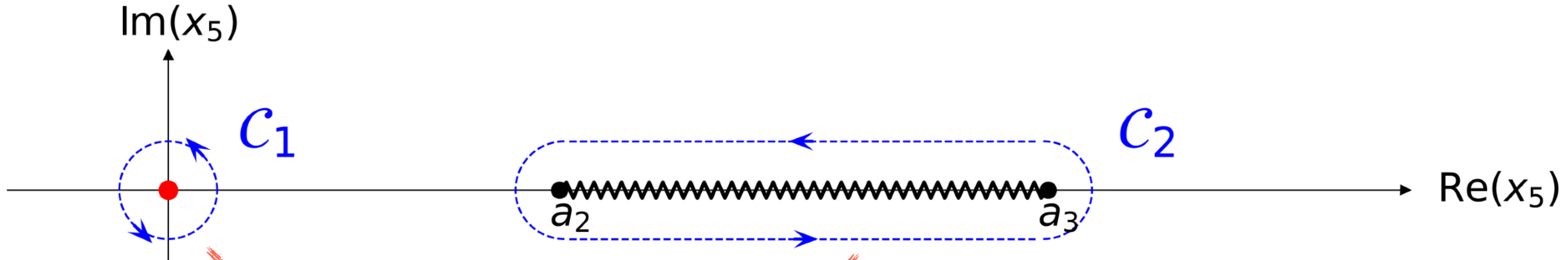

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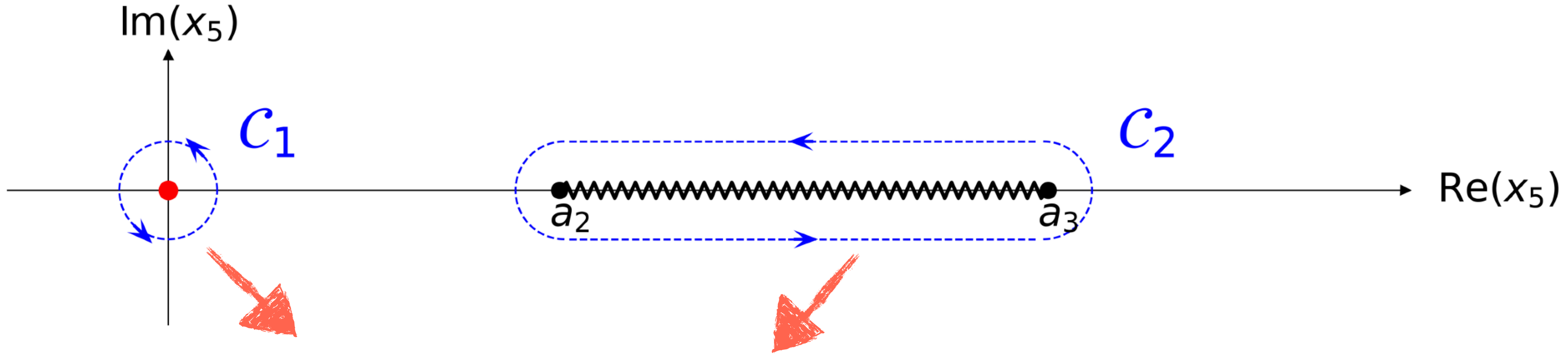

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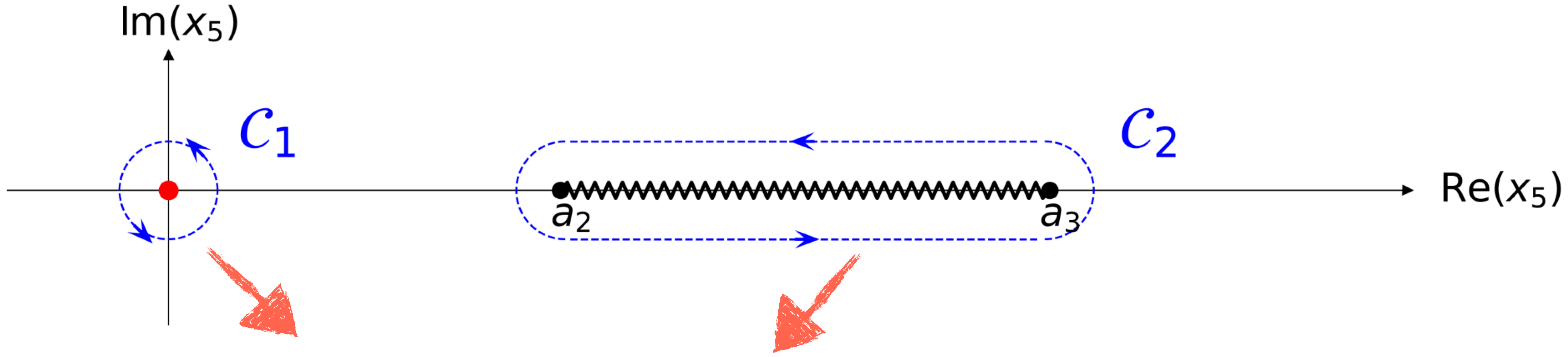

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- $\nu_5 = -1$

Case 1: $M = 0$

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simple pole in z_5

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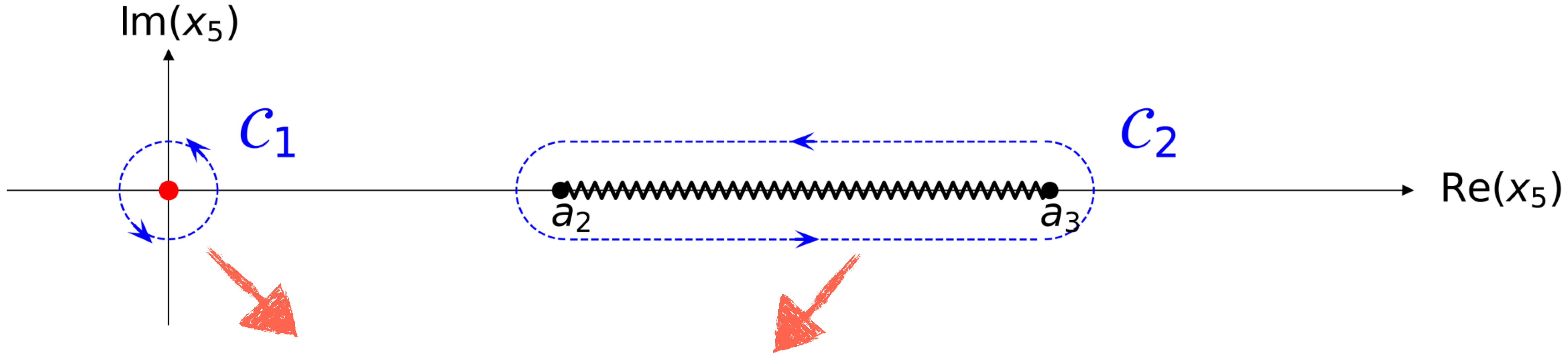

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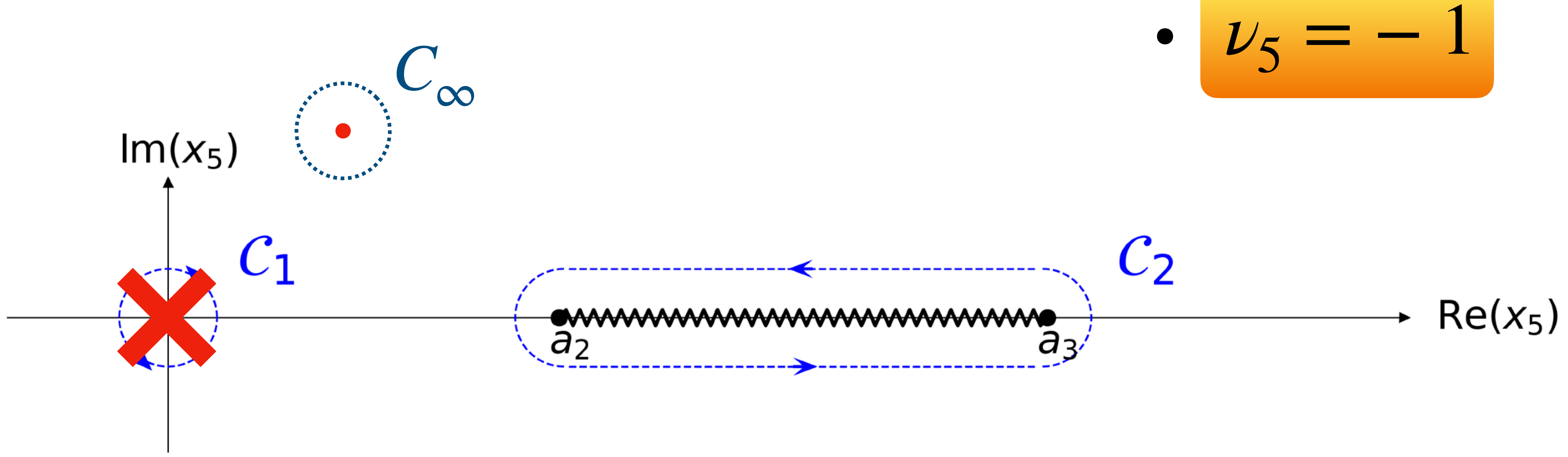

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$$I_{1,1,1,0,0}\big|_{z_1=z_2=z_3=0} \sim \frac{1}{\sqrt{s(s-4m^2)}}\int \mathrm{d}\log g_1 \int \mathrm{d}\log f$$

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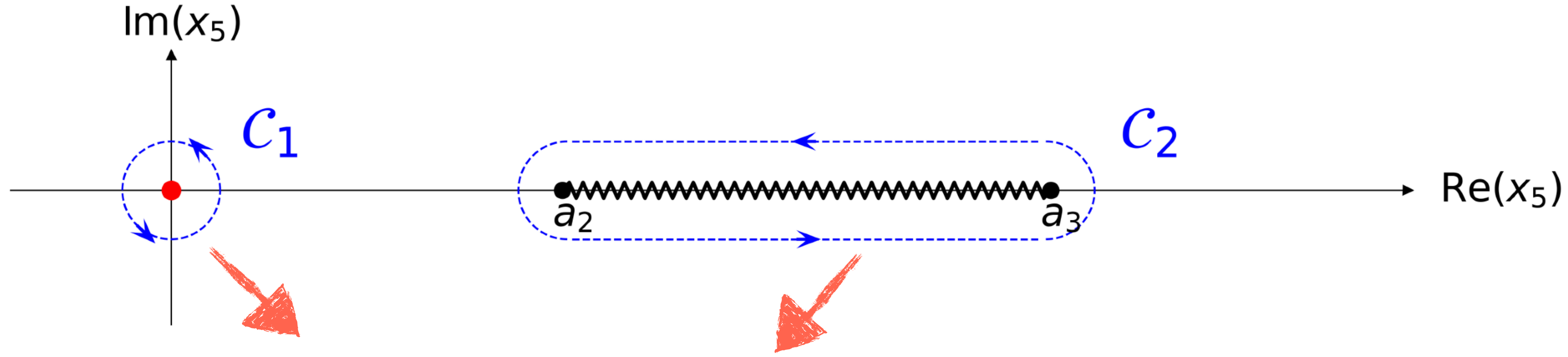

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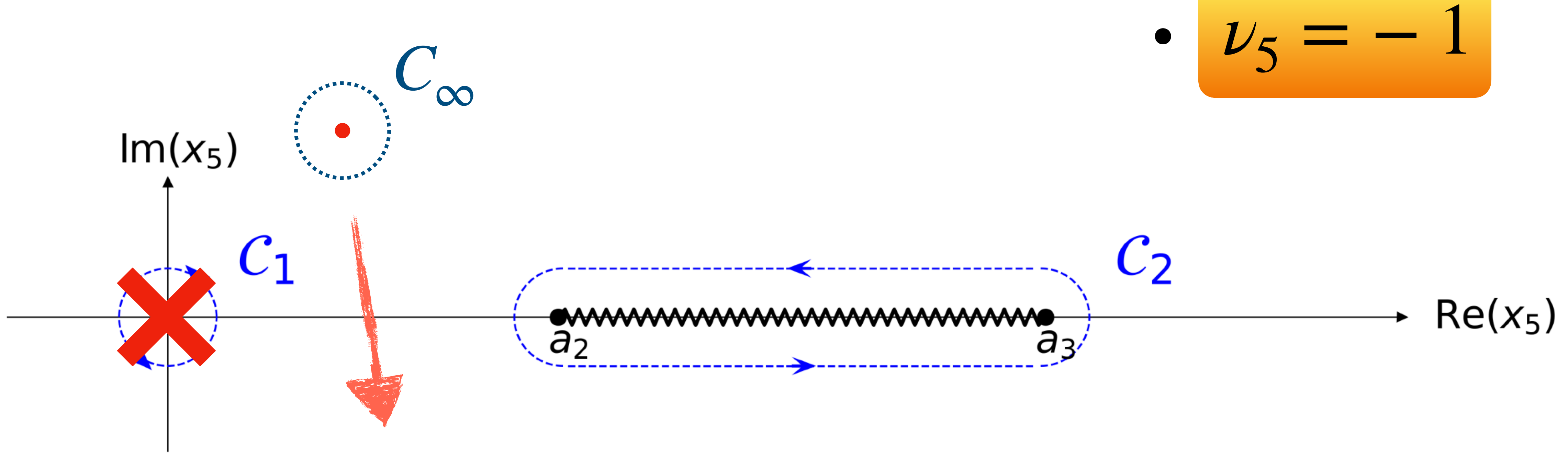

pole at ∞ !

$$\int_{\mathcal{C}_1} \frac{\mathrm{d}z_5}{z_5\sqrt{(2z_5+s)(4m^2-s-2z_5)}} \propto \int_{\mathcal{C}_2} \frac{\mathrm{d}z_5}{z_5\sqrt{(2z_5+s)(4m^2-s-2z_5)}} \propto \frac{1}{\sqrt{s(s-4m^2)}}$$

$$I_{1,1,1,0,0}\big|_{z_1=z_2=z_3=0} \sim \frac{1}{\sqrt{s(s-4m^2)}} \int \mathrm{d}\log g_1 \int \mathrm{d}\log f$$

→ $J_1 = \sqrt{s(s-4m^2)}I_{1,1,1,0,0}$

Case 1: $M = 0$

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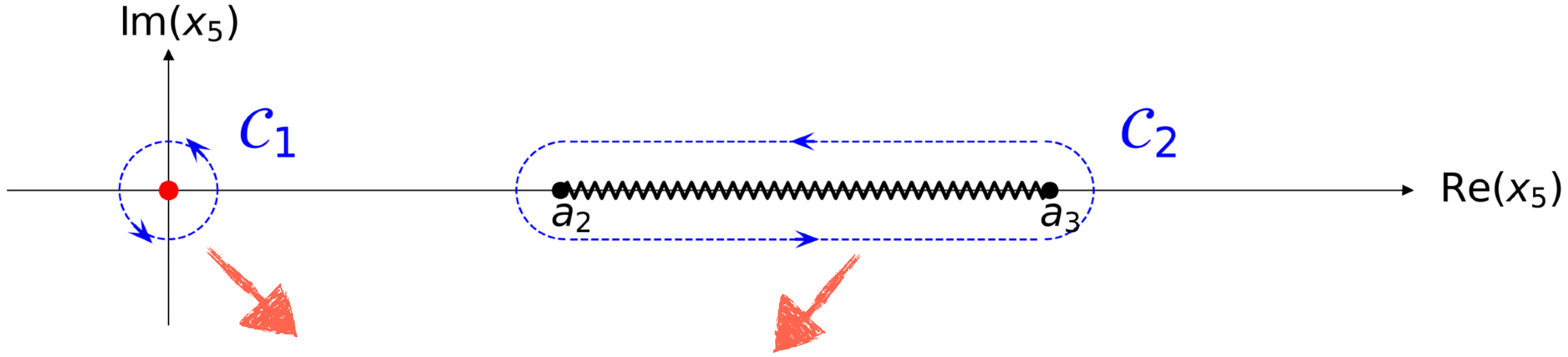

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$$\Rightarrow \boxed{J_1 = \sqrt{s(s-4m^2)}\, I_{1,1,1,0,0}}$$

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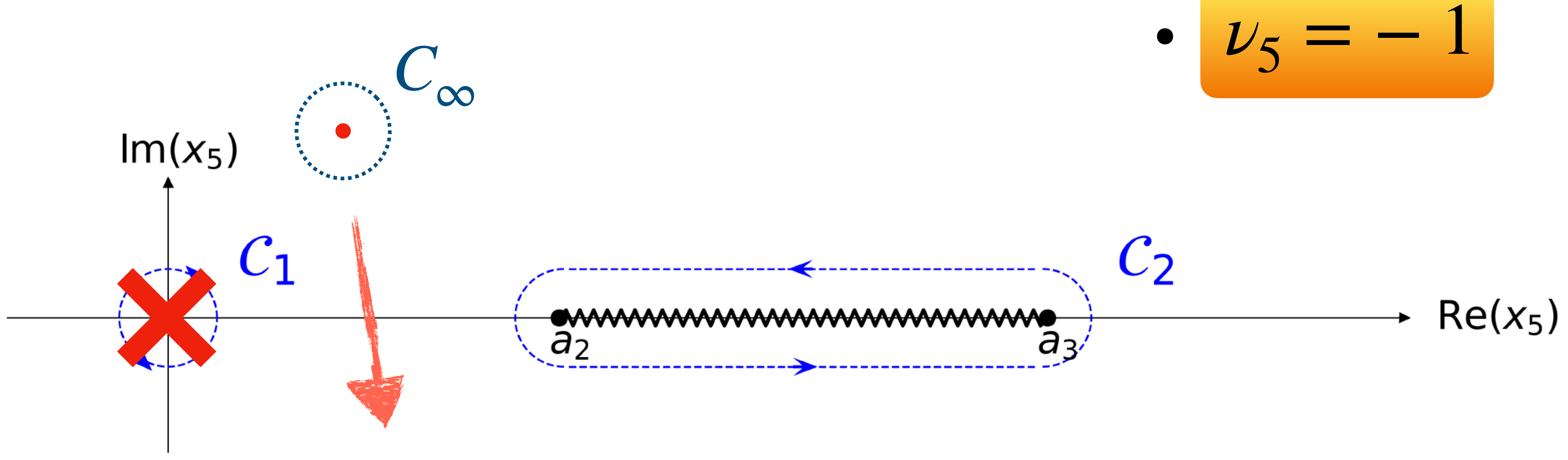

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Case 1: $M = 0$

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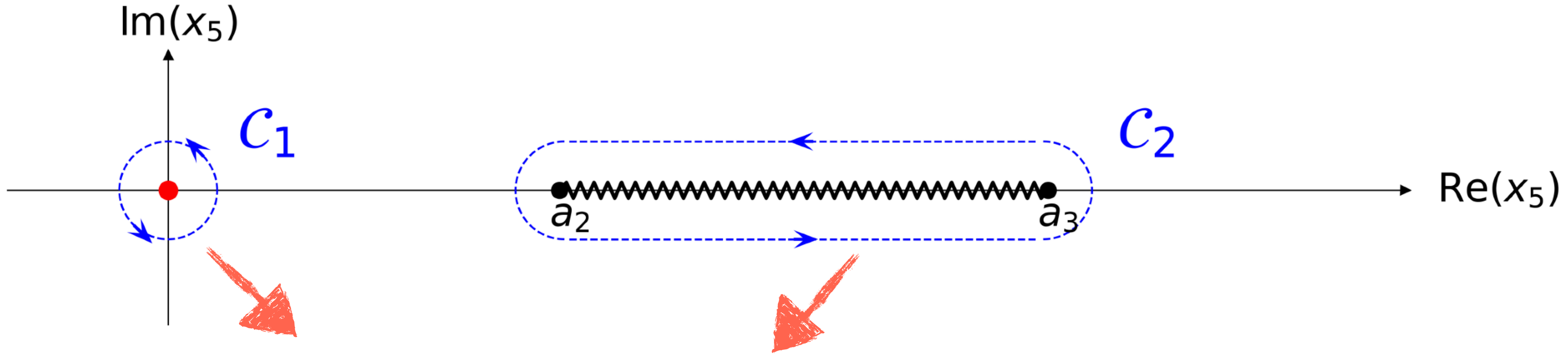

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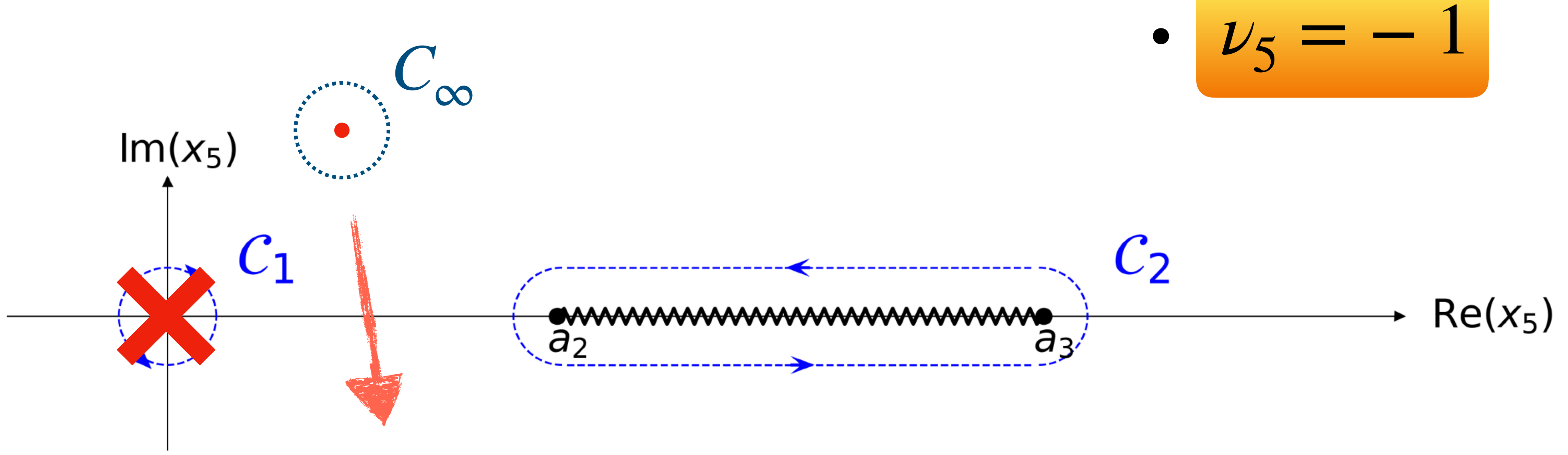

pole at ∞ !

$$\int_{C_\infty} \frac{\mathrm{d}z_5}{\sqrt{(2z_5+s)(4m^2-s-2z_5)}} \propto \int_{C_2} \frac{\mathrm{d}z_5}{\sqrt{(2z_5+s)(4m^2-s-2z_5)}} \propto 1$$

$$I_{1,1,1,0,-1}\big|_{z_1=z_2=z_3=0} \sim \int \mathrm{d}\log g_2 \int \mathrm{d}\log f$$

Case 1: $M = 0$

$$I_{1,1,1,0,\nu_5}\big|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5 z_5^{-\nu_5}}{z_5\sqrt{(2z_5+s)(4m^2-s-2z_5)}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

simple pole in z_5

$\sqrt{P_2(z_5)}$

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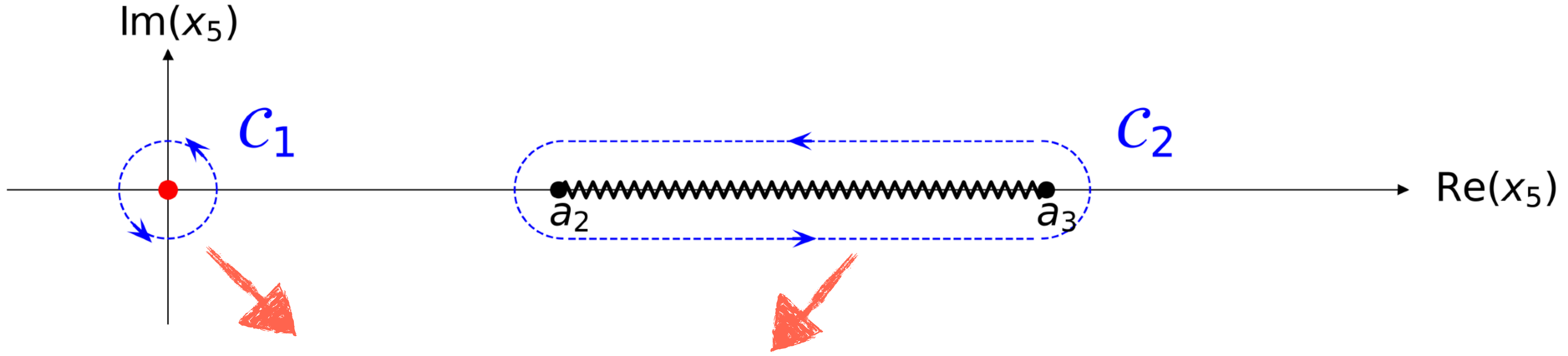

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$$\boxed{J_1 = \sqrt{s(s-4m^2)}\, I_{1,1,1,0,0}}$$

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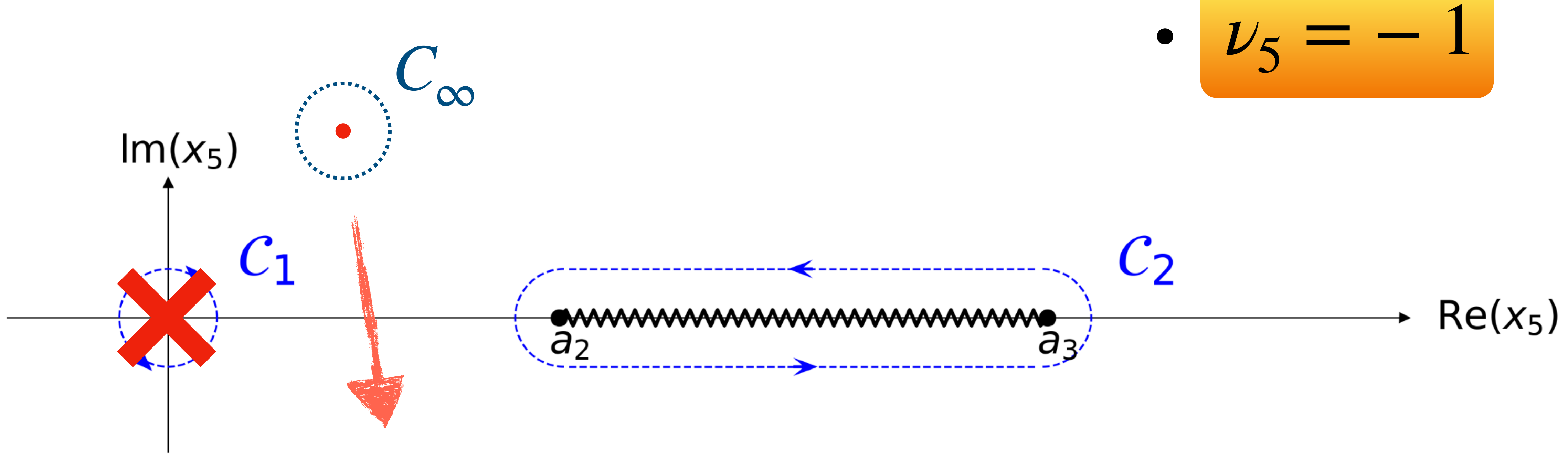

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$$I_{1,1,1,0,-1}\big|_{z_1=z_2=z_3=0} \sim \int \mathrm{d}\log g_2 \int \mathrm{d}\log f$$

$$\boxed{J_2 = I_{1,1,1,0,-1}}$$

Case 1: $M = 0$

$$I_{1,1,1,0,\nu_5}\big|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5 z_5^{-\nu_5}}{z_5\sqrt{(2z_5+s)(4m^2-s-2z_5)}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

simple pole in z_5

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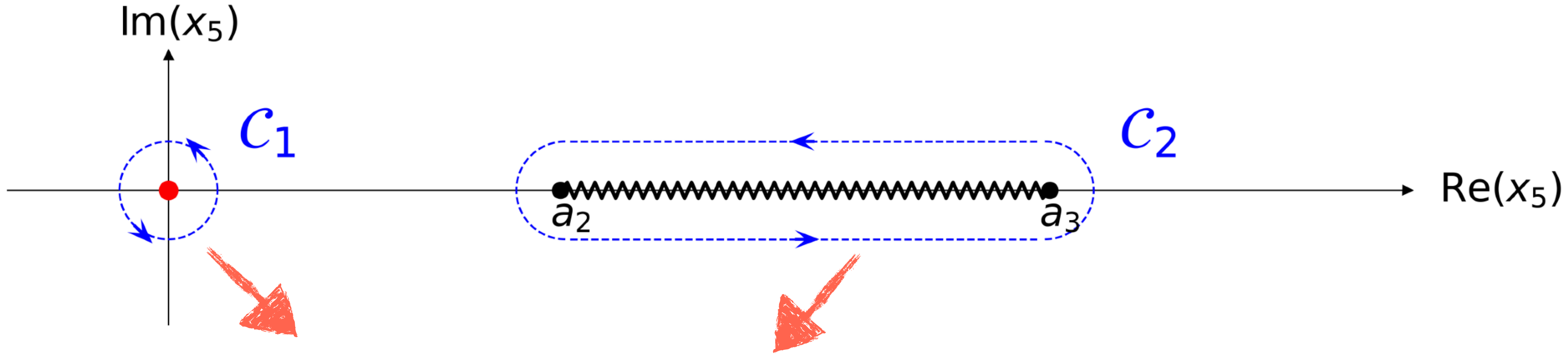

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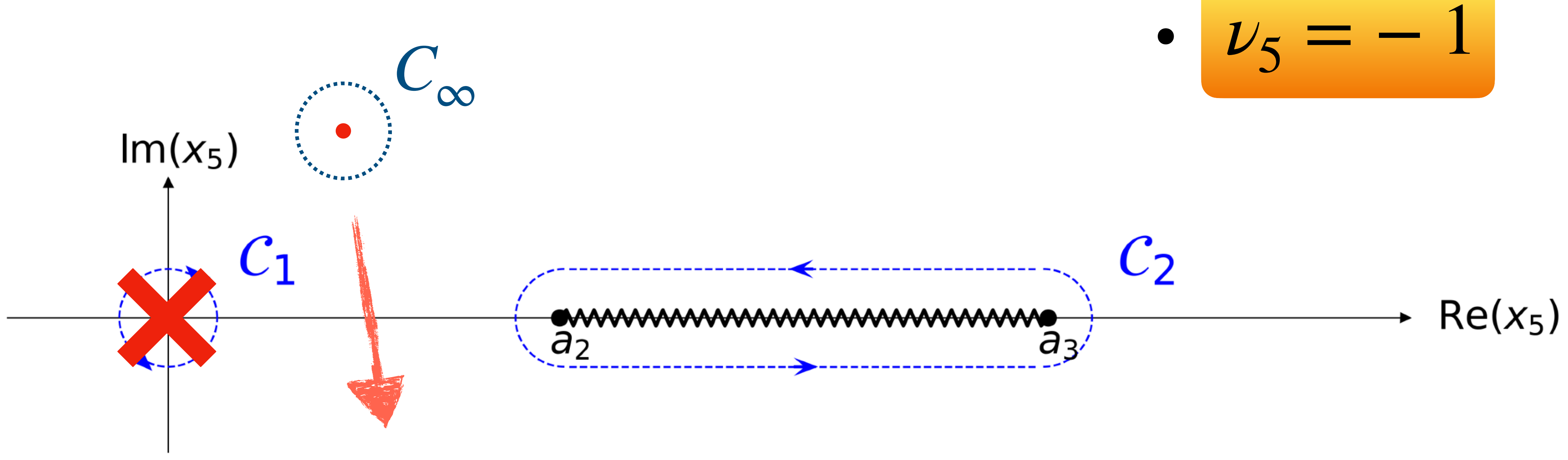

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$$I_{1,1,1,0,0}\big|_{z_1=z_2=z_3=0} \sim \frac{1}{\sqrt{s(s-4m^2)}} \int \mathrm{d}\log g_1 \int \mathrm{d}\log f$$

$$\int_{C_\infty} \frac{\mathrm{d}z_5}{\sqrt{(2z_5+s)(4m^2-s-2z_5)}} \propto \int_{C_2} \frac{\mathrm{d}z_5}{\sqrt{(2z_5+s)(4m^2-s-2z_5)}} \propto 1$$

$$I_{1,1,1,0,-1}\big|_{z_1=z_2=z_3=0} \sim \int \mathrm{d}\log g_2 \int \mathrm{d}\log f$$

$$J_1 = \sqrt{s(s-4m^2)}\, I_{1,1,1,0,0}$$

$$\sum_i R_i(s_{ij}) \int_\gamma \mathrm{d}\log f_n \wedge \ldots \wedge \mathrm{d}\log f_1$$

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Case 2: $M \neq 0$

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$$I_{1,1,1,0,\nu_5}\big|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5 z_5^{-\nu_5}}{\sqrt{(2z_5+s+M^2)(M^2 s - z_5^2)(4m^2-M^2-s-2z_5)}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

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$$\boxed{:= y}$$

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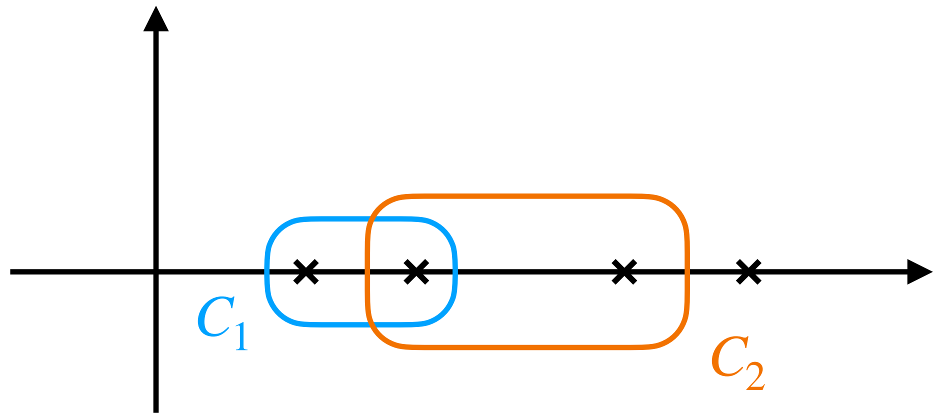

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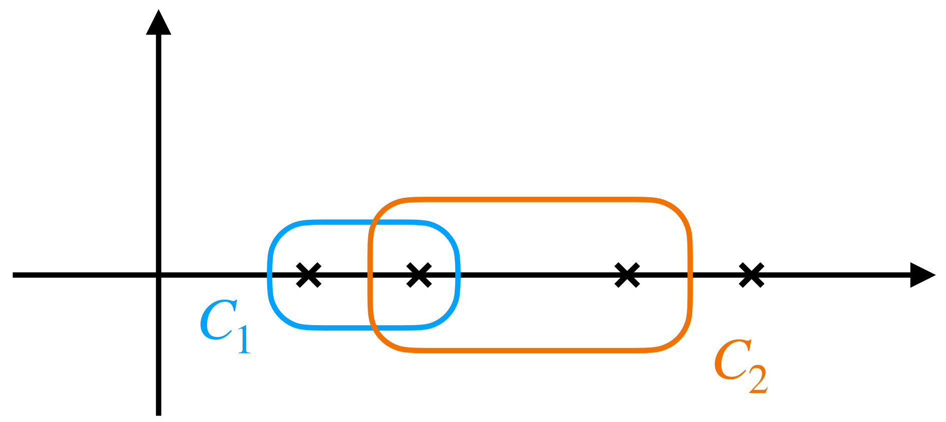

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- $\nu_5 = 0$

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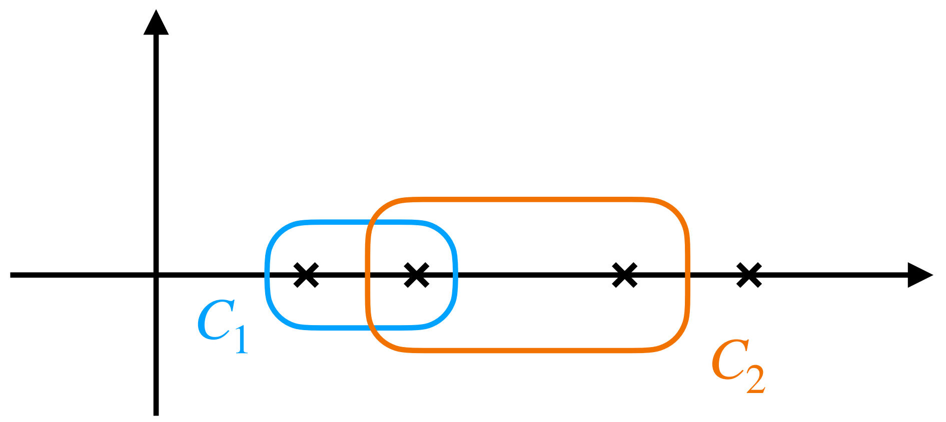
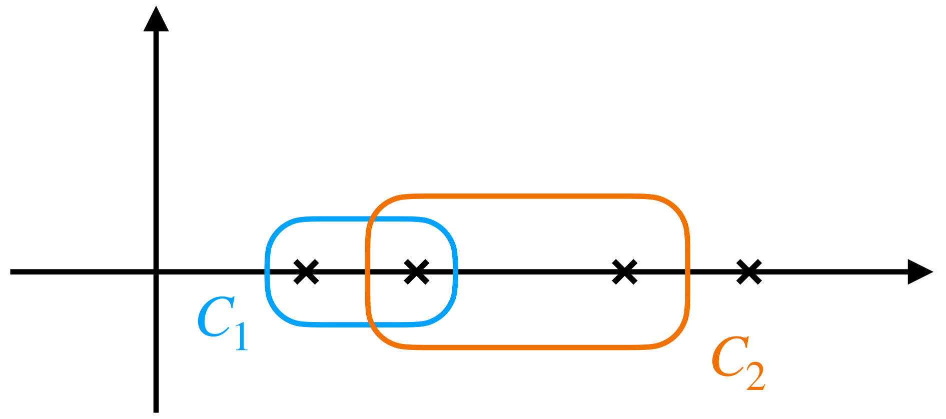


- $\nu_5 = 0$

$$I_{1,1,1,0,0}|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5}{y} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

Case 2: $M \neq 0$

$$I_{1,1,1,0,\nu_5}|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5 z_5^{-\nu_5}}{\underbrace{\boxed{\sqrt{(2z_5+s+M^2)(M^2s-z_5^2)(4m^2-M^2-s-2z_5)}}}_{\boxed{:= y}}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$



- $\nu_5 = 0$ 1st kind integral

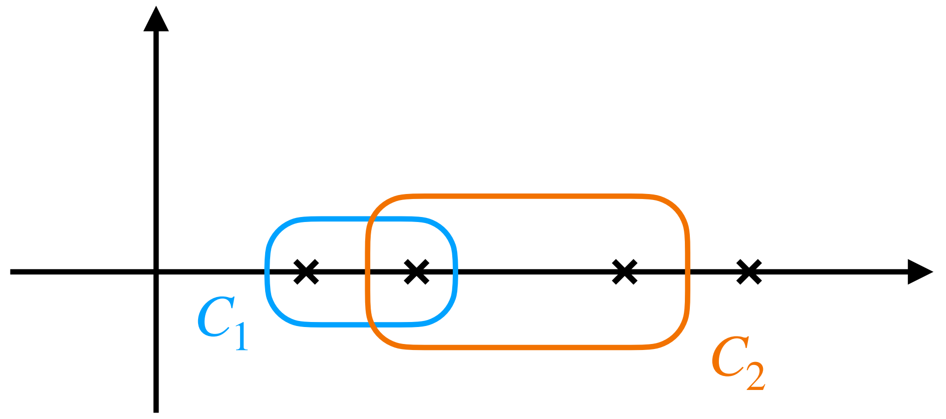
$$I_{1,1,1,0,0}|_{z_1=z_2=z_3=0} \sim \boxed{\int \frac{\mathrm{d}z_5}{y}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

$$\boxed{\int_{C_1} \frac{\mathrm{d}z_5}{y} \propto \psi_0}$$

Case 2: $M \neq 0$

$$I_{1,1,1,0,\nu_5}\big|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5 z_5^{-\nu_5}}{\boxed{\sqrt{(2z_5+s+M^2)(M^2s-z_5^2)(4m^2-M^2-s-2z_5)}}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

$$\boxed{:= y}$$



- $\nu_5 = 0$

1st kind integral

$$I_{1,1,1,0,0}\big|_{z_1=z_2=z_3=0} \sim \boxed{\int \frac{\mathrm{d}z_5}{y}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

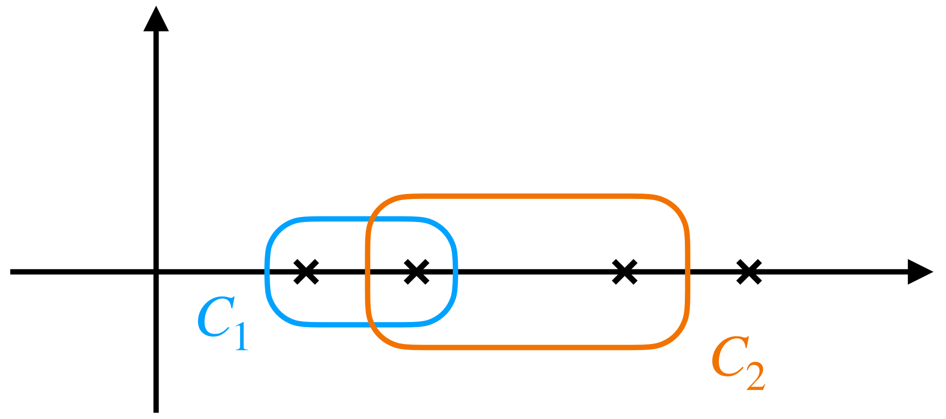
$$\boxed{\int_{C_1} \frac{\mathrm{d}z_5}{y} \propto \psi_0}$$

holomorphic period of the elliptic curve!

Case 2: $M \neq 0$

$$I_{1,1,1,0,\nu_5}\big|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5 z_5^{-\nu_5}}{\boxed{\sqrt{(2z_5+s+M^2)(M^2 s - z_5^2)(4m^2-M^2-s-2z_5)}}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

$\boxed{:= y}$



- $\nu_5 = 0$

1st kind integral

$$I_{1,1,1,0,0}\big|_{z_1=z_2=z_3=0} \sim \boxed{\int \frac{\mathrm{d}z_5}{y}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

$$\boxed{\int_{C_1} \frac{\mathrm{d}z_5}{y} \propto \psi_0}$$

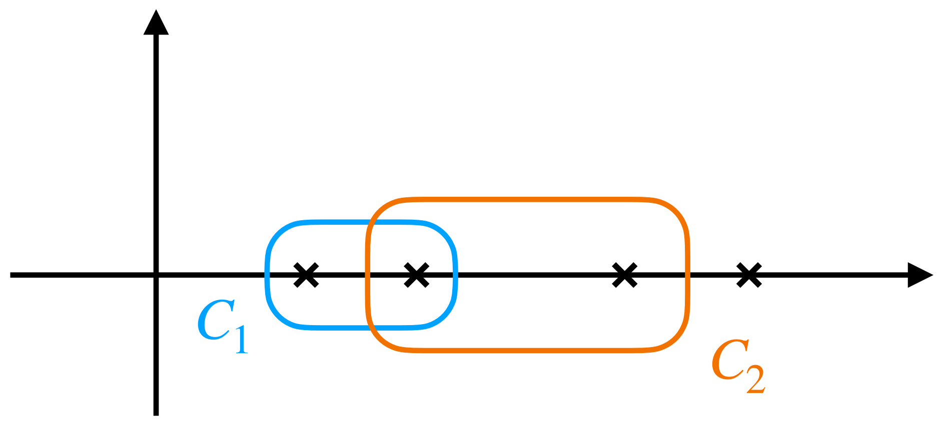
holomorphic period of the elliptic curve!

$$\boxed{J_1 = \frac{1}{\psi_0} I_{1,1,1,0,0}}$$

Case 2: $M \neq 0$

$$I_{1,1,1,0,\nu_5}|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5 z_5^{-\nu_5}}{\boxed{\sqrt{(2z_5+s+M^2)(M^2s-z_5^2)(4m^2-M^2-s-2z_5)}}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

$$\boxed{:= y}$$



- $\nu_5 = 0$ 1st kind integral

$$I_{1,1,1,0,0}|_{z_1=z_2=z_3=0} \sim \boxed{\int \frac{\mathrm{d}z_5}{y}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

$$\boxed{\int_{C_1} \frac{\mathrm{d}z_5}{y} \propto \psi_0}$$

holomorphic period of the elliptic curve!

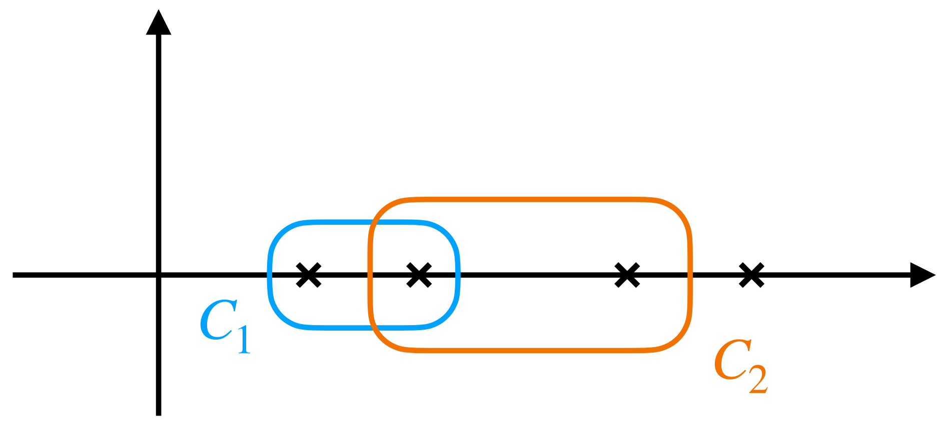
$$\boxed{J_1 = \frac{1}{\psi_0} I_{1,1,1,0,0}}$$

- $\nu_5 = -1$

Case 2: $M \neq 0$

$$I_{1,1,1,0,\nu_5}\big|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5 z_5^{-\nu_5}}{\boxed{\sqrt{(2z_5+s+M^2)(M^2 s - z_5^2)(4m^2-M^2-s-2z_5)}}} \int \mathrm{d}\log f(z_4,z_5,\mathbf{s})$$

$$\boxed{:= y}$$



- $\nu_5 = 0$ 1st kind integral

$$I_{1,1,1,0,0}\big|_{z_1=z_2=z_3=0} \sim \boxed{\int \frac{\mathrm{d}z_5}{y}} \int \mathrm{d}\log f(z_4,z_5,\mathbf{s})$$

$$\boxed{\int_{C_1} \frac{\mathrm{d}z_5}{y} \propto \psi_0}$$

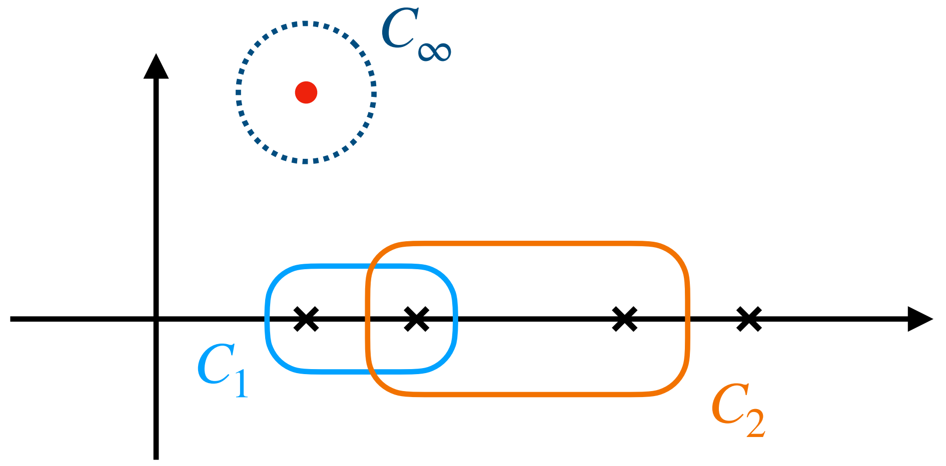
holomorphic period of the elliptic curve!

$$\boxed{J_1 = \frac{1}{\psi_0} I_{1,1,1,0,0}}$$

- $\nu_5 = -1$

$$I_{1,1,1,0,-1}\big|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5\, z_5}{y} \int \mathrm{d}\log f(z_4,z_5,\mathbf{s})$$

Case 2: $M \neq 0$



$$I_{1,1,1,0,\nu_5}|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5 z_5^{-\nu_5}}{\boxed{\sqrt{(2z_5+s+M^2)(M^2s-z_5^2)(4m^2-M^2-s-2z_5)}}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

$$\boxed{:= y}$$

- $\nu_5 = 0$ 1st kind integral

$$I_{1,1,1,0,0}|_{z_1=z_2=z_3=0} \sim \boxed{\int \frac{\mathrm{d}z_5}{y}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

$$\boxed{\int_{C_1} \frac{\mathrm{d}z_5}{y} \propto \psi_0}$$

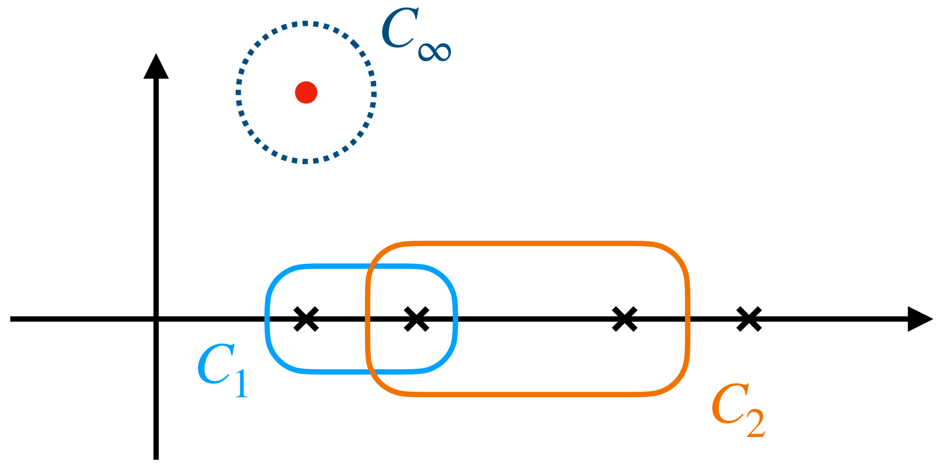
holomorphic period of the elliptic curve!

$$\boxed{J_1 = \frac{1}{\psi_0} I_{1,1,1,0,0}}$$

- $\nu_5 = -1$

$$I_{1,1,1,0,-1}|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5\, z_5}{y} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

Case 2: $M \neq 0$



$$I_{1,1,1,0,\nu_5}\big|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5 z_5^{-\nu_5}}{\boxed{\sqrt{(2z_5+s+M^2)(M^2s-z_5^2)(4m^2-M^2-s-2z_5)}}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

$$\boxed{:= y}$$

- $\nu_5 = 0$ 1st kind integral

$$I_{1,1,1,0,0}\big|_{z_1=z_2=z_3=0} \sim \boxed{\int \frac{\mathrm{d}z_5}{y}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

$$\boxed{\int_{C_1} \frac{\mathrm{d}z_5}{y} \propto \psi_0}$$

holomorphic period of the elliptic curve!

$$\boxed{J_1 = \frac{1}{\psi_0} I_{1,1,1,0,0}}$$

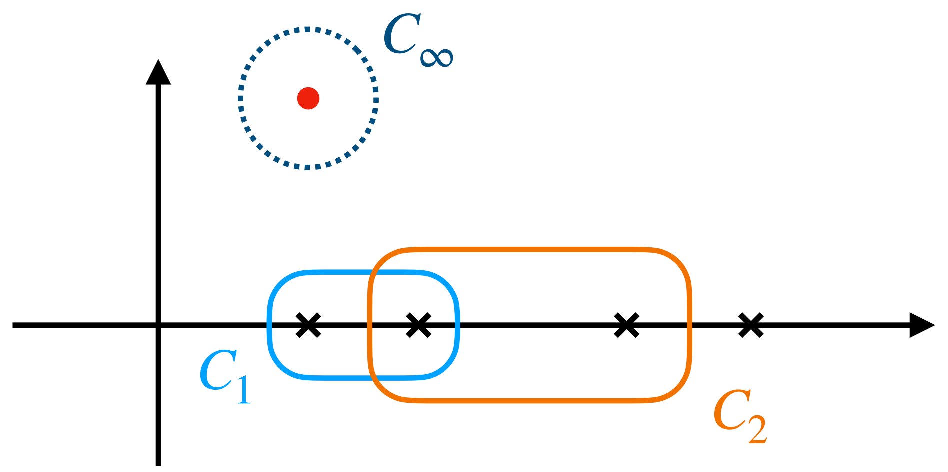
- $\nu_5 = -1$ 3rd kind integral

$$I_{1,1,1,0,-1}\big|_{z_1=z_2=z_3=0} \sim \boxed{\int \frac{\mathrm{d}z_5\, z_5}{y}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

$$\boxed{\int_{C_\infty} \frac{\mathrm{d}z_5\, z_5}{y} \propto 1}$$

simple pole at ∞

Case 2: $M \neq 0$



$$I_{1,1,1,0,\nu_5}\big|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5 z_5^{-\nu_5}}{\boxed{\sqrt{(2z_5+s+M^2)(M^2 s - z_5^2)(4m^2-M^2-s-2z_5)}}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

$$\boxed{:= y}$$

- $\nu_5 = 0$ 1st kind integral

$$I_{1,1,1,0,0}\big|_{z_1=z_2=z_3=0} \sim \boxed{\int \frac{\mathrm{d}z_5}{y}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

$$\boxed{\int_{C_1} \frac{\mathrm{d}z_5}{y} \propto \psi_0}$$

holomorphic period of the elliptic curve!

$$\boxed{J_1 = \frac{1}{\psi_0} I_{1,1,1,0,0}}$$

- $\nu_5 = -1$ 3rd kind integral

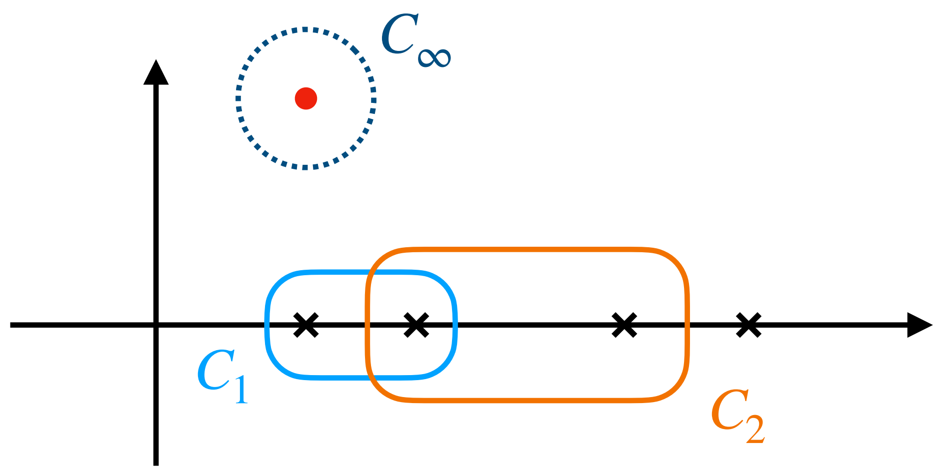
$$I_{1,1,1,0,-1}\big|_{z_1=z_2=z_3=0} \sim \boxed{\int \frac{\mathrm{d}z_5\, z_5}{y}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

$$\boxed{\int_{C_\infty} \frac{\mathrm{d}z_5\, z_5}{y} \propto 1}$$

simple pole at ∞

$$\boxed{J_2 = I_{1,1,1,0,-1}}$$

Case 2: $M \neq 0$



$$I_{1,1,1,0,\nu_5}\big|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5 z_5^{-\nu_5}}{\boxed{\sqrt{(2z_5+s+M^2)(M^2s-z_5^2)(4m^2-M^2-s-2z_5)}}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

$$\boxed{:= y}$$

- $\nu_5 = 0$ 1st kind integral

$$I_{1,1,1,0,0}\big|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5}{y} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

$$\int_{C_1} \frac{\mathrm{d}z_5}{y} \propto \psi_0$$

holomorphic period of the elliptic curve!

$$J_1 = \frac{1}{\psi_0} I_{1,1,1,0,0}$$

- $\nu_5 = -1$ 3rd kind integral

$$I_{1,1,1,0,-1}\big|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5\, z_5}{y} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

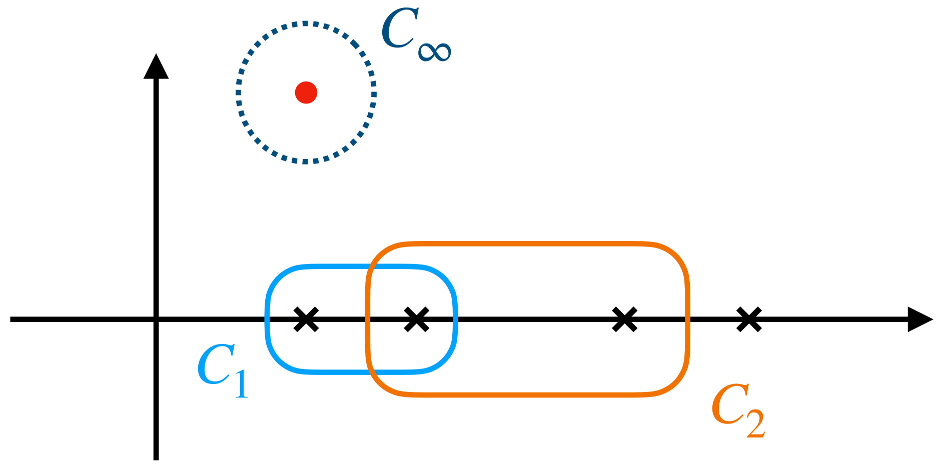
$$\int_{C_\infty} \frac{\mathrm{d}z_5\, z_5}{y} \propto 1$$

simple pole at ∞

$$J_2 = I_{1,1,1,0,-1}$$

- $\nu_5 = -2$

Case 2: $M \neq 0$



$$I_{1,1,1,0,\nu_5}\big|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5 z_5^{-\nu_5}}{\boxed{\sqrt{(2z_5+s+M^2)(M^2s-z_5^2)(4m^2-M^2-s-2z_5)}}} \int \mathrm{d}\log f(z_4,z_5,\mathbf{s})$$

$\boxed{:= y}$

- $\nu_5 = 0$ 1st kind integral

$$I_{1,1,1,0,0}\big|_{z_1=z_2=z_3=0} \sim \boxed{\int \frac{\mathrm{d}z_5}{y}} \int \mathrm{d}\log f(z_4,z_5,\mathbf{s})$$

$\boxed{\int_{C_1} \frac{\mathrm{d}z_5}{y} \propto \psi_0}$ holomorphic period of the elliptic curve!

$$\boxed{J_1 = \frac{1}{\psi_0} I_{1,1,1,0,0}}$$

- $\nu_5 = -1$ 3rd kind integral

$$I_{1,1,1,0,-1}\big|_{z_1=z_2=z_3=0} \sim \boxed{\int \frac{\mathrm{d}z_5\, z_5}{y}} \int \mathrm{d}\log f(z_4,z_5,\mathbf{s})$$

$\boxed{\int_{C_\infty} \frac{\mathrm{d}z_5\, z_5}{y} \propto 1}$ simple pole at ∞

$$\boxed{J_2 = I_{1,1,1,0,-1}}$$

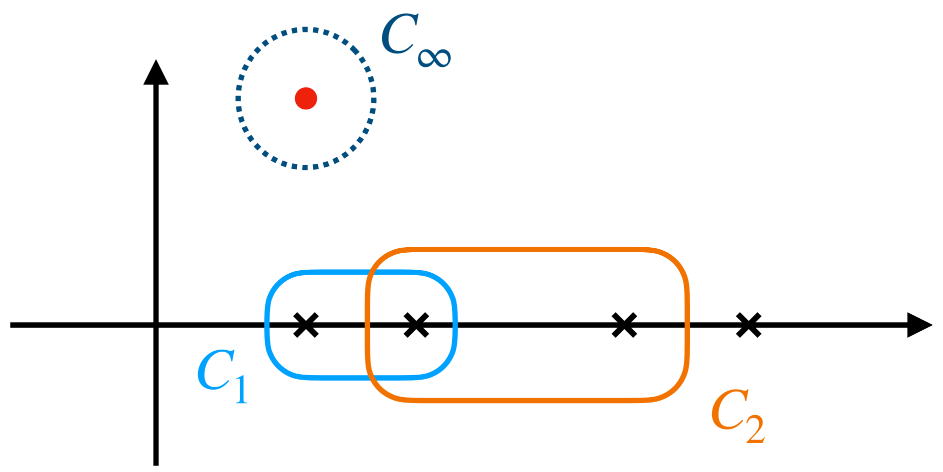
- $\nu_5 = -2$

$$I_{1,1,1,0,-2}\big|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5\, z_5^2}{y} \int \mathrm{d}\log f(z_4,z_5,\mathbf{s})$$

Case 2: $M \neq 0$

$$I_{1,1,1,0,\nu_5}|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5 z_5^{-\nu_5}}{\boxed{\sqrt{(2z_5+s+M^2)(M^2s-z_5^2)(4m^2-M^2-s-2z_5)}}} \int \mathrm{d}\log f(z_4,z_5,\mathbf{s})$$

$$\boxed{:= y}$$



- $\nu_5 = 0$ 1st kind integral

$$I_{1,1,1,0,0}|_{z_1=z_2=z_3=0} \sim \boxed{\int \frac{\mathrm{d}z_5}{y}} \int \mathrm{d}\log f(z_4,z_5,\mathbf{s})$$

$$\boxed{\int_{C_1} \frac{\mathrm{d}z_5}{y} \propto \psi_0}$$

holomorphic period of the elliptic curve!

$$\boxed{J_1 = \frac{1}{\psi_0} I_{1,1,1,0,0}}$$

- $\nu_5 = -1$ 3rd kind integral

$$I_{1,1,1,0,-1}|_{z_1=z_2=z_3=0} \sim \boxed{\int \frac{\mathrm{d}z_5\, z_5}{y}} \int \mathrm{d}\log f(z_4,z_5,\mathbf{s})$$

$$\boxed{\int_{C_\infty} \frac{\mathrm{d}z_5\, z_5}{y} \propto 1}$$

simple pole at ∞

$$\boxed{J_2 = I_{1,1,1,0,-1}}$$

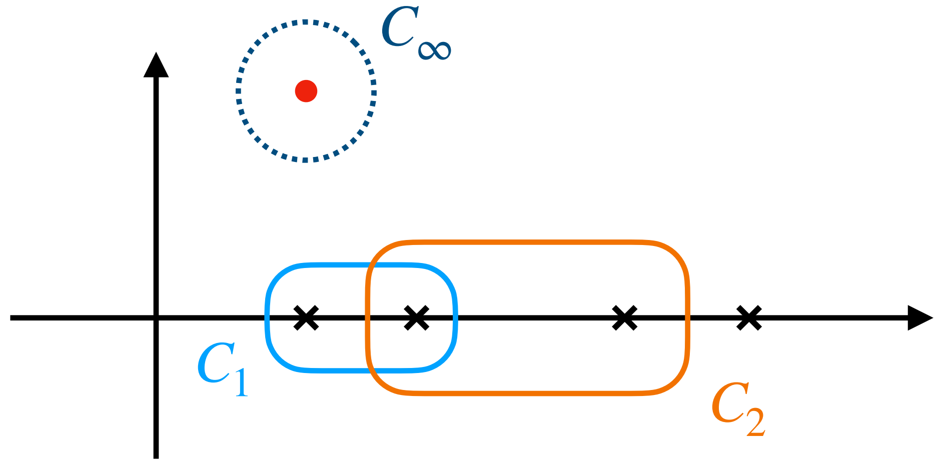
- $\nu_5 = -2$ 2nd kind integral

$$I_{1,1,1,0,-2}|_{z_1=z_2=z_3=0} \sim \boxed{\int \frac{\mathrm{d}z_5\, z_5^2}{y}} \int \mathrm{d}\log f(z_4,z_5,\mathbf{s})$$

$$\boxed{\int \frac{\mathrm{d}z_5\, z_5^2}{y}}$$

double pole at ∞ (no residue)

Case 2: $M \neq 0$



$$I_{1,1,1,0,\nu_5}\big|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5 z_5^{-\nu_5}}{\boxed{\sqrt{(2z_5+s+M^2)(M^2s-z_5^2)(4m^2-M^2-s-2z_5)}}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

$$\boxed{:= y}$$

- $\nu_5 = 0$ 1st kind integral

$$I_{1,1,1,0,0}\big|_{z_1=z_2=z_3=0} \sim \boxed{\int \frac{\mathrm{d}z_5}{y}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

$$\boxed{\int_{C_1} \frac{\mathrm{d}z_5}{y} \propto \psi_0}$$

holomorphic period of the elliptic curve!

$$\boxed{J_1 = \frac{1}{\psi_0} I_{1,1,1,0,0}}$$

- $\nu_5 = -1$ 3rd kind integral

$$I_{1,1,1,0,-1}\big|_{z_1=z_2=z_3=0} \sim \boxed{\int \frac{\mathrm{d}z_5\, z_5}{y}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

$$\boxed{\int_{C_\infty} \frac{\mathrm{d}z_5\, z_5}{y} \propto 1}$$

simple pole at ∞

$$\boxed{J_2 = I_{1,1,1,0,-1}}$$

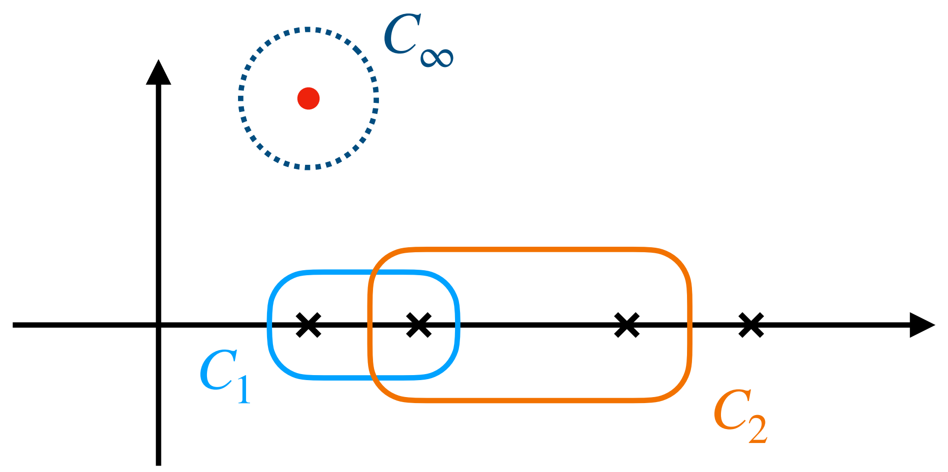
- $\nu_5 = -2$ 2nd kind integral

$$I_{1,1,1,0,-2}\big|_{z_1=z_2=z_3=0} \sim \boxed{\int \frac{\mathrm{d}z_5\, z_5^2}{y}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

$$\boxed{\int \frac{\mathrm{d}z_5\, z_5^2}{y}}$$

double pole at ∞ (no residue)

Case 2: $M \neq 0$



$$I_{1,1,1,0,\nu_5}\big|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5 z_5^{-\nu_5}}{\underbrace{\sqrt{(2z_5+s+M^2)(M^2 s - z_5^2)(4m^2-M^2-s-2z_5)}}_{:= y}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

- $\nu_5 = 0$ 1st kind integral

$$I_{1,1,1,0,0}\big|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5}{y} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

$$\int_{C_1} \frac{\mathrm{d}z_5}{y} \propto \psi_0$$

holomorphic period of the elliptic curve!

$$J_1 = \frac{1}{\psi_0} I_{1,1,1,0,0}$$

- $\nu_5 = -1$ 3rd kind integral

$$I_{1,1,1,0,-1}\big|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5\, z_5}{y} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

$$\int_{C_\infty} \frac{\mathrm{d}z_5\, z_5}{y} \propto 1$$

simple pole at ∞

$$J_2 = I_{1,1,1,0,-1}$$

- $\nu_5 = -2$ 2nd kind integral

$$I_{1,1,1,0,-2}\big|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5\, z_5^2}{y} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

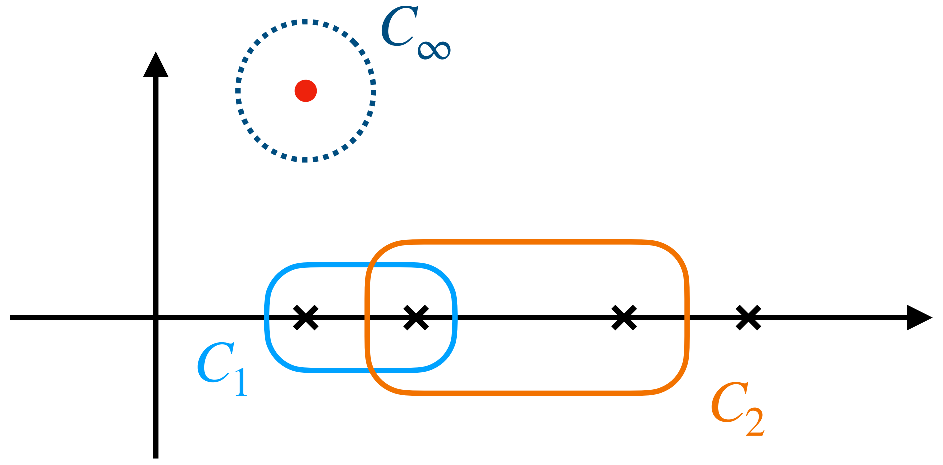
$$\int \frac{\mathrm{d}z_5\, z_5^2}{y}$$

double pole at ∞ (no residue)

$\downarrow$

no log singularities?!

Case 2: $M \neq 0$



$$I_{1,1,1,0,\nu_5}\big|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5 z_5^{-\nu_5}}{\boxed{\sqrt{(2z_5+s+M^2)(M^2 s-z_5^2)(4m^2-M^2-s-2z_5)}}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

$$\boxed{:= y}$$

- $\nu_5 = 0$ 1st kind integral

$$I_{1,1,1,0,0}\big|_{z_1=z_2=z_3=0} \sim \boxed{\int \frac{\mathrm{d}z_5}{y}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

$$\boxed{\int_{C_1} \frac{\mathrm{d}z_5}{y} \propto \psi_0}$$

holomorphic period of the elliptic curve!

$$\boxed{J_1 = \frac{1}{\psi_0} I_{1,1,1,0,0}}$$

- $\nu_5 = -1$ 3rd kind integral

$$I_{1,1,1,0,-1}\big|_{z_1=z_2=z_3=0} \sim \boxed{\int \frac{\mathrm{d}z_5\, z_5}{y}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

$$\boxed{\int_{C_\infty} \frac{\mathrm{d}z_5\, z_5}{y} \propto 1}$$

simple pole at ∞

$$\boxed{J_2 = I_{1,1,1,0,-1}}$$

- $\nu_5 = -2$ 2nd kind integral

$$I_{1,1,1,0,-2}\big|_{z_1=z_2=z_3=0} \sim \boxed{\int \frac{\mathrm{d}z_5\, z_5^2}{y}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

$$\boxed{\int \frac{\mathrm{d}z_5\, z_5^2}{y}}$$

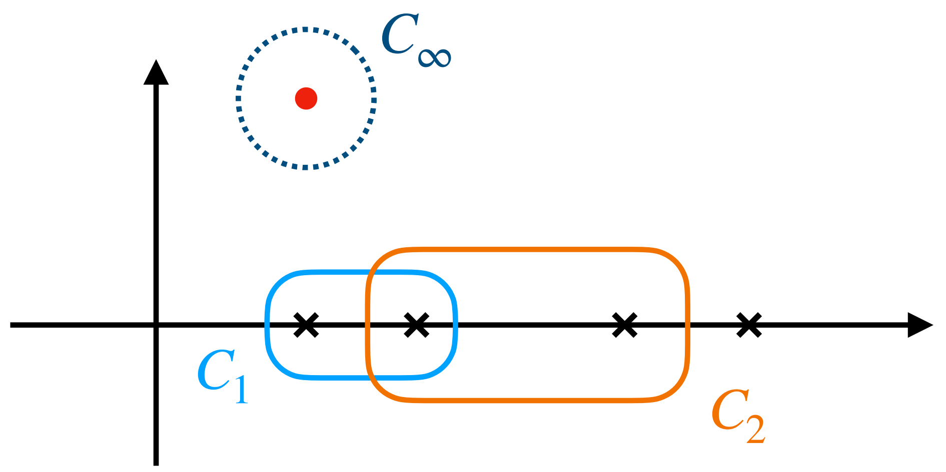
double pole at ∞ (no residue)

no log singularities?!

[Görges, Nega, Tancredi, Wagner '23]
[Duhr, **SM**, Nega, Sauer, Tancredi, Wagner '25]

Case 2: $M \neq 0$

$$I_{1,1,1,0,\nu_5}|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5 z_5^{-\nu_5}}{\underbrace{\sqrt{(2z_5+s+M^2)(M^2s-z_5^2)(4m^2-M^2-s-2z_5)}}_{:= y}} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$



- $\nu_5 = 0$ 1st kind integral

$$I_{1,1,1,0,0}|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5}{y} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

$$\int_{C_1} \frac{\mathrm{d}z_5}{y} \propto \psi_0$$

holomorphic period of the elliptic curve!

$$J_1 = \frac{1}{\psi_0} I_{1,1,1,0,0}$$

- $\nu_5 = -1$ 3rd kind integral

$$I_{1,1,1,0,-1}|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5\, z_5}{y} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

$$\int_{C_\infty} \frac{\mathrm{d}z_5\, z_5}{y} \propto 1$$

simple pole at ∞

$$J_2 = I_{1,1,1,0,-1}$$

- $\nu_5 = -2$ 2nd kind integral

$$I_{1,1,1,0,-2}|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5\, z_5^2}{y} \int \mathrm{d}\log f(z_4, z_5, \mathbf{s})$$

$$\int \frac{\mathrm{d}z_5\, z_5^2}{y}$$

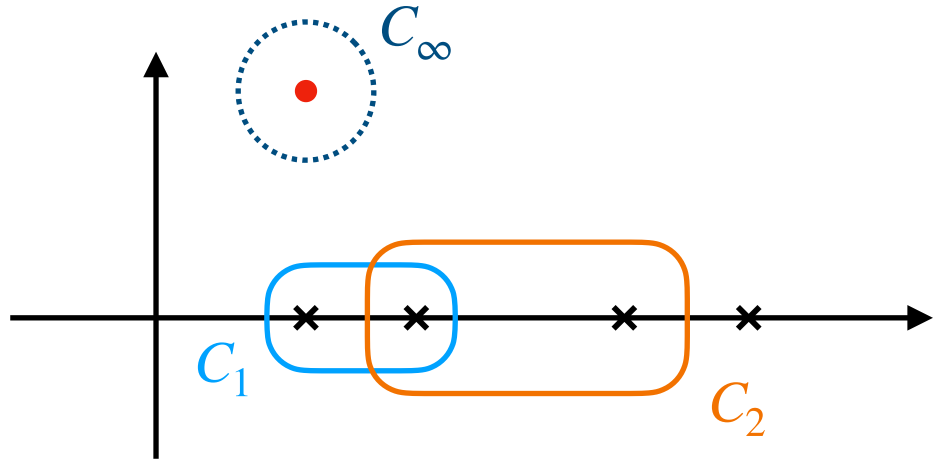
double pole at ∞ (no residue)

no log singularities?!

[Görges, Nega, Tancredi, Wagner '23]
[Duhr, **SM**, Nega, Sauer, Tancredi, Wagner '25]

$$\tilde{I}_3 = \partial I_{1,1,1,0,0}$$

Case 2: $M \neq 0$



$$I_{1,1,1,0,\nu_5}\big|_{z_1=z_2=z_3=0} \sim \int \frac{\mathrm{d}z_5 z_5^{-\nu_5}}{\boxed{\sqrt{(2z_5+s+M^2)(M^2 s-z_5^2)(4m^2-M^2-s-2z_5)}}} \int \mathrm{d}\log f(z_4,z_5,\mathbf{s})$$

$$\boxed{:= y}$$

- $\nu_5 = 0$ 1st kind integral

$$I_{1,1,1,0,0}\big|_{z_1=z_2=z_3=0} \sim \boxed{\int \frac{\mathrm{d}z_5}{y}} \int \mathrm{d}\log f(z_4,z_5,\mathbf{s})$$

$$\boxed{\int_{C_1} \frac{\mathrm{d}z_5}{y} \propto \psi_0}$$ holomorphic period of the elliptic curve!

$$\boxed{J_1 = \frac{1}{\psi_0} I_{1,1,1,0,0}}$$

- $\nu_5 = -1$ 3rd kind integral

$$I_{1,1,1,0,-1}\big|_{z_1=z_2=z_3=0} \sim \boxed{\int \frac{\mathrm{d}z_5\, z_5}{y}} \int \mathrm{d}\log f(z_4,z_5,\mathbf{s})$$

$$\boxed{\int_{C_\infty} \frac{\mathrm{d}z_5\, z_5}{y} \propto 1}$$ simple pole at ∞

$$\boxed{J_2 = I_{1,1,1,0,-1}}$$

- $\nu_5 = -2$ 2nd kind integral

$$I_{1,1,1,0,-2}\big|_{z_1=z_2=z_3=0} \sim \boxed{\int \frac{\mathrm{d}z_5\, z_5^2}{y}} \int \mathrm{d}\log f(z_4,z_5,\mathbf{s})$$

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→

- $\varepsilon-$factorised differential equations: $\partial_{s_{ij}} \mathbf{J} = \varepsilon\, \mathbf{A}(s_{ij})\, \mathbf{J}$

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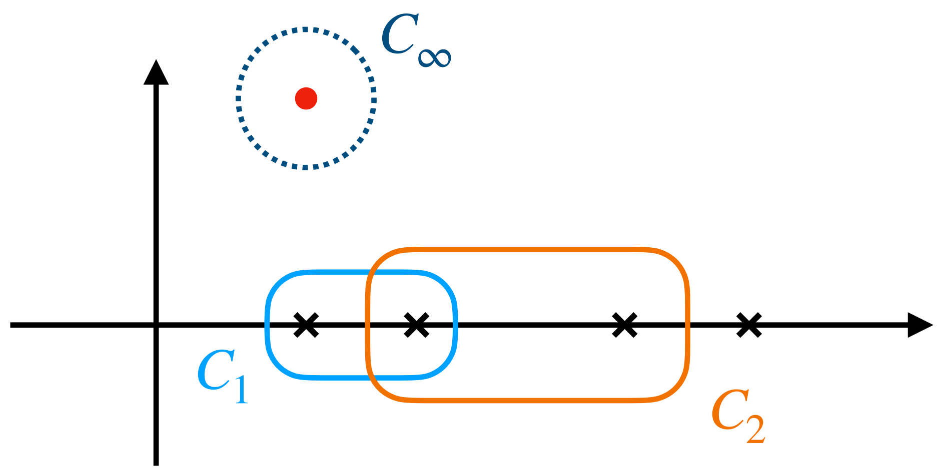

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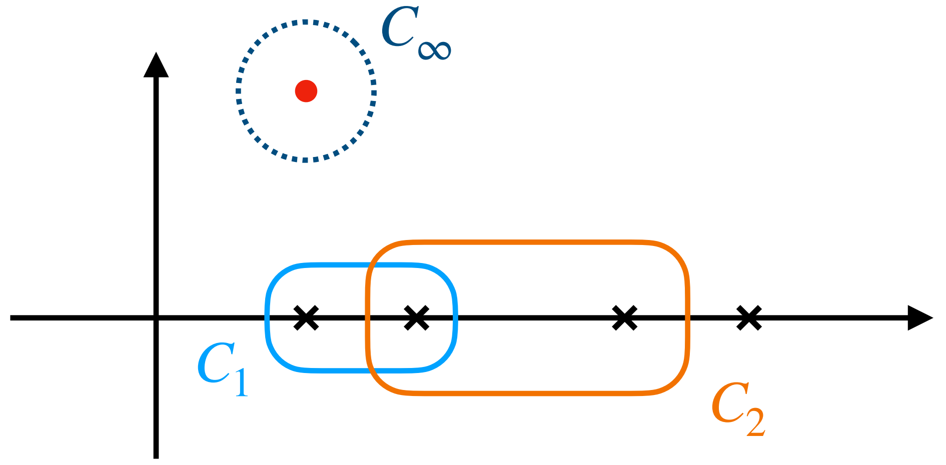

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✓