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Threshold Effects on the Massless Neutrino in the Canonical Seesaw Mechanism

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Based on DZ, JHEP 10 (2024) 002

Synergies Towards the Future Standard Model
DESY Theory Workshop
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Status of Neutrino Masses

Oscillation

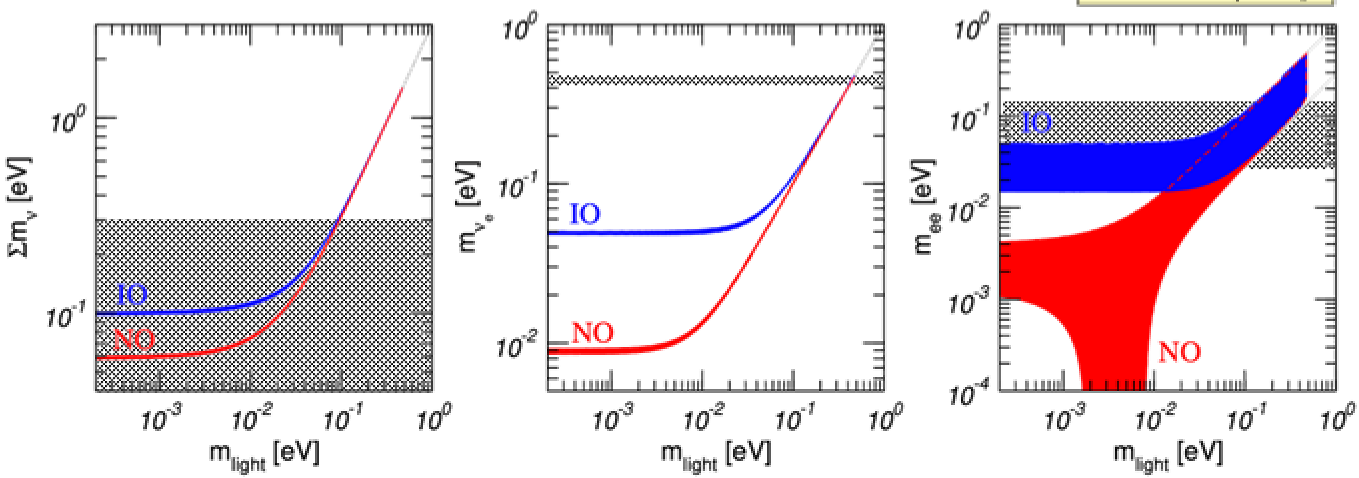
NuFIT 6.0 (2024)

	Normal Ordering ($\Delta\chi^2 = 0.6$)		Inverted Ordering (best fit)	
	bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range
$\frac{\Delta m^2_{21}}{10^{-5}\ \mathrm{eV}^2}$	$7.49^{+0.19}_{-0.19}$	$6.92 \to 8.05$	$7.49^{+0.19}_{-0.19}$	$6.92 \to 8.05$
$\frac{\Delta m^2_{3\ell}}{10^{-3}\ \mathrm{eV}^2}$	$+2.513^{+0.021}_{-0.019}$	$+2.451 \to +2.578$	$-2.484^{+0.020}_{-0.020}$	$-2.547 \to -2.421$

Ivan Esteban et al., 2024

NuFIT 6.0 (2024)

Non-oscillation



$\sum m_\nu \lesssim 0.04 - 0.3$ eV

J. Q. Jiang et al., 2024

D. Naredo-Tuero et al., 2024

$m_{\nu_e} < 0.45$ eV (90% CL)

KATRIN collaboration, 2024

$m_{ee} \lesssim 0.028 - 0.122$ eV (90% CL)

KAMLAND-ZEN collaboration, 2024

$m_{ee} \lesssim 0.079 - 0.180$ eV (90% CL)

GERDA collaboration, 2020

No lower limit on the lightest neutrino mass allowing one massless neutrino

Neutrino Masses in Seesaw Mechanism

Can we have an exactly massless neutrino?

Neutrino Masses in Seesaw Mechanism

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⬇

Is the vanishing neutrino mass stable against quantum corrections if no extra symmetry protects it? If not, it provides theoretical lower limit

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The (minimal) type-I seesaw mechanism

P. Minkowski, 1977; T. Yanagida, 1979; M. Gell-Mann, P. Ramond, R. Slansky, 1979;
S. L. Glashow, 1980; R. N. Mohapatra, G. Senjanovic, 1980

Extending the SM with three (or two) right-handed neutrinos (RHNs):

$$\mathcal{L} = \mathcal{L}_{\rm SM} + \overline{N_{\rm R}} {\rm i} \partial\!\!\!/ N_{\rm R} - \left(\frac{1}{2} \overline{N^{\rm c}_{\rm R}} M_N N_{\rm R} + \overline{\ell_{\rm L}} Y_\nu \widetilde{H} N_{\rm R} + {\rm h.c.} \right)$$

$$\langle H \rangle = v/\sqrt{2}$$

$$\mathcal{L}_\nu = -\frac{1}{2} \overline{\nu_{\rm L}} M_\nu \nu^{\rm c}_{\rm L} + {\rm h.c.}$$ **with** $$M_\nu = -v^2 Y_\nu M_N^{-1} Y_\nu^{\rm T}/2$$

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One massless neutrino
- **a) Two RHNs**
- **b) Three RHNs and Rank-2 neutrino Yukawa matrix**

Neutrino Masses in Seesaw Mechanism

Can we have an exactly massless neutrino?

↓

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One massless neutrino {
- **a) Two RHNs**
- **b) Three RHNs and Rank-2 neutrino Yukawa matrix**

This vanishing neutrino mass is stable against one-loop Renormalization Group (RG) running effects without threshold effects

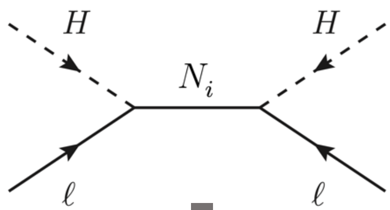
J. A. Casas, J. R. Espinosa, A. Ibarra, 2000; S. Antusch et al., 2001;2003;2005; J. W. Mei, Z. Z. Xing, 2004;
Z. Z. Xing, 2005; J. W. Mei, 2005; T. Ohlsson, H. Zhang, S. Zhou, 2013; T. Ohlsson, S. Zhou, 2014;...

But the case with hierarchical RHN masses is still unclear!!!

S. Antusch et al., 2002; N. J. Benoit et al, 2022

Threshold Effects in Seesaw Mechanism

In the mass-independent renormalization scheme, e.g., the MS/$\overline{\rm MS}$ scheme:

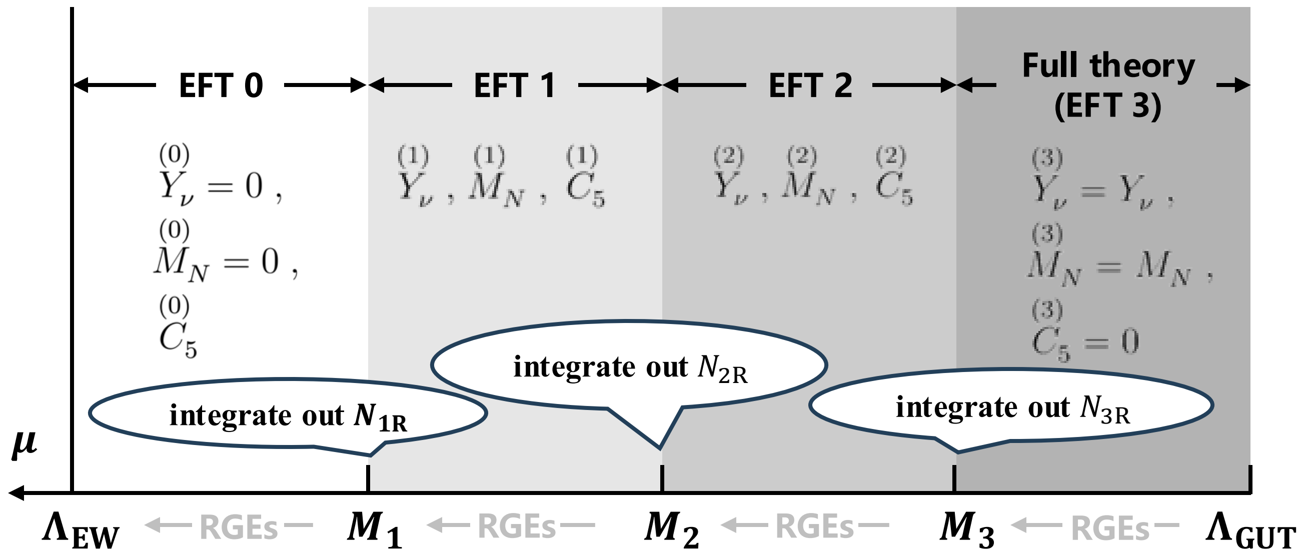


The Weinberg Operator

$$\frac{1}{2} C_5^{\alpha\beta} \overline{\ell_{\alpha \rm L}} \widetilde{H} \widetilde{H}^{\rm T} \ell^{\rm c}_{\beta \rm L}$$

$$(C_5)_{\alpha\beta} = (Y_\nu)_{\alpha i} M_i^{-1} (Y_\nu)_{\beta i}$$

S. Weinberg, 1979



Matching conditions at each threshold scale $\mu = M_{n+1}$ **:**

- $\overset{(n)}{(C_5)}_{\alpha\beta} = \overset{(n+1)}{(C_5)}_{\alpha\beta} + \overset{(n+1)}{(Y_\nu)}_{\alpha(n+1)} M_{n+1}^{-1} \overset{(n+1)}{(Y_\nu)}_{\beta(n+1)}$
- $\overset{(n)}{Y_\nu}$ **removing the last column of** $\overset{(n+1)}{Y_\nu}$
- $\overset{(n)}{M_N}$ **removing both the last column and the last row of** $\overset{(n+1)}{M_N}$

$$\overset{(n_{\rm max})}{C_5} = \overset{(0)}{Y_\nu} = \overset{(0)}{M_N} = 0$$

$$\overset{(n_{\rm max})}{Y_\nu} = Y_\nu \qquad \overset{(n_{\rm max})}{M_N} = M_N$$

One-loop RGEs and Threshold Effects

One-loop RGEs for non-degenerate seesaw scales: S. Antusch et al., 2003; 2005

$$\mu\frac{\mathrm{d}\overset{(n)}{M}_N}{\mathrm{d}\mu} = \frac{1}{16\pi^2}\left[\overset{(n)}{M}_N \overset{(n)}{Y}{}_\nu^\dagger \overset{(n)}{Y}_\nu + \left(\overset{(n)}{Y}{}_\nu^\dagger \overset{(n)}{Y}_\nu\right)^{\mathrm{T}} \overset{(n)}{M}_N\right]$$

$$\overset{(n)}{T} = \mathrm{Tr}\left(\overset{(n)}{Y}_\nu \overset{(n)}{Y}{}_\nu^\dagger + Y_l Y_l^\dagger + 3 Y_\mathrm{u} Y_\mathrm{u}^\dagger + 3 Y_\mathrm{d} Y_\mathrm{d}^\dagger\right)$$

$$\mu\frac{\mathrm{d}\overset{(n)}{Y}_\nu}{\mathrm{d}\mu} = \frac{1}{16\pi^2}\left[\left(\overset{(n)}{T} - \frac{3}{4}g_1^2 - \frac{9}{4}g_2^2\right)\overset{(n)}{Y}_\nu + \frac{3}{2}\left(\overset{(n)}{Y}_\nu \overset{(n)}{Y}{}_\nu^\dagger - Y_l Y_l^\dagger\right)\overset{(n)}{Y}_\nu\right]$$

$$\mu\frac{\mathrm{d}\overset{(n)}{C}_5}{\mathrm{d}\mu} = \frac{1}{16\pi^2}\left[\left(4\lambda + 2\overset{(n)}{T} - 3g_2^2\right)\overset{(n)}{C}_5 + \left(\frac{1}{2}\overset{(n)}{Y}_\nu \overset{(n)}{Y}{}_\nu^\dagger - \frac{3}{2}Y_l Y_l^\dagger\right)\overset{(n)}{C}_5 + \overset{(n)}{C}_5\left(\frac{1}{2}\overset{(n)}{Y}_\nu \overset{(n)}{Y}{}_\nu^\dagger - \frac{3}{2}Y_l Y_l^\dagger\right)^{\mathrm{T}}\right]$$

The effect neutrino mass matrix:

$$\overset{(n)}{M}_\nu = -v^2 \overset{(n)}{\kappa}_\nu / 2$$ **with** $\overset{(n)}{\kappa}_\nu = \overset{(n)}{C}_5 + \overset{(n)}{\kappa}_D$ **and** $\overset{(n)}{\kappa}_D = \overset{(n)}{Y}_\nu \overset{(n)}{M}{}_N^{-1} \overset{(n)}{Y}{}_\nu^{\mathrm{T}}$

$$\mu\frac{\mathrm{d}\overset{(n)}{\kappa}_D}{\mathrm{d}\mu} = \frac{1}{16\pi^2}\left[\left(2\overset{(n)}{T} - \frac{3}{2}g_1^2 - \frac{9}{2}g_2^2\right)\overset{(n)}{\kappa}_D + \left(\frac{1}{2}\overset{(n)}{Y}_\nu \overset{(n)}{Y}{}_\nu^\dagger - \frac{3}{2}Y_l Y_l^\dagger\right)\overset{(n)}{\kappa}_D + \overset{(n)}{\kappa}_D\left(\frac{1}{2}\overset{(n)}{Y}_\nu \overset{(n)}{Y}{}_\nu^\dagger - \frac{3}{2}Y_l Y_l^\dagger\right)^{\mathrm{T}}\right]$$

$$\mu\frac{\mathrm{d}\overset{(n)}{\kappa}_\nu}{\mathrm{d}\mu} = \frac{1}{16\pi^2}\left[\left(2\overset{(n)}{T} - \frac{3}{2}g_1^2 - \frac{9}{2}g_2^2\right)\overset{(n)}{\kappa}_\nu + \left(\frac{1}{2}\overset{(n)}{Y}_\nu \overset{(n)}{Y}{}_\nu^\dagger - \frac{3}{2}Y_l Y_l^\dagger\right)\overset{(n)}{\kappa}_\nu + \overset{(n)}{\kappa}_\nu\left(\frac{1}{2}\overset{(n)}{Y}_\nu \overset{(n)}{Y}{}_\nu^\dagger - \frac{3}{2}Y_l Y_l^\dagger\right)^{\mathrm{T}} + \left(4\lambda + \frac{3}{2}g_1^2 + \frac{3}{2}g_2^2\right)\overset{(n)}{C}_5\right]$$

One-loop RGEs and Threshold Effects

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$$\mu\frac{\mathrm{d}\overset{(n)}{M}_N}{\mathrm{d}\mu} = \frac{1}{16\pi^2}\left[\overset{(n)}{M}_N \overset{(n)}{Y}{}_\nu^\dagger \overset{(n)}{Y}_\nu + \left(\overset{(n)}{Y}{}_\nu^\dagger \overset{(n)}{Y}_\nu\right)^{\mathrm{T}} \overset{(n)}{M}_N\right]$$

$$\overset{(n)}{T} = \mathrm{Tr}\left(\overset{(n)}{Y}_\nu \overset{(n)}{Y}{}_\nu^\dagger + Y_l Y_l^\dagger + 3Y_{\mathrm{u}} Y_{\mathrm{u}}^\dagger + 3Y_{\mathrm{d}} Y_{\mathrm{d}}^\dagger\right)$$

Additional terms

$$\mu\frac{\mathrm{d}\overset{(n)}{Y}_\nu}{\mathrm{d}\mu} = \frac{1}{16\pi^2}\left[\left(\overset{(n)}{T} - \frac{3}{4}g_1^2 - \frac{9}{4}g_2^2\right)\overset{(n)}{Y}_\nu + \frac{3}{2}\left(\overset{(n)}{Y}_\nu \overset{(n)}{Y}{}_\nu^\dagger - Y_l Y_l^\dagger\right)\overset{(n)}{Y}_\nu\right]$$

$$\mu\frac{\mathrm{d}\overset{(n)}{C}_5}{\mathrm{d}\mu} = \frac{1}{16\pi^2}\left[\left(4\lambda + 2\overset{(n)}{T} - 3g_2^2\right)\overset{(n)}{C}_5 + \left(\frac{1}{2}\overset{(n)}{Y}_\nu \overset{(n)}{Y}{}_\nu^\dagger - \frac{3}{2}Y_l Y_l^\dagger\right)\overset{(n)}{C}_5 + \overset{(n)}{C}_5\left(\frac{1}{2}\overset{(n)}{Y}_\nu \overset{(n)}{Y}{}_\nu^\dagger - \frac{3}{2}Y_l Y_l^\dagger\right)^{\mathrm{T}}\right]$$

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$$\overset{(n)}{M}_\nu = -v^2 \overset{(n)}{\kappa}_\nu/2$$ **with** $$\overset{(n)}{\kappa}_\nu = \overset{(n)}{C}_5 + \overset{(n)}{\kappa}_D$$ **and** $$\overset{(n)}{\kappa}_D = \overset{(n)}{Y}_\nu \overset{(n)}{M}{}_N^{-1} \overset{(n)}{Y}{}_\nu^{\mathrm{T}}$$

$$\mu\frac{\mathrm{d}\overset{(n)}{\kappa}_D}{\mathrm{d}\mu} = \frac{1}{16\pi^2}\left[\left(2\overset{(n)}{T} - \frac{3}{2}g_1^2 - \frac{9}{2}g_2^2\right)\overset{(n)}{\kappa}_D + \left(\frac{1}{2}\overset{(n)}{Y}_\nu \overset{(n)}{Y}{}_\nu^\dagger - \frac{3}{2}Y_l Y_l^\dagger\right)\overset{(n)}{\kappa}_D + \overset{(n)}{\kappa}_D\left(\frac{1}{2}\overset{(n)}{Y}_\nu \overset{(n)}{Y}{}_\nu^\dagger - \frac{3}{2}Y_l Y_l^\dagger\right)^{\mathrm{T}}\right]$$

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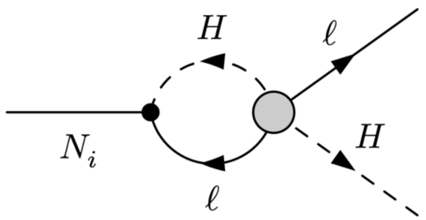
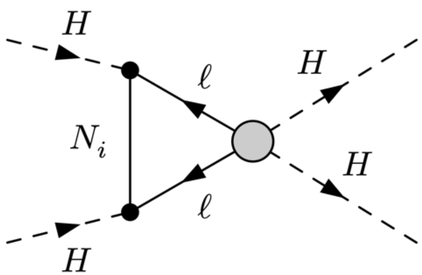
Revisiting the One-loop RGEs

Overlooked UV divergent 1PI diagrams: DZ, 2024



→ Y_ν, C_5



→ λ

$$R^{\alpha\beta}_{\ell HN1} = \overline{\ell_{\alpha \rm L}}\widetilde{H}\partial\!\!\!/ N^{\rm c}_{\beta \rm R} \qquad \delta G_{\ell HN1} = \frac{3{\rm i}C_5 Y^*_\nu}{32\pi^2\varepsilon}$$

$$\delta G^{\alpha\beta}_{\ell HN1}\mathcal{R}^{\alpha\beta}_{\ell HN1} = -{\rm i}\left(\delta G_{\ell HN1}M_N\right)_{\alpha\beta}\overline{\ell_{\alpha \rm L}}\widetilde{H}N_{\beta \rm R} - {\rm i}\left(\delta G_{\ell HN1}Y^{\rm T}_\nu\right)_{\alpha\beta}\overline{\ell_{\alpha \rm L}}\widetilde{H}\widetilde{H}^{\rm T}\ell^{\rm c}_{\beta \rm L}$$

New contributions to the RGEs among seesaw scales:

$$\mu\frac{{\rm d}\lambda}{{\rm d}\mu} = \frac{1}{16\pi^2}\left[4\lambda\overset{(n)}{T} + \frac{3}{8}(g1^2+g2^2)^2 + \frac{3}{4}g_2^4 - 3\lambda(g_1^2+3g_2^2) + 24\lambda^2 - 2\overset{(n)}{T'} + 2{\rm Tr}\left(\overset{(n)}{C_5}\overset{(n)}{Y^*_\nu}\overset{(n)}{M_N}\overset{(n)}{Y^\dagger_\nu} + \overset{(n)}{Y_\nu}\overset{(n)}{M^\dagger_N}\overset{(n)}{Y^{\rm T}_\nu}\overset{(n)}{C^\dagger_5}\right)\right],$$

$$\mu\frac{{\rm d}\overset{(n)}{Y_\nu}}{{\rm d}\mu} = \frac{1}{16\pi^2}\left[\left(\overset{(n)}{T} - \frac{3}{4}g_1^2 - \frac{9}{4}g_2^2\right)\overset{(n)}{Y_\nu} + \frac{3}{2}\left(\overset{(n)}{Y_\nu}\overset{(n)}{Y^\dagger_\nu} - Y_l Y_l^\dagger\right)\overset{(n)}{Y_\nu} - 3\overset{(n)}{C_5}\overset{(n)}{Y^*_\nu}\overset{(n)}{M_N}\right],$$

$$\mu\frac{{\rm d}\overset{(n)}{C_5}}{{\rm d}\mu} = \frac{1}{16\pi^2}\left[\left(4\lambda + 2\overset{(n)}{T} - 3g_2^2\right)\overset{(n)}{C_5} + \left(\frac{7}{2}\overset{(n)}{Y_\nu}\overset{(n)}{Y^\dagger_\nu} - \frac{3}{2}Y_l Y_l^\dagger\right)\overset{(n)}{C_5} + \overset{(n)}{C_5}\left(\frac{7}{2}\overset{(n)}{Y_\nu}\overset{(n)}{Y^\dagger_\nu} - \frac{3}{2}Y_l Y_l^\dagger\right)^{\rm T}\right]$$

Revisiting the One-loop RGEs

$$\mu\frac{\mathrm{d}\overset{(n)}{\kappa}_D}{\mathrm{d}\mu} = \frac{1}{16\pi^2}\left[\left(2\overset{(n)}{T} - \frac{3}{2}g_1^2 - \frac{9}{2}g_2^2\right)\overset{(n)}{\kappa}_D + \left(\frac{1}{2}\overset{(n)}{Y}_\nu\overset{(n)}{Y}{}_\nu^\dagger - \frac{3}{2}Y_l Y_l^\dagger\right)\overset{(n)}{\kappa}_D + \overset{(n)}{\kappa}_D\left(\frac{1}{2}\overset{(n)}{Y}_\nu\overset{(n)}{Y}{}_\nu^\dagger - \frac{3}{2}Y_l Y_l^\dagger\right)^{\mathrm{T}} - 3\overset{(n)}{Y}_\nu\overset{(n)}{Y}{}_\nu^\dagger\overset{(n)}{C}_5 - 3\overset{(n)}{C}_5\left(\overset{(n)}{Y}_\nu\overset{(n)}{Y}{}_\nu^\dagger\right)^{\mathrm{T}}\right],$$

$$\overset{(n)}{\kappa}_\nu = \overset{(n)}{C}_5 + \overset{(n)}{\kappa}_D$$

$$\overset{(n)}{\kappa}_D = \overset{(n)}{Y}_\nu\overset{(n)}{M}{}_N^{-1}\overset{(n)}{Y}{}_\nu^{\mathrm{T}}$$

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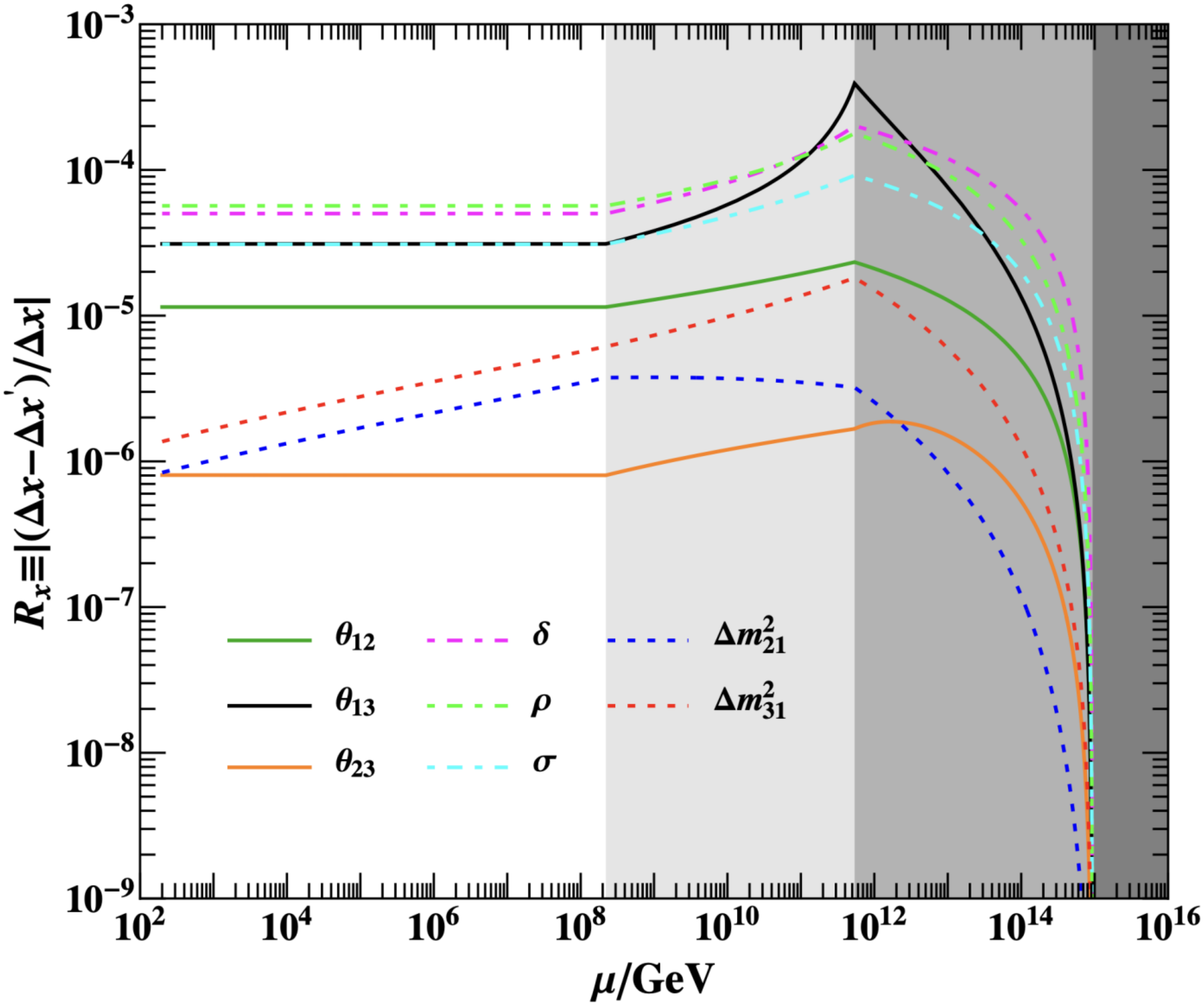
$$\mu\frac{\mathrm{d}\overset{(n)}{\kappa}_\nu}{\mathrm{d}\mu} = \frac{1}{16\pi^2}\left[\left(2\overset{(n)}{T} - \frac{3}{2}g_1^2 - \frac{9}{2}g_2^2\right)\overset{(n)}{\kappa}_\nu + \left(\frac{1}{2}\overset{(n)}{Y}_\nu\overset{(n)}{Y}{}_\nu^\dagger - \frac{3}{2}Y_l Y_l^\dagger\right)\overset{(n)}{\kappa}_\nu + \overset{(n)}{\kappa}_\nu\left(\frac{1}{2}\overset{(n)}{Y}_\nu\overset{(n)}{Y}{}_\nu^\dagger - \frac{3}{2}Y_l Y_l^\dagger\right)^{\mathrm{T}} + \left(4\lambda + \frac{3}{2}g_1^2 + \frac{3}{2}g_2^2\right)\overset{(n)}{C}_5\right].$$

- **The counteraction of new contributions in κ_ν leaves κ_ν in the same form as before**
- **New contributions affect the running behaviors of λ, κ_D, and C_5**
- **The running behavior of κ_ν is indirectly influenced by those of λ, κ_D, and C_5**

Revisiting the One-loop RGEs

Relative differences between the results with/without new contributions



Threshold Effects on the Massless Neutrino

$$\mu\frac{\mathrm{d}\overset{(n)}{\kappa}_\nu}{\mathrm{d}\mu} = \frac{1}{16\pi^2}\left[\left(2\overset{(n)}{T} - \frac{3}{2}g_1^2 - \frac{9}{2}g_2^2\right)\overset{(n)}{\kappa}_\nu + \left(\frac{1}{2}\overset{(n)}{Y}_\nu\overset{(n)}{Y}{}_\nu^\dagger - \frac{3}{2}Y_l Y_l^\dagger\right)\overset{(n)}{\kappa}_\nu + \overset{(n)}{\kappa}_\nu\left(\frac{1}{2}\overset{(n)}{Y}_\nu\overset{(n)}{Y}{}_\nu^\dagger - \frac{3}{2}Y_l Y_l^\dagger\right)^{\mathrm{T}} + \left(4\lambda + \frac{3}{2}g_1^2 + \frac{3}{2}g_2^2\right)\overset{(n)}{C}_5\right].$$

- **Above the highest or below the lowest seesaw scale:**

$$\frac{\mathrm{d}\kappa_\nu}{\mathrm{d}t} = \alpha\kappa_\nu + \beta\kappa_\nu + \kappa_\nu\beta^{\mathrm{T}} \qquad U^\dagger\kappa_\nu U^* = \mathrm{Diag}\{\chi_1, \chi_2, \chi_3\} \qquad \widetilde{\beta} = U^\dagger\beta U$$

$$\frac{\mathrm{d}\chi_i}{\mathrm{d}t} = \left(\alpha + 2\widetilde{\beta}_{ii}\right)\chi_i \qquad \boxed{\chi_i(t) = \chi_i(t_0)\cdot\exp\left[\int_{t_0}^{t}\left(\alpha + 2\widetilde{\beta}_{ii}\right)\mathrm{d}t\right]} \qquad t \equiv \ln\mu$$

Neutrino masses are proportional to their initial values, therefore, they can not be generated if they are initially zero

- **Among seesaw scales (i.e., including threshold effects):**

$$\frac{\mathrm{d}\kappa_\nu}{\mathrm{d}t} = \alpha\kappa_\nu + \beta\kappa_\nu + \kappa_\nu\beta^{\mathrm{T}} + \alpha' C_5$$

It becomes much more complicated due to the coexistence of κ_ν and C_5

Threshold Effects on the Massless Neutrino

$$\mu\frac{\mathrm{d}\overset{(n)}{\kappa_\nu}}{\mathrm{d}\mu} = \frac{1}{16\pi^2}\left[\left(2\overset{(n)}{T} - \frac{3}{2}g_1^2 - \frac{9}{2}g_2^2\right)\overset{(n)}{\kappa_\nu} + \left(\frac{1}{2}\overset{(n)}{Y_\nu}\overset{(n)}{Y_\nu^\dagger} - \frac{3}{2}Y_l Y_l^\dagger\right)\overset{(n)}{\kappa_\nu} + \overset{(n)}{\kappa_\nu}\left(\frac{1}{2}\overset{(n)}{Y_\nu}\overset{(n)}{Y_\nu^\dagger} - \frac{3}{2}Y_l Y_l^\dagger\right)^{\mathrm{T}} + \left(4\lambda + \frac{3}{2}g_1^2 + \frac{3}{2}g_2^2\right)\overset{(n)}{C_5}\right] .$$

$$\mathrm{Re}[\mathrm{Det}(\overset{(n)}{\kappa_\nu})] = \overset{(n)}{\chi_1}\overset{(n)}{\chi_2}\overset{(n)}{\chi_3}$$

$$\frac{\mathrm{dDet}(M)}{\mathrm{d}x} = \mathrm{Tr}\left(\mathrm{Adj}(M)\frac{\mathrm{d}M}{\mathrm{d}x}\right)$$

$$M\mathrm{Adj}(M) = \mathrm{Adj}(M)M = \mathrm{Det}(M)\mathbf{1}$$

$$\mu\frac{\mathrm{dDet}(\overset{(n)}{\kappa_\nu})}{\mathrm{d}\mu} = \frac{1}{16\pi^2}\left[\left(3\overset{(n)}{\alpha_1} + 2\overset{(n)}{\beta}\right)\mathrm{Det}(\overset{(n)}{\kappa_\nu}) + \left(\overset{(n)}{\alpha_2} - \overset{(n)}{\alpha_1}\right)\mathrm{Tr}\left(\mathrm{Adj}(\overset{(n)}{\kappa_\nu})\overset{(n)}{C_5}\right)\right]$$

$$\overset{(n)}{\alpha_1} = 2\overset{(n)}{T} - \frac{3}{2}g_1^2 - \frac{9}{2}g_2^2\,,\quad \overset{(n)}{\alpha_2} = 4\lambda + 2\overset{(n)}{T} - 3g_2^2\,,\quad \overset{(n)}{\beta} = \frac{1}{2}\mathrm{Tr}\left(\overset{(n)}{Y_\nu}\overset{(n)}{Y_\nu^\dagger} - 3Y_l Y_l^\dagger\right)$$

Above the highest seesaw scale: $\mathrm{Tr}\left(\mathrm{Adj}(\overset{(n_{\max})}{\kappa_\nu})\overset{(n_{\max})}{C_5}\right) = 0$

Below the lowest seesaw scale: $\mathrm{Tr}\left(\mathrm{Adj}(\overset{(0)}{\kappa_\nu})\overset{(0)}{C_5}\right) = 3\mathrm{Det}(\overset{(0)}{\kappa_\nu})$

Among seesaw scales: $\mathrm{Tr}\left(\mathrm{Adj}(\overset{(n)}{\kappa_\nu})\overset{(n)}{C_5}\right)$? $\mathrm{Det}(\overset{(n)}{\kappa_\nu})$

Threshold Effects on the Massless Neutrino

$$\mu\frac{\mathrm{d}\overset{(n)}{\kappa}_\nu}{\mathrm{d}\mu} = \frac{1}{16\pi^2}\left[\left(2\overset{(n)}{T} - \frac{3}{2}g_1^2 - \frac{9}{2}g_2^2\right)\overset{(n)}{\kappa}_\nu + \left(\frac{1}{2}\overset{(n)}{Y}_\nu\overset{(n)}{Y}{}_\nu^\dagger - \frac{3}{2}Y_l Y_l^\dagger\right)\overset{(n)}{\kappa}_\nu + \overset{(n)}{\kappa}_\nu\left(\frac{1}{2}\overset{(n)}{Y}_\nu\overset{(n)}{Y}{}_\nu^\dagger - \frac{3}{2}Y_l Y_l^\dagger\right)^{\mathrm{T}} + \left(4\lambda + \frac{3}{2}g_1^2 + \frac{3}{2}g_2^2\right)\overset{(n)}{C}_5\right] .$$

$$\mathrm{Re}[\mathrm{Det}(\overset{(n)}{\kappa}_\nu)] = \overset{(n)}{\chi}_1\overset{(n)}{\chi}_2\overset{(n)}{\chi}_3$$

$$\frac{\mathrm{dDet}(M)}{\mathrm{d}x} = \mathrm{Tr}\left(\mathrm{Adj}(M)\frac{\mathrm{d}M}{\mathrm{d}x}\right)$$

$$M\mathrm{Adj}(M) = \mathrm{Adj}(M)M = \mathrm{Det}(M)\mathbf{1}$$

$$\mu\frac{\mathrm{dDet}(\overset{(n)}{\kappa}_\nu)}{\mathrm{d}\mu} = \frac{1}{16\pi^2}\left[\left(3\overset{(n)}{\alpha}_1 + 2\overset{(n)}{\beta}\right)\mathrm{Det}(\overset{(n)}{\kappa}_\nu) + \left(\overset{(n)}{\alpha}_2 - \overset{(n)}{\alpha}_1\right)\mathrm{Tr}\left(\mathrm{Adj}(\overset{(n)}{\kappa}_\nu)\overset{(n)}{C}_5\right)\right]$$

$$\overset{(n)}{\alpha}_1 = 2\overset{(n)}{T} - \frac{3}{2}g_1^2 - \frac{9}{2}g_2^2 , \quad \overset{(n)}{\alpha}_2 = 4\lambda + 2\overset{(n)}{T} - 3g_2^2 , \quad \overset{(n)}{\beta} = \frac{1}{2}\mathrm{Tr}\left(\overset{(n)}{Y}_\nu\overset{(n)}{Y}{}_\nu^\dagger - 3Y_l Y_l^\dagger\right)$$

Above the highest seesaw scale: $\mathrm{Tr}\left(\mathrm{Adj}(\overset{(n_{\max})}{\kappa_\nu})\overset{(n_{\max})}{C_5}\right) = 0$

Below the lowest seesaw scale: $\mathrm{Tr}\left(\mathrm{Adj}(\overset{(0)}{\kappa}_\nu)\overset{(0)}{C}_5\right) = 3\mathrm{Det}(\overset{(0)}{\kappa}_\nu)$

Among seesaw scales: $\mathrm{Tr}\left(\mathrm{Adj}(\overset{(n)}{\kappa}_\nu)\overset{(n)}{C}_5\right) = (3-n)\mathrm{Det}(\overset{(n)}{\kappa}_\nu)$

See backup or DZ, 2024 for more details

Threshold Effects on the Massless Neutrino

$$\mathrm{Tr}\left(\mathrm{Adj}(\overset{(n)}{\kappa_\nu})\overset{(n)}{C_5}\right) = (3-n)\mathrm{Det}(\overset{(n)}{\kappa_\nu}) \quad \Rightarrow \quad \mu\frac{\mathrm{dDet}(\overset{(n)}{\kappa_\nu})}{\mathrm{d}\mu} = \frac{1}{16\pi^2}\left[3\overset{(n)}{\alpha_1} + (3-n)\left(\overset{(n)}{\alpha_2} - \overset{(n)}{\alpha_1}\right) + 2\overset{(n)}{\beta}\right]\mathrm{Det}(\overset{(n)}{\kappa_\nu})$$

Solution for the determinant:

$$\mathrm{Det}(\kappa_\nu(t)) = I_{\rm Int}(t)\cdot \mathrm{Det}(\kappa_\nu(t_4))$$

$$I_{\rm Int}(t) = \begin{cases} \overset{(3)}{I'_{\rm Int}}(t,t_4)\ , & t_3 < t \leq t_4 \\ \overset{(2)}{I'_{\rm Int}}(t,t_3)\,\overset{(3)}{I'_{\rm Int}}(t_3,t_4)\ , & t_2 < t \leq t_3 \\ \overset{(1)}{I'_{\rm Int}}(t,t_2)\,\overset{(2)}{I'_{\rm Int}}(t_2,t_3)\,\overset{(3)}{I'_{\rm Int}}(t_3,t_4)\ , & t_1 < t \leq t_2 \\ \overset{(0)}{I'_{\rm Int}}(t,t_1)\,\overset{(1)}{I'_{\rm Int}}(t_1,t_2)\,\overset{(2)}{I'_{\rm Int}}(t_2,t_3)\,\overset{(3)}{I'_{\rm Int}}(t_3,t_4)\ , & t \leq t_1 \end{cases}$$

$$t_{(n)} \equiv \ln(\mu_{(n)})$$

$$\mu_n = M_n$$

$$\mu_4 = \Lambda_{\rm GUT}$$

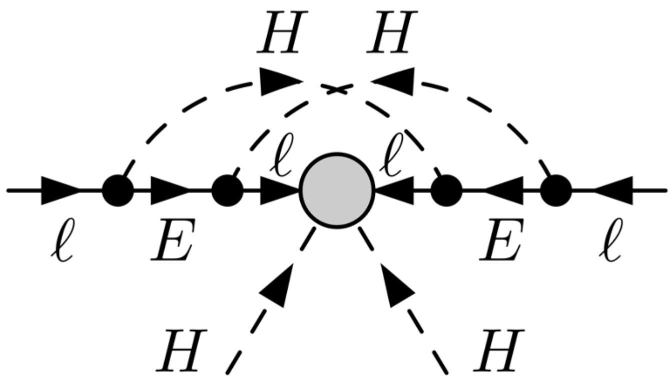
$$\overset{(n)}{I'_{\rm Int}}(t,t_{n+1}) = \exp\left\{\frac{1}{16\pi^2}\int_{t_{n+1}}^{t}\left[3\overset{(n)}{\alpha_1} + (3-n)\left(\overset{(n)}{\alpha_2} - \overset{(n)}{\alpha_1}\right) + 2\overset{(n)}{\beta}\right]\mathrm{d}t\right\}$$

Massless neutrino at the GUT scale → **Zero determinant at the GUT scale** → **Zero determinant at the EW scale** → **Massless at the EW scale**

The vanishing neutrino mass is stable against one-loop RG running effects EVEN with threshold effects

Two-loop RG Running Effects

Rank-increase diagram for renormalization of the Weinberg operator:



$$\frac{\mathrm{d}C_5}{\mathrm{d}t} \ni \frac{1}{(16\pi^2)^2} \cdot 2Y_l Y_l^\dagger C_5 \left(Y_l Y_l^\dagger\right)^{\mathrm{T}}$$

$$m_1\left(\Lambda_{\mathrm{EW}}\right) \sim \frac{2y_\tau^4}{\left(16\pi^2\right)^2} \ln\left(\frac{\Lambda}{\Lambda_{\mathrm{EW}}}\right) \cdot \sum_{i=2,3} m_i \,\mathrm{Re}\left(U_{\tau 1}^* U_{\tau i}\right)^2 \quad \textbf{(NMO)}$$

$$m_3\left(\Lambda_{\mathrm{EW}}\right) \sim \frac{2y_\tau^4}{\left(16\pi^2\right)^2} \ln\left(\frac{\Lambda}{\Lambda_{\mathrm{EW}}}\right) \cdot \sum_{i=1,2} m_i \,\mathrm{Re}\left(U_{\tau 3}^* U_{\tau i}\right)^2 \quad \textbf{(IMO)}$$

$$m_{\text{lightest}} \sim 10^{-13}\ \mathrm{eV}$$

S. Davidson, G. Isidori, A. Strumia, 2007; Z. Z. Xing, DZ, 2020; A. Ibarra, N. Leister, DZ, 2024

Conclusions

- We revisit the one-loop RGEs in the canonical seesaw mechanism among seesaw scales and obtain the missed terms
- We achieve the running behavior of the determinant of the effect neutrino mass matrix against the energy scale
- We strictly prove that if the lightest neutrino is initially massless, it remains massless at the one-loop level even threshold effects are taken into account.
- Nevertheless, two-loop RG running effects can generate a non-zero mass for the initially massless neutrino

THANKS FOR YOUR ATTENTION

Backup

For 3x3 matrix M, its adjugate can be written as

$$\mathrm{Adj}(M) = \frac{1}{2}\left[(\mathrm{Tr}(M))^2 - \mathrm{Tr}\left(M^2\right)\right]\mathbf{1} - M\mathrm{Tr}(M) + M^2$$

$$\begin{aligned}\mathrm{Adj}(M+W) = {} & \mathrm{Adj}(M) + \mathrm{Adj}(W) - M\mathrm{Tr}(W) - W\mathrm{Tr}(M) + MW + WM \\ & + [\mathrm{Tr}(M)\mathrm{Tr}(W) - \mathrm{Tr}(MW)]\,\mathbf{1}\,.\end{aligned}$$

$$\mathrm{Tr}\left(\mathrm{Adj}(M+W)W\right) = 3\mathrm{Det}(W) + 2\mathrm{Tr}\left(\mathrm{Adj}(W)M\right) + \mathrm{Tr}\left(\mathrm{Adj}(M)W\right)$$

$$\mathrm{Tr}\left(\mathrm{Adj}(\kappa_\nu^{(n)})C_5^{(n)}\right) = 3\mathrm{Det}(C_5^{(n)}) + 2\mathrm{Tr}\left(\mathrm{Adj}(C_5^{(n)})\kappa_D^{(n)}\right) + \mathrm{Tr}\left(\mathrm{Adj}(\kappa_D^{(n)})C_5^{(n)}\right)$$

$$\mathrm{Tr}\left(\mathrm{Adj}(\kappa_\nu^{(n)})\kappa_D^{(n)}\right) = 3\mathrm{Det}(\kappa_D^{(n)}) + 2\mathrm{Tr}\left(\mathrm{Adj}(\kappa_D^{(n)})C_5^{(n)}\right) + \mathrm{Tr}\left(\mathrm{Adj}(C_5^{(n)})\kappa_D^{(n)}\right)$$

$$\mathrm{Det}(\kappa_\nu^{(n)}) = \mathrm{Det}(C_5^{(n)}) + \mathrm{Det}(\kappa_D^{(n)}) + \mathrm{Tr}\left(\mathrm{Adj}(C_5^{(n)})\kappa_D^{(n)}\right) + \mathrm{Tr}\left(\mathrm{Adj}(\kappa_D^{(n)})C_5^{(n)}\right)$$

$$\mathrm{Tr}\left[\mathrm{Adj}(\kappa_\nu^{(n)})C_5^{(n)}\right] = \mathrm{Det}(\kappa_\nu^{(n)}) + 2\mathrm{Det}(C_5^{(n)}) - \mathrm{Det}(\kappa_D^{(n)}) + \mathrm{Tr}\left[\mathrm{Adj}(C_5^{(n)})\kappa_D^{(n)}\right]$$

$$\mathrm{Tr}\left[\mathrm{Adj}(\kappa_\nu^{(n)})C_5^{(n)}\right] = 2\mathrm{Det}(\kappa_\nu^{(n)}) + \mathrm{Det}(C_5^{(n)}) - 2\mathrm{Det}(\kappa_D^{(n)}) - \mathrm{Tr}\left[\mathrm{Adj}(\kappa_D^{(n)})C_5^{(n)}\right]$$

Backup

Some essential points used for discussion:

- **For 3-by-3 matrices, one can prove**

$$\mathrm{Tr}\left(\mathrm{Adj}(M+W)W\right) = 3\mathrm{Det}(W) + 2\mathrm{Tr}\left(\mathrm{Adj}(W)M\right) + \mathrm{Tr}\left(\mathrm{Adj}(M)W\right)$$

Linear algebra relation

$$\mathrm{Tr}\left(\mathrm{Adj}(\overset{(n)}{\kappa}_\nu)\overset{(n)}{C}_5\right) = \mathrm{Det}(\overset{(n)}{\kappa}_\nu) + 2\mathrm{Det}(\overset{(n)}{C}_5) - \mathrm{Det}(\overset{(n)}{\kappa}_D) + \mathrm{Tr}\left(\mathrm{Adj}(\overset{(n)}{C}_5)\overset{(n)}{\kappa}_D\right)$$

$$\mathrm{Tr}\left(\mathrm{Adj}(\overset{(n)}{\kappa}_\nu)\overset{(n)}{C}_5\right) = 2\mathrm{Det}(\overset{(n)}{\kappa}_\nu) + \mathrm{Det}(\overset{(n)}{C}_5) - 2\mathrm{Det}(\overset{(n)}{\kappa}_D) - \mathrm{Tr}\left(\mathrm{Adj}(\overset{(n)}{\kappa}_D)\overset{(n)}{C}_5\right)$$

Independent of the energy scale

- **The differential equation of C_5 has the form** $\mu\frac{\mathrm{d}X}{\mathrm{d}\mu} = \alpha X + \beta X + X\beta^{\mathrm{T}}$ **, leading to that its rank will not be changed by running effects**

- **The rank of** $\overset{(n)}{\kappa}_D = \overset{(n)}{Y}_\nu \overset{(n)}{M}{}_N^{-1} \overset{(n)}{Y}{}_\nu^{\mathrm{T}}$ **is determined by those of Y_ν and M_N, i.e.,**

$$\mathrm{Rank}(\overset{(n)}{\kappa}_D) = \min\{\mathrm{Rank}(\overset{(n)}{Y}_\nu), \mathrm{Rank}(\overset{(n)}{M}_N)\}$$

- **If Rank(M) ≤ n − 2 with n being the dimension of M, then Adj(M) = 0**

- **Subadditivity: Rank(M+W) ≤ Rank(M) + Rank(W)**

Backup

- **Two right-handed neutrinos: trivial**

$$\mathrm{Rank}(\overset{(2)}{Y}_\nu(\mu)) = 2 \;\Rightarrow\; \mathrm{Det}(\overset{(2)}{\kappa}_\nu(\mu)) = 0$$

M_2

$$(\overset{(1)}{C}_5)_{\alpha\beta} = (\overset{(2)}{Y}_\nu)_{\alpha 2}(\overset{(2)}{Y}_\nu)_{\beta 2}/M_2 \;\Rightarrow\; \mathrm{Rank}(\overset{(1)}{C}_5(M_2)) = 1$$

$$\mathrm{Rank}(\overset{(1)}{C}_5(\mu)) = 1 \qquad \overset{(1)}{\kappa}_D(\mu) = \overset{(1)}{Y}_\nu(\mu)\overset{(1)}{Y}{}_\nu^{\mathrm{T}}(\mu)/M_1(\mu) \;\Rightarrow\; \mathrm{Rank}(\overset{(1)}{\kappa}_D(\mu)) = 1$$

$$\mathrm{Rank}(\overset{(1)}{\kappa}_\nu(\mu)) \leq \mathrm{Rank}(\overset{(1)}{C}_5(\mu)) + \mathrm{Rank}(\overset{(1)}{\kappa}_D(\mu)) = 2 \;\Rightarrow\; \mathrm{Det}(\overset{(1)}{\kappa}_\nu(\mu)) = 0$$

M_1

$$\overset{(0)}{\kappa}_\nu(M_1) = \overset{(1)}{\kappa}_\nu(M_1) \;\Rightarrow\; \mathrm{Det}(\overset{(0)}{\kappa}_\nu(M_1)) = 0$$

$$\mathrm{Det}(\overset{(0)}{\kappa}_\nu(\mu)) = 0$$

The determinant of the effect neutrino mass matrix remains vanishing against RG running effects ➡ **Massless neutrino**

Backup

- **Three right-handed neutrinos:**

$$\mathrm{Tr}\left(\mathrm{Adj}(\overset{(n)}{\kappa}_\nu)\overset{(n)}{C}_5\right) = \mathrm{Det}(\overset{(n)}{\kappa}_\nu) + 2\mathrm{Det}(\overset{(n)}{C}_5) - \mathrm{Det}(\overset{(n)}{\kappa}_D) + \mathrm{Tr}\left(\mathrm{Adj}(\overset{(n)}{C}_5)\overset{(n)}{\kappa}_D\right)$$

$$\mathrm{Tr}\left(\mathrm{Adj}(\overset{(n)}{\kappa}_\nu)\overset{(n)}{C}_5\right) = 2\mathrm{Det}(\overset{(n)}{\kappa}_\nu) + \mathrm{Det}(\overset{(n)}{C}_5) - 2\mathrm{Det}(\overset{(n)}{\kappa}_D) - \mathrm{Tr}\left(\mathrm{Adj}(\overset{(n)}{\kappa}_D)\overset{(n)}{C}_5\right)$$

$$\mathrm{Rank}(\overset{(3)}{\kappa}_\nu(\mu)) = \mathrm{Rank}(\overset{(3)}{Y}_\nu(\mu)) \leq 3$$

M_3

$$(\overset{(2)}{C}_5)_{\alpha\beta} = (\overset{(3)}{Y}_\nu)_{\alpha 3}(\overset{(3)}{Y}_\nu)_{\beta 3}/M_3 \Rightarrow \mathrm{Rank}(\overset{(2)}{C}_5(M_3)) = 1$$

$$\mathrm{Rank}(\overset{(2)}{C}_5(\mu)) = \mathrm{Rank}(\overset{(2)}{C}_5(M_3)) = 1 \qquad \mathrm{Rank}(\overset{(2)}{\kappa}_D(\mu)) = \mathrm{Rank}(\overset{(2)}{Y}_\nu(\mu)) \leq 2$$

$$\mathrm{Det}(\overset{(2)}{C}_5(\mu)) = \mathrm{Det}(\overset{(2)}{\kappa}_D(\mu)) = 0 \qquad \mathrm{Adj}(\overset{(2)}{C}_5(\mu)) = 0$$

$$\boxed{\mathrm{Tr}\left(\mathrm{Adj}(\overset{(2)}{\kappa}_\nu(\mu))\overset{(2)}{C}_5(\mu)\right) = \mathrm{Det}(\overset{(2)}{\kappa}_\nu(\mu))}$$

M_2

$$(\overset{(1)}{C}_5)_{\alpha\beta} = (\overset{(2)}{C}_5)_{\alpha\beta} + (\overset{(2)}{Y}_\nu)_{\alpha 2}(\overset{(2)}{Y}_\nu)_{\beta 2}/M_2 \Rightarrow \mathrm{Rank}(\overset{(1)}{C}_5(M_2)) \leq \mathrm{Rank}(\overset{(2)}{C}_5(M_2)) + \mathrm{Rank}(\overset{(2)}{\mathbf{y}}_2(M_2)\overset{(2)}{\mathbf{y}}{}_2^{\mathrm{T}}(M_2)/M_2) = 2$$

$$\mathrm{Rank}(\overset{(1)}{C}_5(\mu)) = \mathrm{Rank}(\overset{(1)}{C}_5(M_2)) \leq 2 \Rightarrow \mathrm{Det}(\overset{(1)}{C}_5(\mu)) = 0$$

$$\mathrm{Rank}(\overset{(1)}{\kappa}_D(\mu)) = \mathrm{Rank}(\overset{(1)}{Y}_\nu(\mu)) = 1 \Rightarrow \mathrm{Det}(\overset{(1)}{\kappa}_D(\mu)) = \mathrm{Adj}(\overset{(1)}{\kappa}_D(\mu)) = 0$$

$$\boxed{\mathrm{Tr}\left(\mathrm{Adj}(\overset{(1)}{\kappa}_\nu(\mu))\overset{(1)}{C}_5(\mu)\right) = 2\mathrm{Det}(\overset{(1)}{\kappa}_\nu(\mu))}$$

M_1

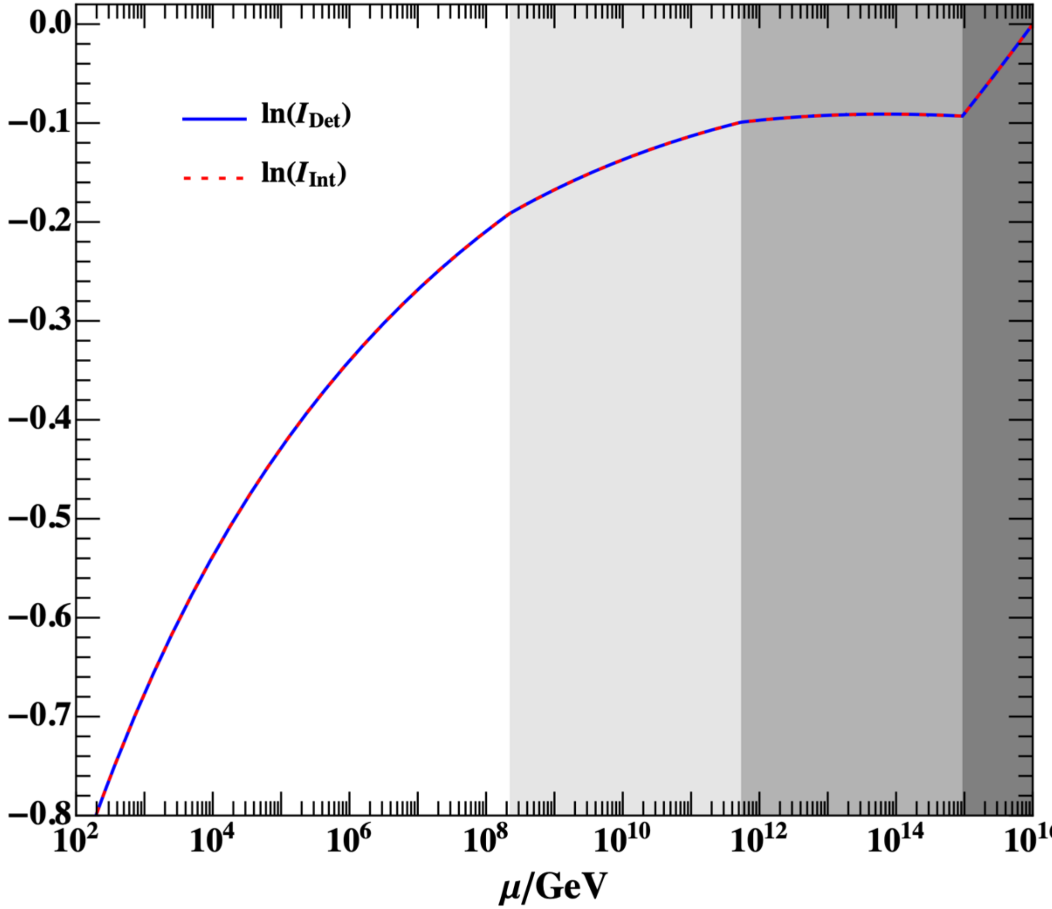
$$\overset{(0)}{\kappa}_\nu(M_1) = \overset{(0)}{C}_5(M_1) = \overset{(1)}{\kappa}_\nu(M_1)$$

$$\boxed{\mathrm{Tr}\left(\mathrm{Adj}(\overset{(0)}{\kappa}_\nu(\mu))\overset{(0)}{C}_5(\mu)\right) = 3\mathrm{Det}(\overset{(0)}{\kappa}_\nu(\mu))}$$

$$\boxed{\mathrm{Tr}\left(\mathrm{Adj}(\overset{(n)}{\kappa}_\nu)\overset{(n)}{C}_5\right) = (3-n)\mathrm{Det}(\overset{(n)}{\kappa}_\nu)}$$

Backup

Running of the determinant against μ : $\mathrm{Det}(\kappa_\nu(t)) = I_{\mathrm{Int}}(t) \cdot \mathrm{Det}(\kappa_\nu(t_4))$



$I_{\mathrm{Det}}(t) = \mathrm{Det}(\kappa_\nu(t))/\mathrm{Det}(\kappa_\nu(t_4))$ **is calculated directly from κ_ν**

$I_{\mathrm{Int}}(t)$ **is calculated from its expression with Higgs, gauge and Yukawa couplings**

Backup

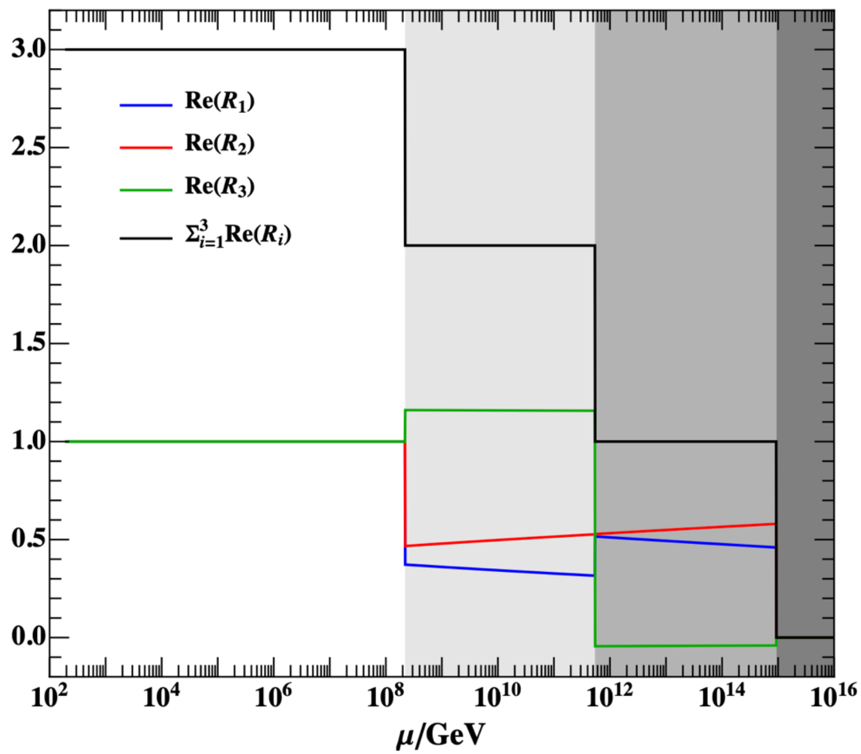
A little more discussion on $\mathrm{Tr}\left(\mathrm{Adj}(\overset{(n)}{\kappa_\nu})\overset{(n)}{C_5}\right) = (3-n)\mathrm{Det}(\overset{(n)}{\kappa_\nu})$

$$(3-n)\overset{(n)}{\chi_1}\overset{(n)}{\chi_2}\overset{(n)}{\chi_3} = \overset{(n)}{\chi_2}\overset{(n)}{\chi_3}(\overset{(n)}{\mathcal{C}_5})_{11} + \overset{(n)}{\chi_1}\overset{(n)}{\chi_3}(\overset{(n)}{\mathcal{C}_5})_{22} + \overset{(n)}{\chi_1}\overset{(n)}{\chi_2}(\overset{(n)}{\mathcal{C}_5})_{33}$$

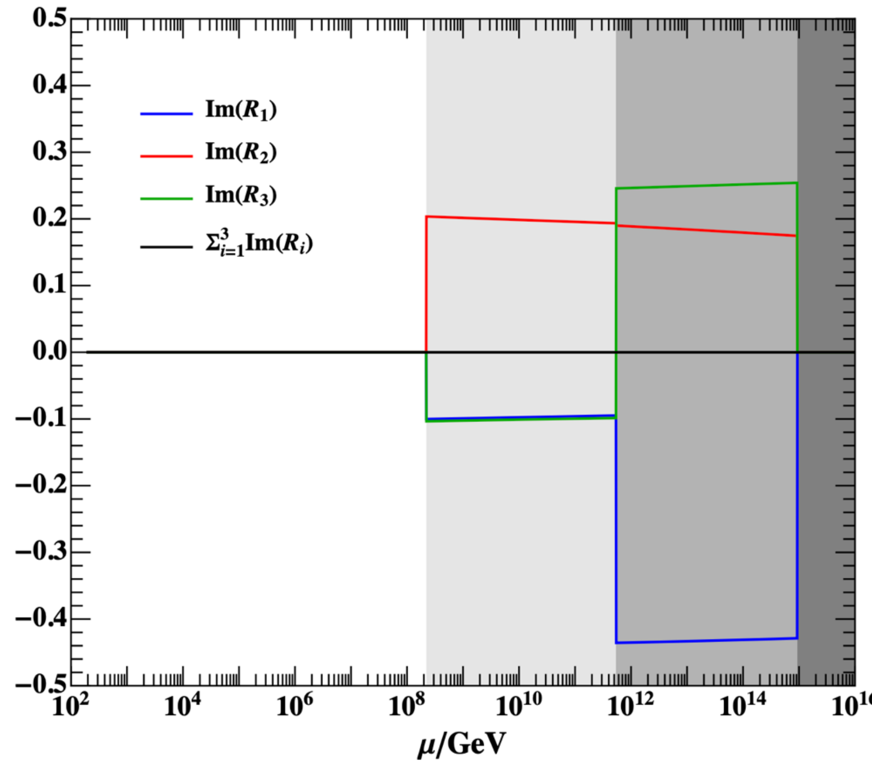
$$\overset{(n)}{U_\nu^\dagger}\overset{(n)}{\kappa_\nu}\overset{(n)}{U_\nu^*} = \mathrm{Diag}\{\overset{(n)}{\chi_1}, \overset{(n)}{\chi_2}, \overset{(n)}{\chi_3}\}$$

$$\overset{(n)}{\mathcal{C}_5} \equiv \overset{(n)}{U_\nu^\dagger}\overset{(n)}{C_5}\overset{(n)}{U_\nu^*}$$

Running of $R_i \equiv (\overset{(n)}{\mathcal{C}_5})_{ii}/\overset{(n)}{\chi_i}$ **against the energy scale:**



$$\sum_i \mathrm{Re}[(\overset{(n)}{\mathcal{C}_5})_{ii}/\overset{(n)}{\chi_i}] = 3-n$$



$$\sum_i \mathrm{Im}[(\overset{(n)}{\mathcal{C}_5})_{ii}/\overset{(n)}{\chi_i}] = 0$$