Electroweak Double-Box Integrals for Møller Scattering with Three Z Bosons

Dmytro Melnichenko

based on work with **I.Bree** and **S.Weinzierl**

24.09.25

Synergies Towards
the Future Standard Model

JOHANNES GUTENBERG UNIVERSITÄT MAINZ

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Introduction

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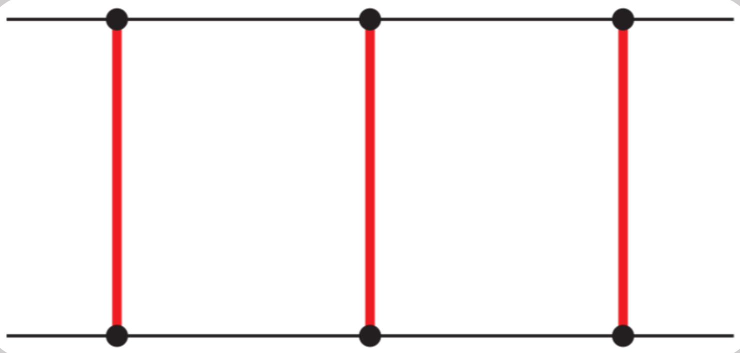

- **35** masters & **20** sectors

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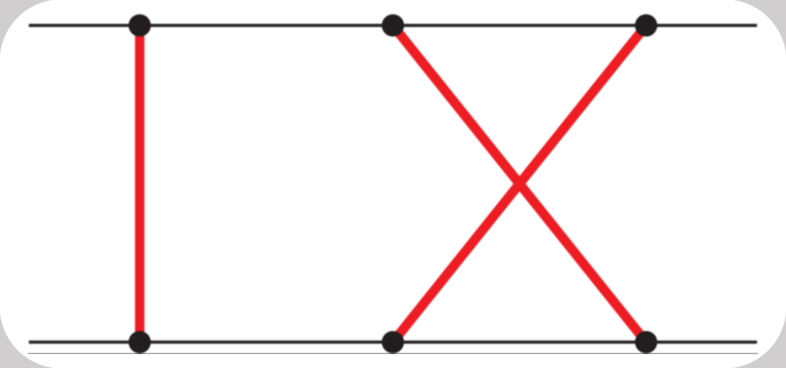

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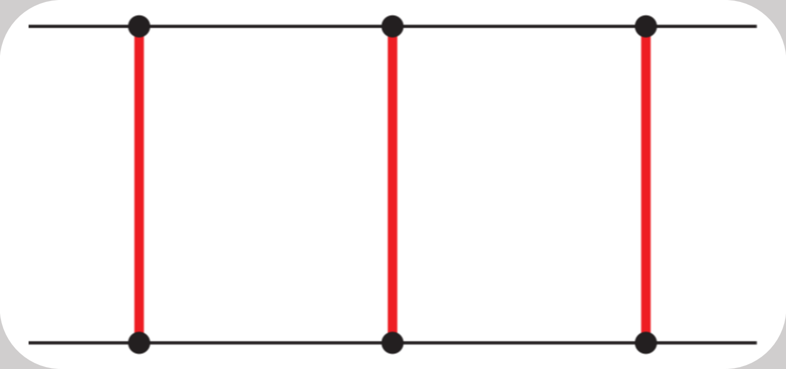

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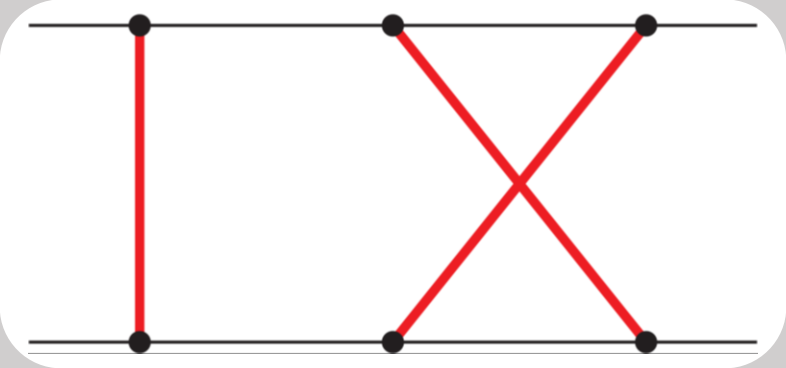

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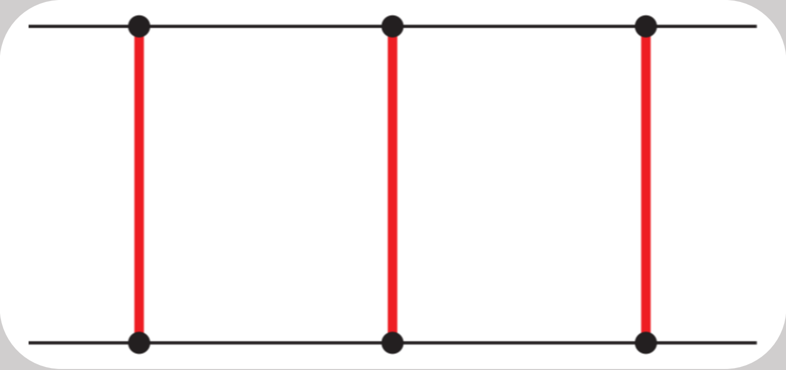

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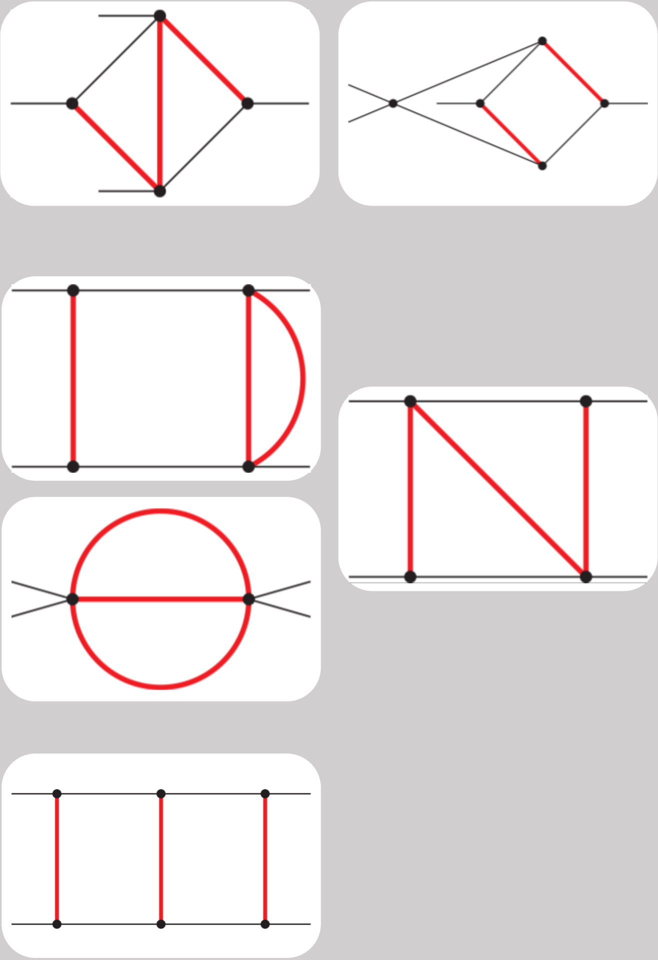

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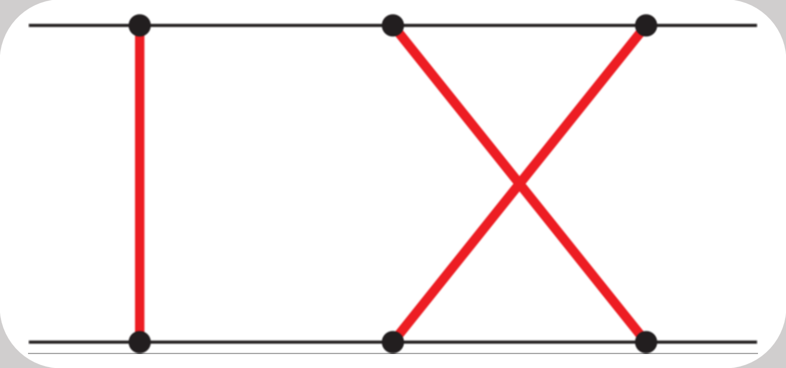

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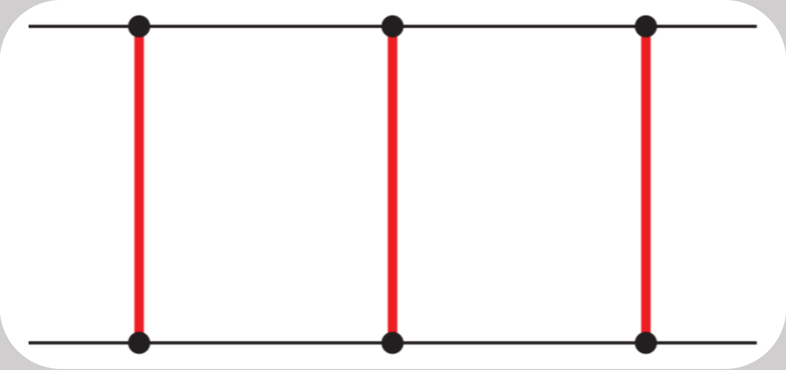

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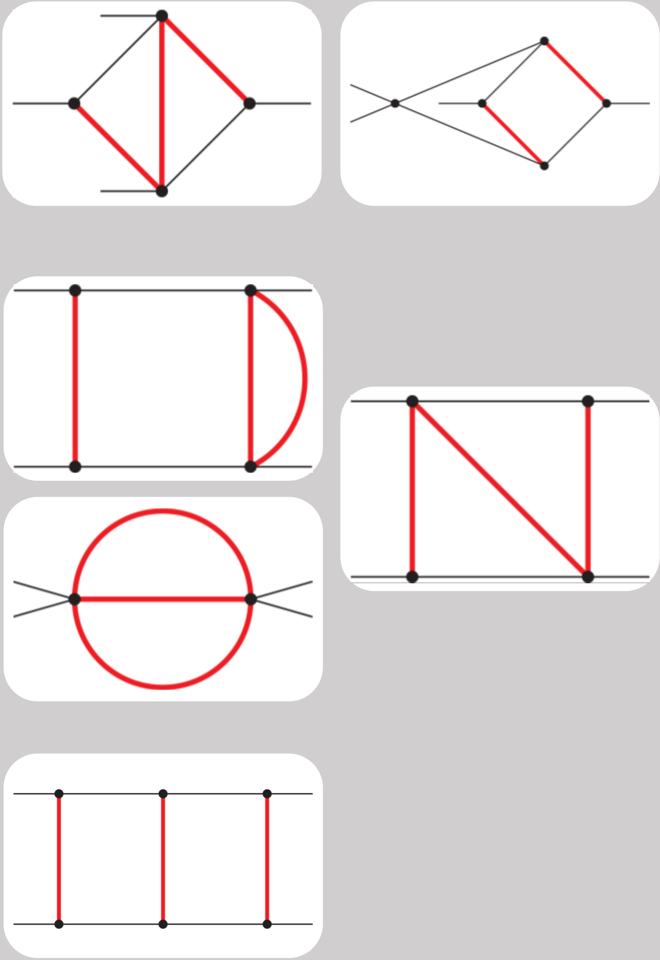
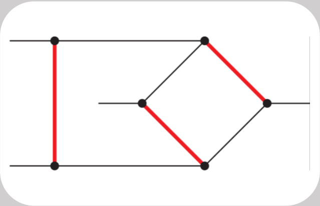


Genus 2



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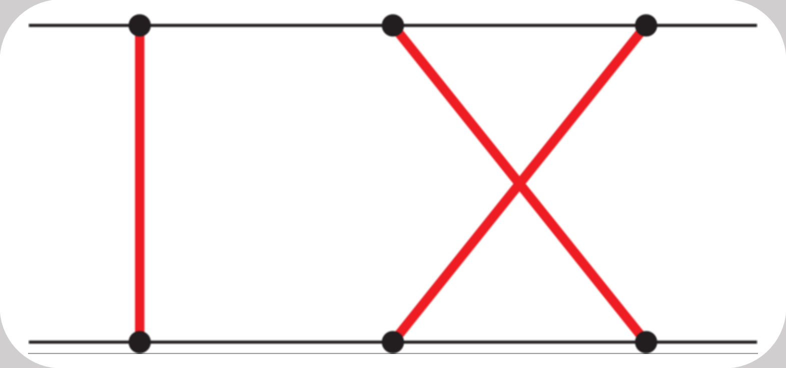

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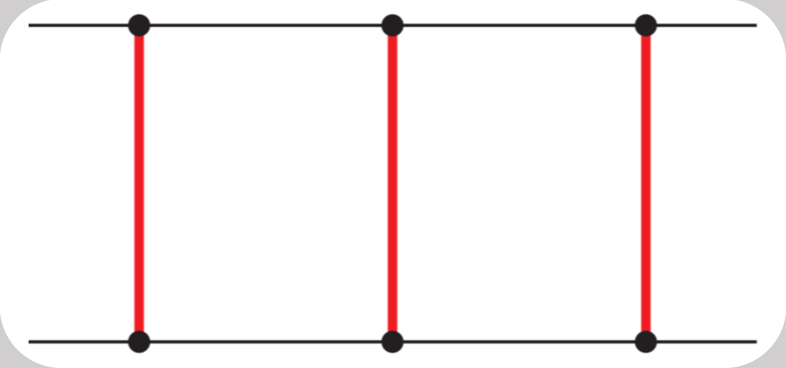

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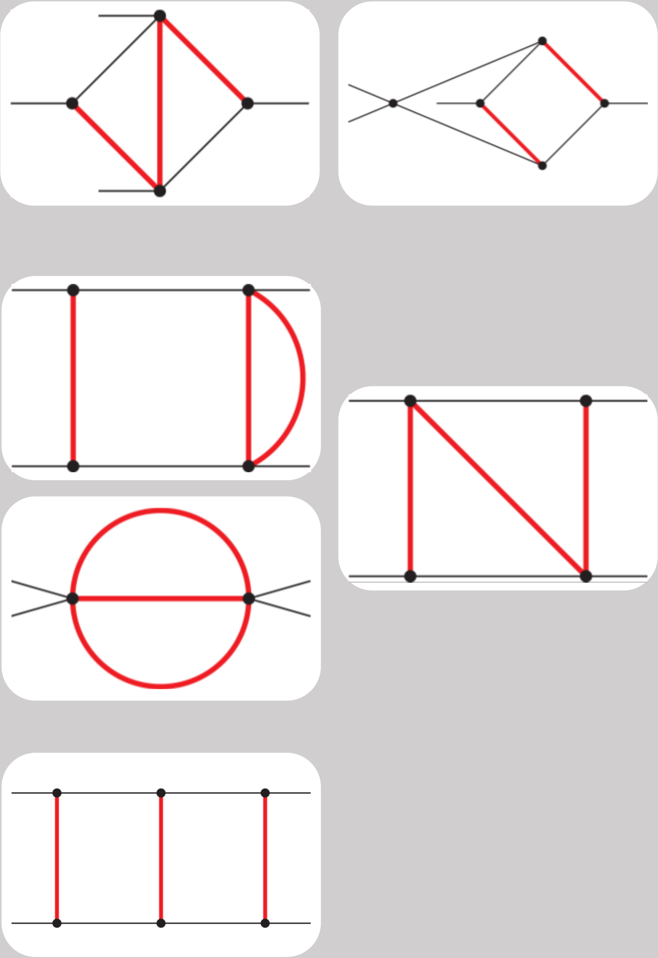

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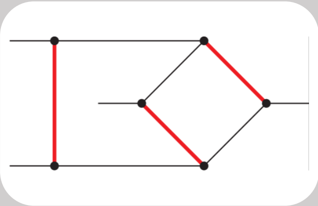
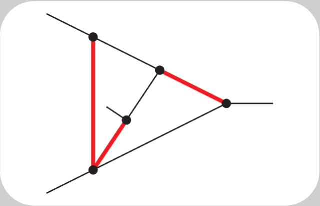


K3 Surface



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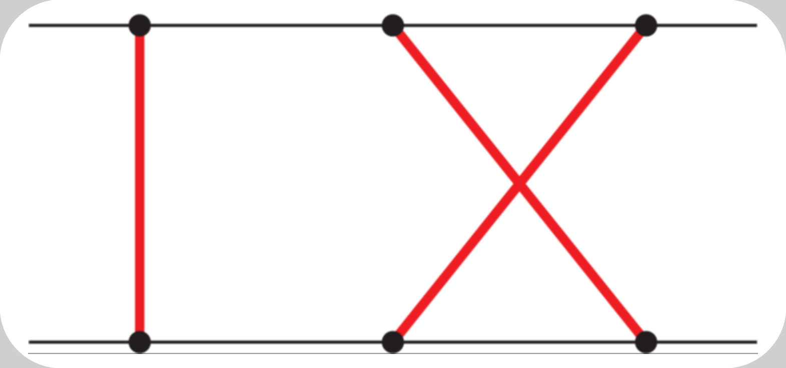

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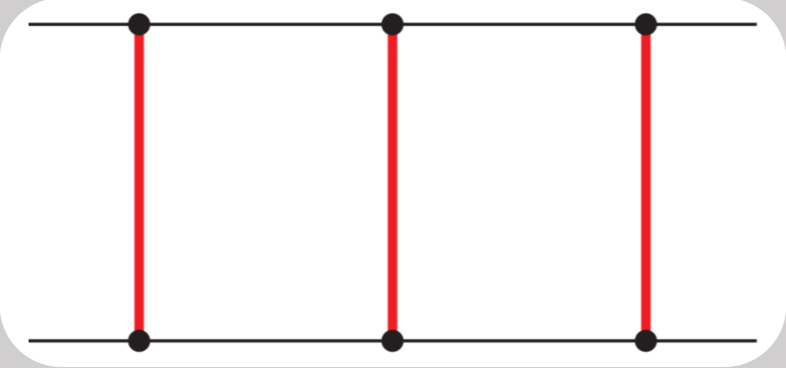

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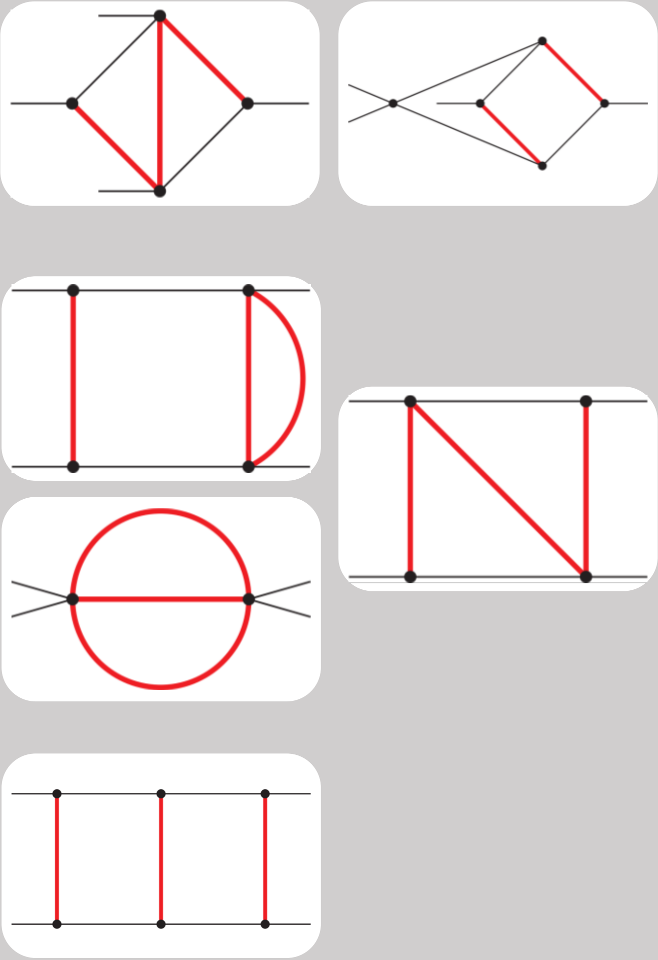

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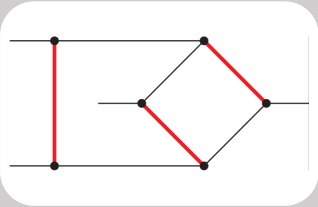

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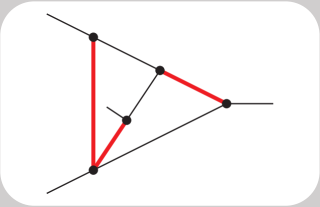

Many non-trivial geometries appear!

Differential Equations

$$\frac{d}{dx} I_1 = \tilde{I}_1 = c_1^1(x,\varepsilon) I_1 + \ldots + c_{N_F}^1(x,\varepsilon) I_{N_F}$$

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$$\frac{d}{dx} \vec{I} = \tilde{A}(x,\varepsilon) \vec{I}$$

$$\vec{K} = \tilde{R}^{-1}(\varepsilon, x) \vec{I}$$

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Find preliminary $\vec{J}$-basis using *clever ordering:*

$$\boxed{\frac{d}{dx}\vec{J} = \sum_{k=k_{\min}}^{k=1} \varepsilon^k A^{(k)}(x)\vec{J}}$$

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Step 2

Construct a *rotation* into ε-factorized $\vec{K}$-basis:

$$\boxed{\vec{K} = R^{-1}(\varepsilon, x)\vec{J}}$$

Building Blocks

Feynman Integrand
(in Baikov representation)

$$\Psi_{\mu_0 \ldots \mu_{N_D}}[\mathrm{Q}] = C_\varepsilon U(z) \widehat{\Phi}_{\mu_0 \ldots \mu_{N_D}}[\mathrm{Q}] \eta$$

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Feynman Integrand
(in Baikov representation)

We work in homogeneous coordinates

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Recipe

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Recipe

- Appropriate pre-factor C_ε
- Twist $U(z)$

$$U(z_0,\ldots,z_n) = \prod_{i,\mathrm{odd}} P_i^{-\frac{1}{2}+\frac{1}{2}b_i\varepsilon} \prod_{j,\mathrm{even}} P_j^{\frac{1}{2}b_j\varepsilon}$$

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Recipe

- Appropriate pre-factor C_ε
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- Stripped differential form $\widehat{\Phi}$

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Recipe

- Appropriate pre-factor C_ε
- Twist $U(z)$
- Rational polynomials P_i
- Stripped differential form $\hat{\Phi}$
- Measure η

$$U(z_0, \ldots, z_n) = \prod_{i,\mathrm{odd}} P_i^{-\frac{1}{2}+\frac{1}{2}b_i\varepsilon} \prod_{j,\mathrm{even}} P_j^{\frac{1}{2}b_j\varepsilon}$$

$$\hat{\Phi}_{\mu_0 \ldots \mu_{N_D}}[\mathrm{Q}] = \frac{\mathrm{Q}}{\prod_i P_i^{\mu_i}}$$

$$\eta = \sum_{j=0}^{n} (-1)^j z_j dz_0 \wedge \cdots \wedge \cancel{dz_j} \wedge \cdots dz_n$$

Ordering

Find **basis** according to

(Laporta algorithm with modified ordering)

$$(a, r, o, |\mu|, \ldots)$$

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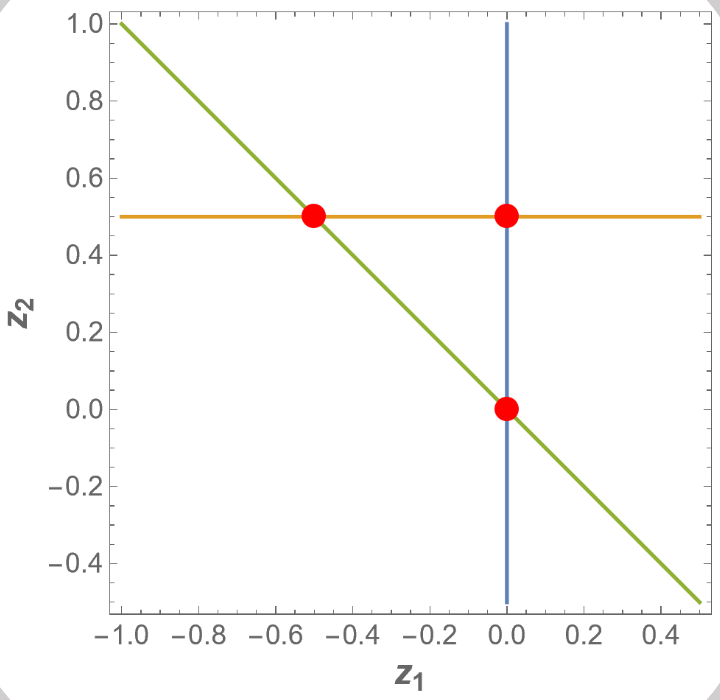
Example

$$\frac{\eta}{z_1(z_2 - x)(z_1 + z_2)}$$



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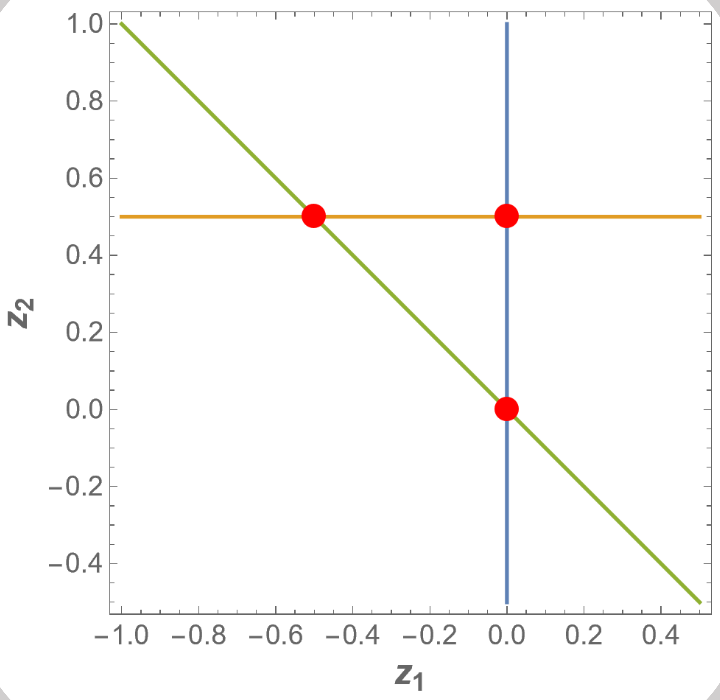

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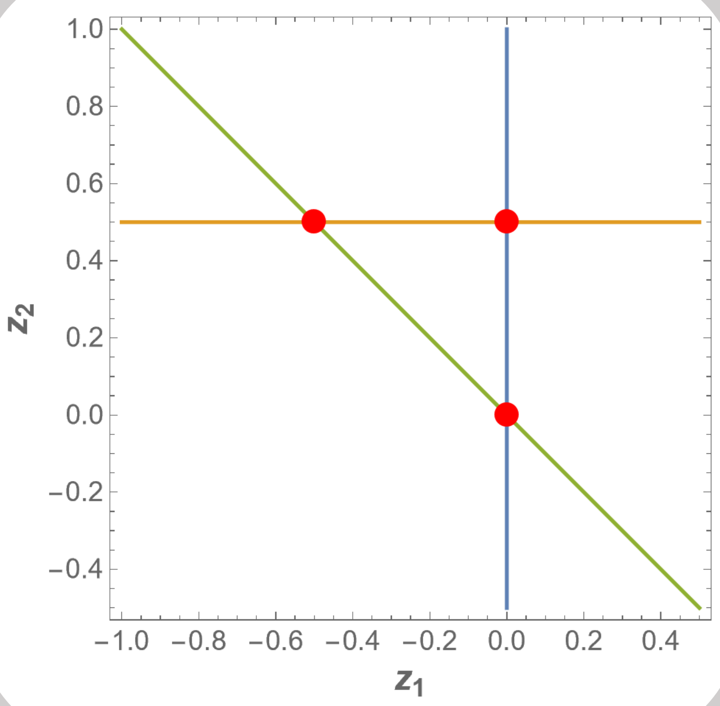

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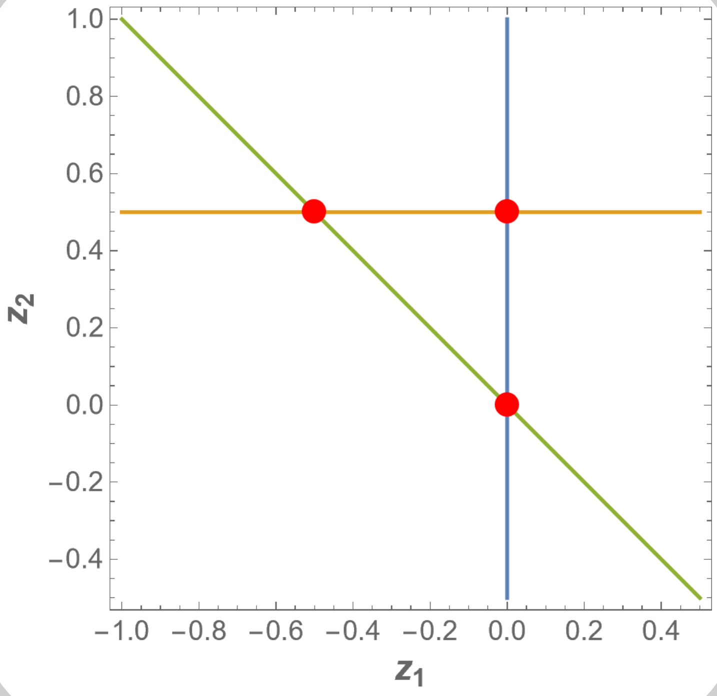
2 consecutive residues at

$(z_1, z_2) = \{(0,0), (0,x), (-x,0)\}$



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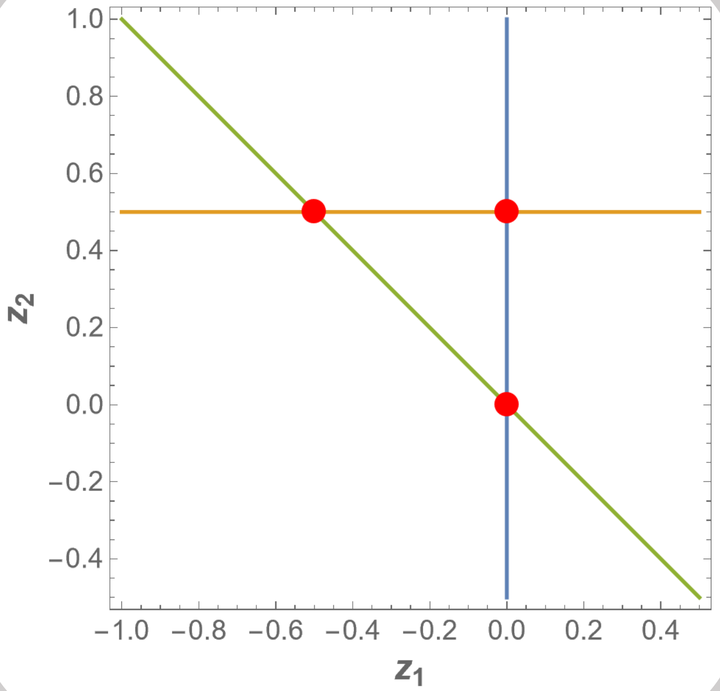

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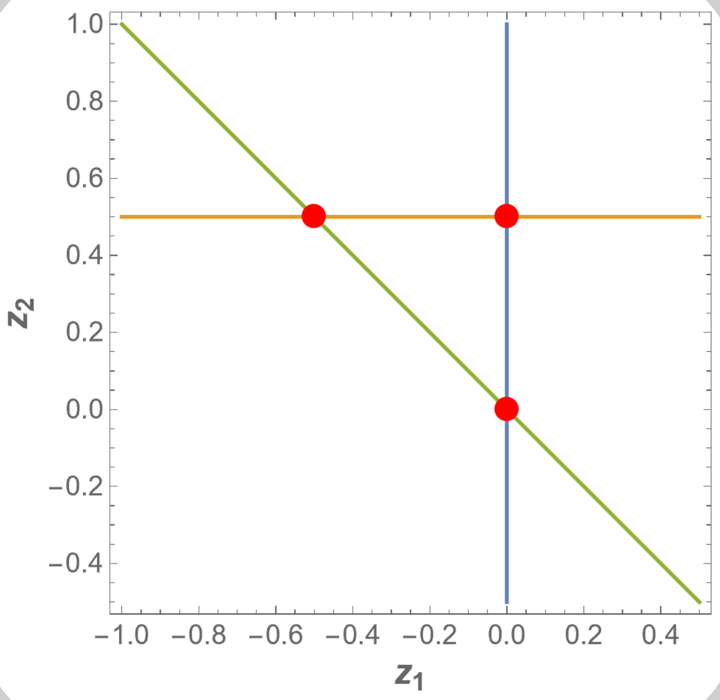

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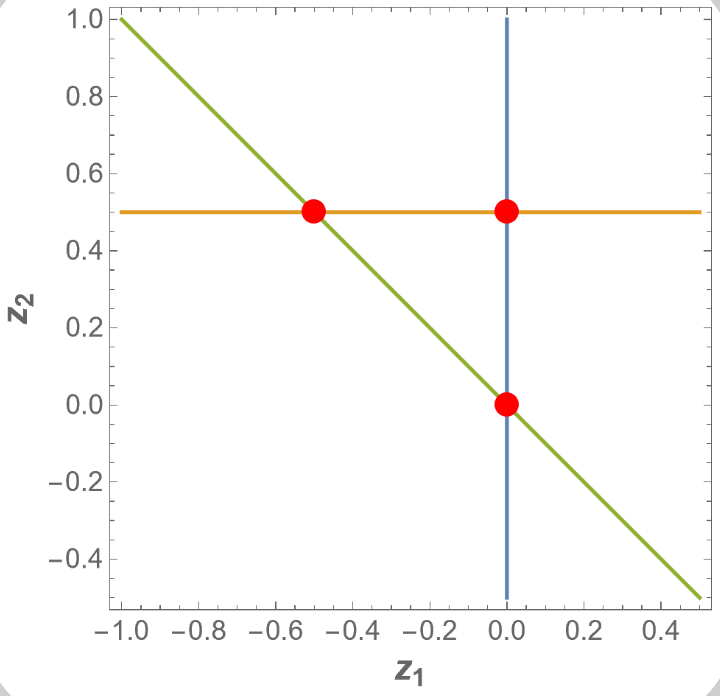

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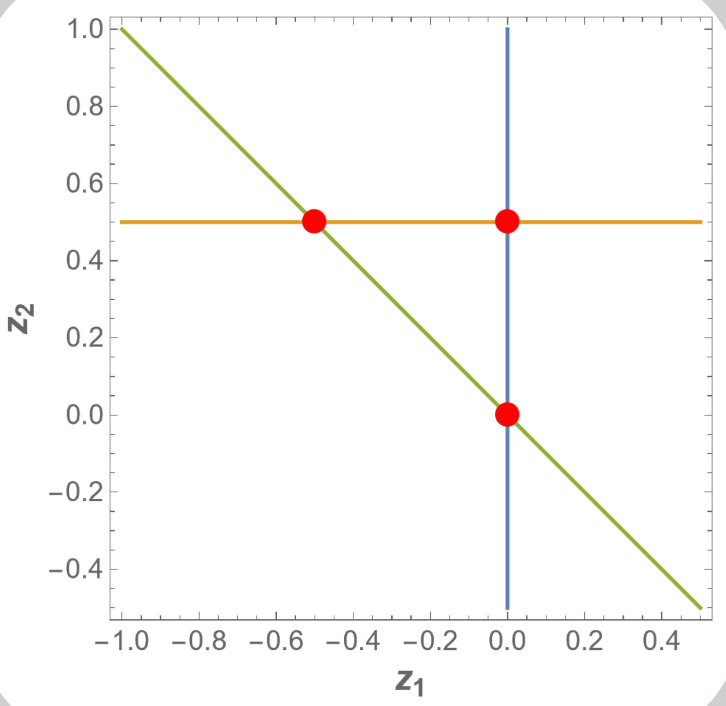
$|\boldsymbol{\mu}| = 1 + 1 + 1 = 3$



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Localization level of $\Psi^0_{\mu_0 \dots \mu_{N_D}}[Q]$

Number of residues of $\Psi^0_{\mu_0 \dots \mu_{N_D}}[Q]$

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Pole order of $\Psi^0_{\mu_0 \dots \mu_{N_D}}[Q]$

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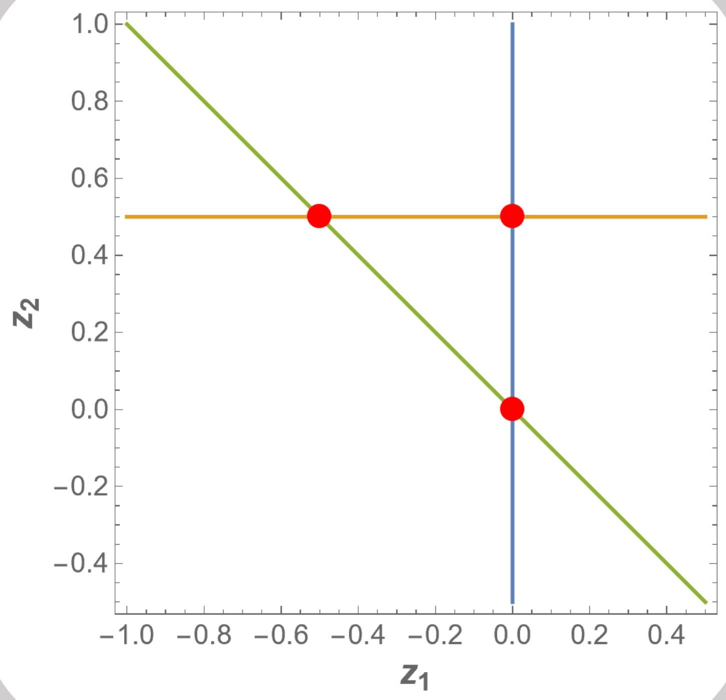

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Step 1

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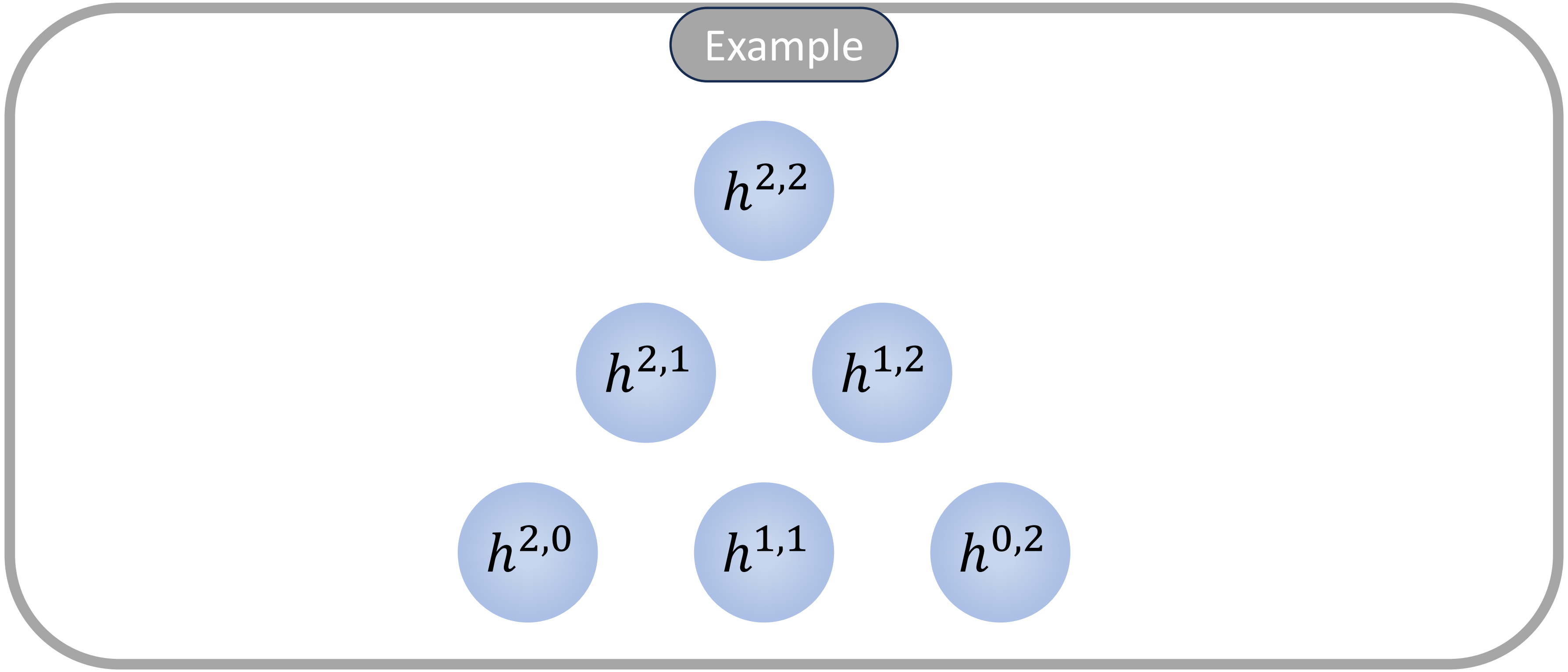

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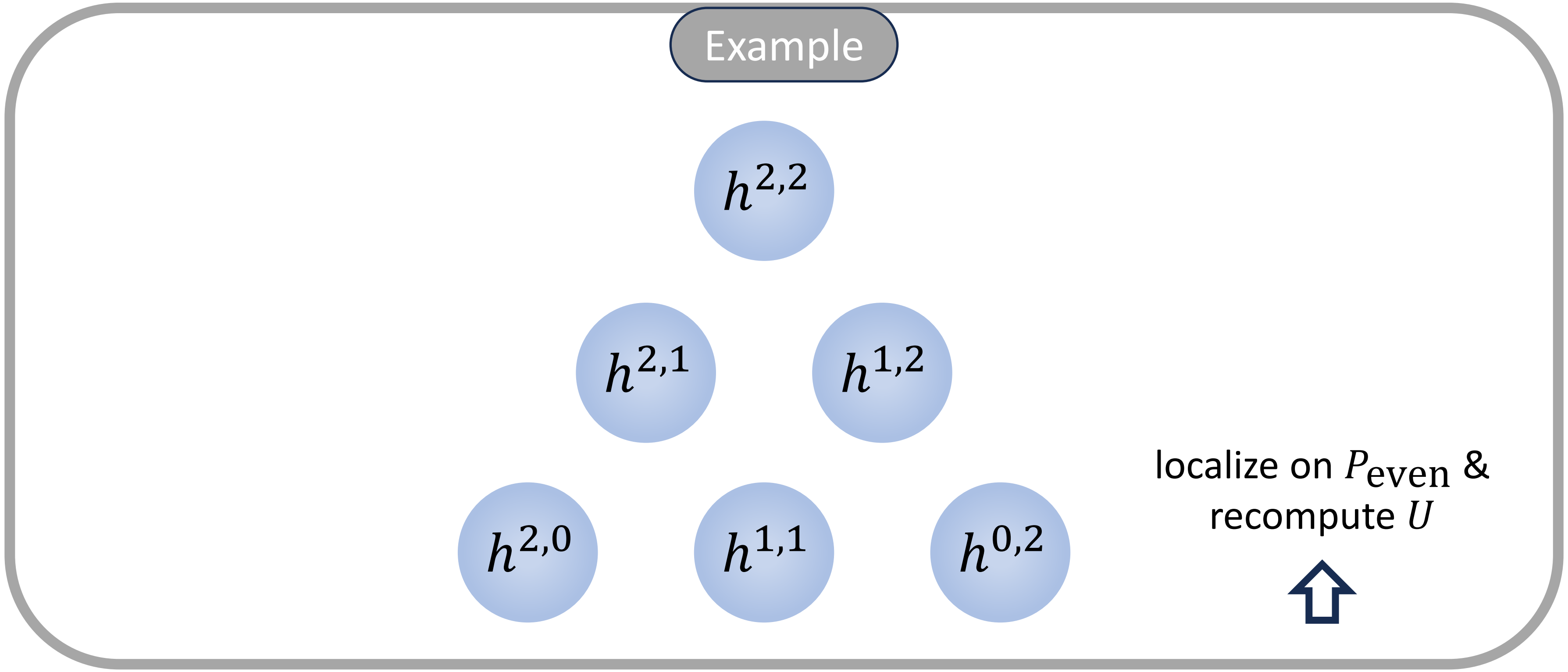

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Step 1

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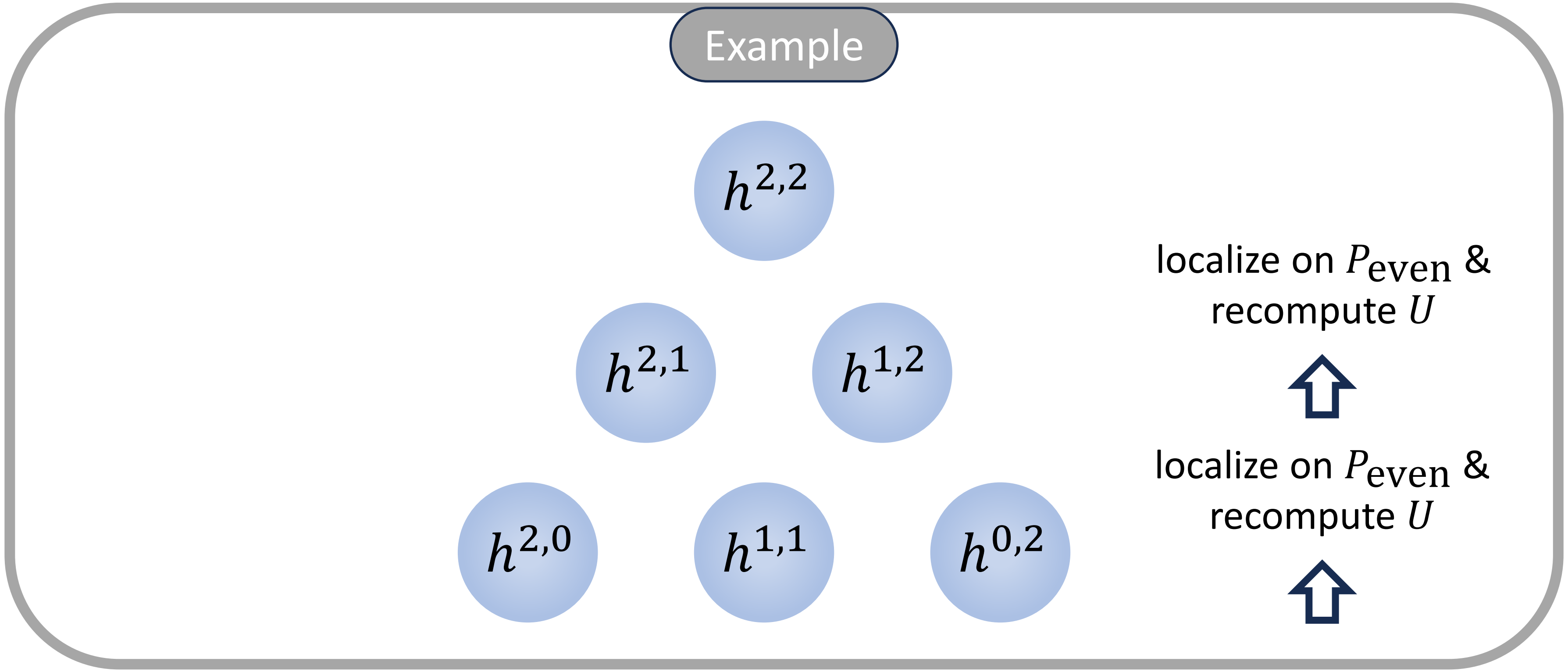
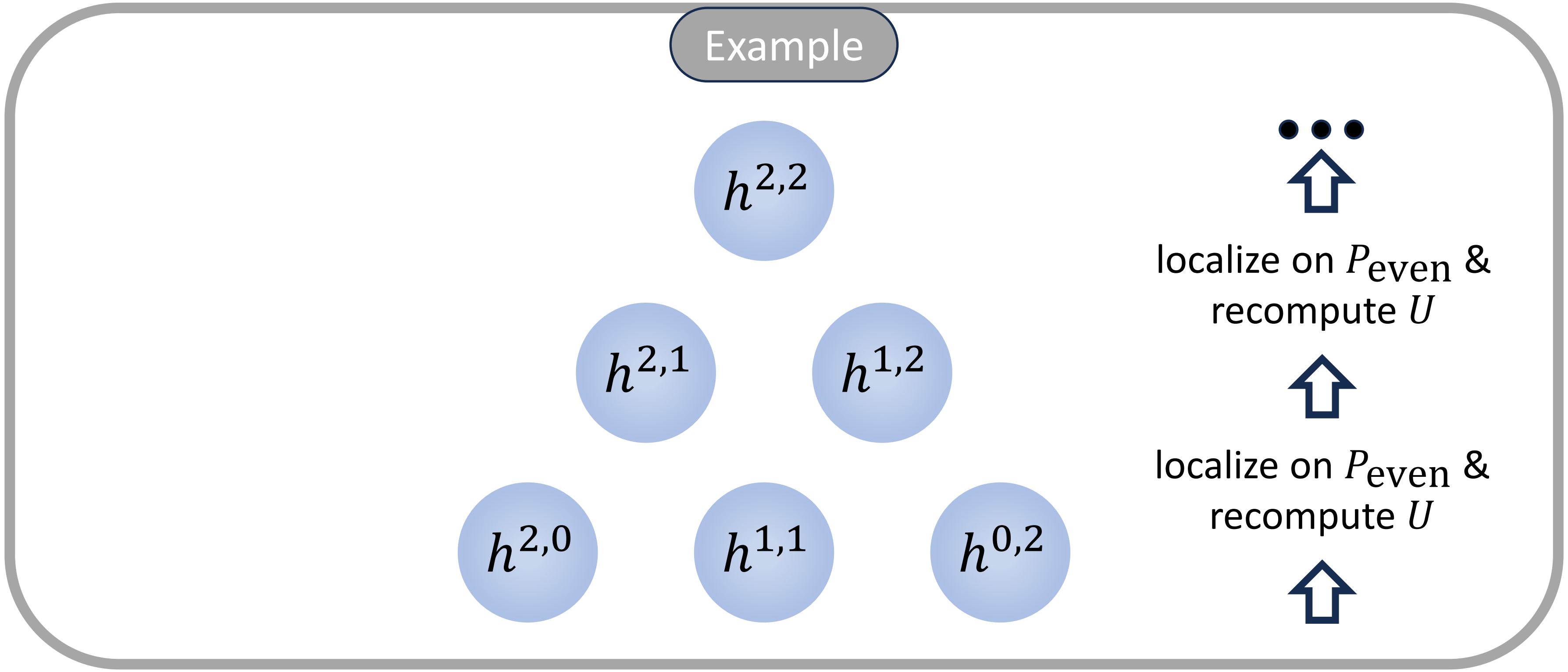


Localization Level

Step 1

Find local basis of the **lowest** subsystem and proceed recursively to the **global** system



Localization Level

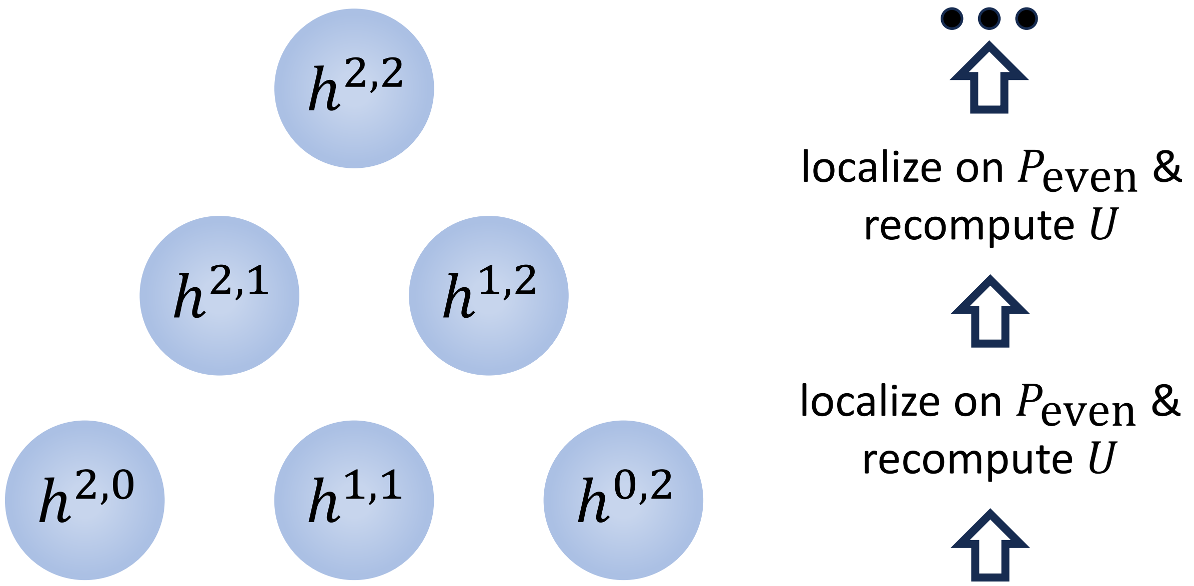
Step 1

Find local basis of the **lowest** subsystem and proceed recursively to the **global** system

$\{\Psi^{(2)}\}$

Example

assign $a = -2$
to the local
basis



Localization Level

Find local basis of the **lowest** subsystem and proceed recursively to the **global** system

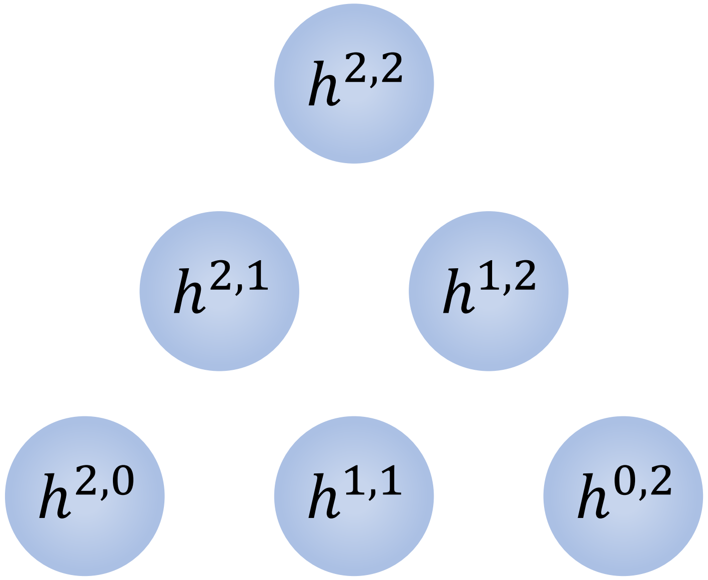
$\{\Psi^{(2)}\}$

$\{\Psi^{(1)}\}$

Example

assign a $= -2$ to the local basis

assign a $= -1$ to the new basis elements



•••

localize on P_{even} & recompute U

localize on P_{even} & recompute U

Localization Level

Step 1

Find local basis of the **lowest** subsystem and proceed recursively to the **global** system

$\{\Psi^{(2)}\}$

$\{\Psi^{(1)}\}$

$\{\Psi\}$

Example

assign a $= -2$ to the local basis

assign a $= -1$ to the new basis elements

run the full system

$h^{2,2}$

$h^{2,1}$ $h^{1,2}$

$h^{2,0}$ $h^{1,1}$ $h^{0,2}$

…

localize on P_{even} & recompute U

localize on P_{even} & recompute U

Localization Level

Find local basis of the **lowest** subsystem and proceed recursively to the **global** system

$\{\Psi^{(2)}\}$

$\{\Psi^{(1)}\}$

$\{\Psi\}$

$\vec{J}$-basis

Example

assign a $= -2$ to the local basis

$h^{2,2}$

assign a $= -1$ to the new basis elements

$h^{2,1}$ $h^{1,2}$

run the full system

$h^{2,0}$ $h^{1,1}$ $h^{0,2}$

• • •

⇧

localize on P_{even} & recompute U

⇧

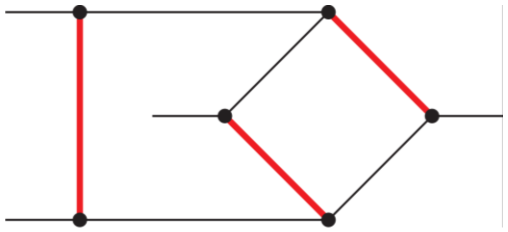
localize on P_{even} & recompute U

⇧

Translation to FI

$\{\Psi\} \longrightarrow \{\,\mathrm{J}\,\}$

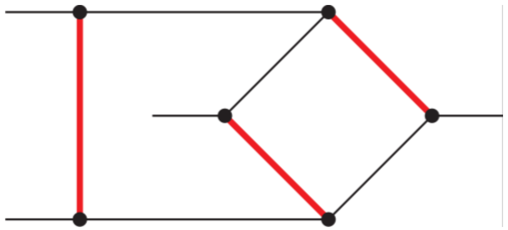
Top Sector: Non-Planar



Twist U is derived from a **one-dimensional** LBL Baikov representation:

$$U(z_0, z_1) = [P_0(z_0)]^{4\varepsilon}[P_1(z_0, z_1)]^{-\frac{1}{2}}[P_1(z_0, z_1)]^{-\frac{1}{2}}[P_3(z_0, z_1)]^{-\frac{1}{2}-\varepsilon}[P_4(z_0, z_1)]^{-\frac{1}{2}-\varepsilon}$$

Top Sector: Non-Planar



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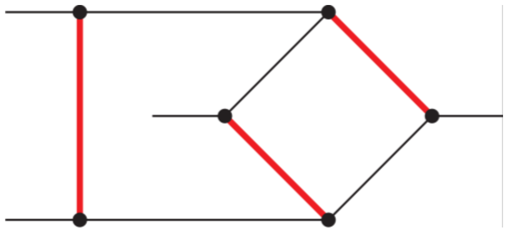

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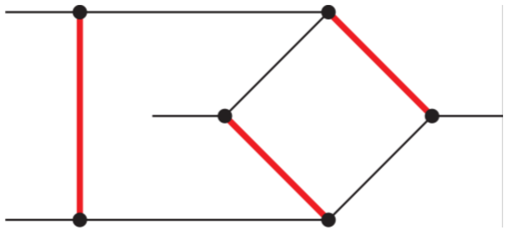

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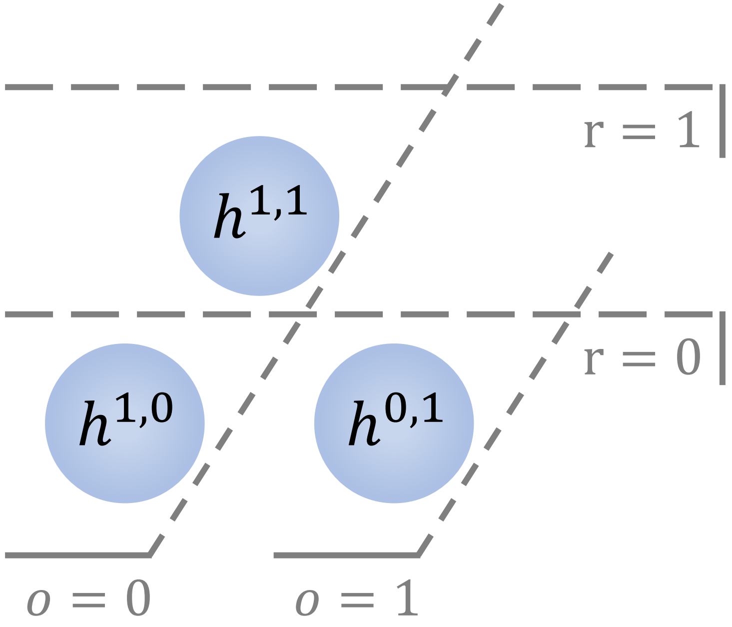

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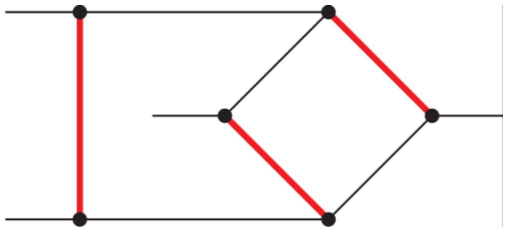

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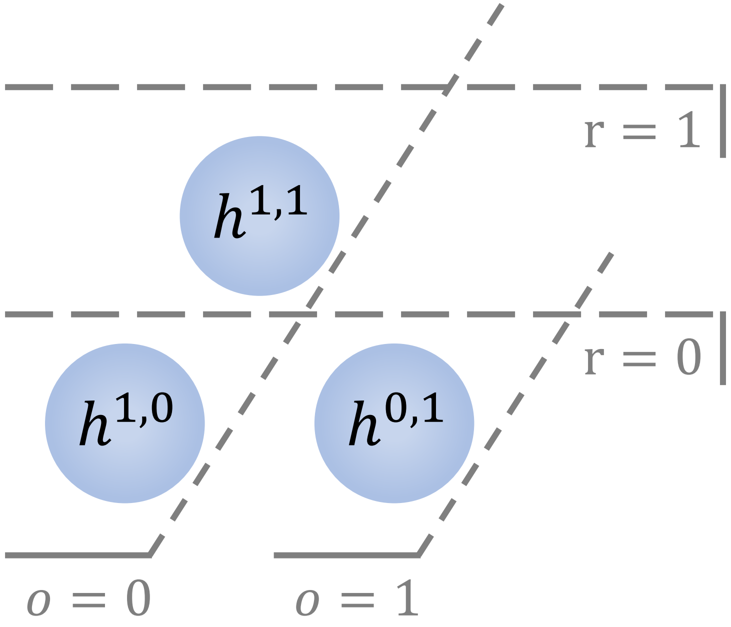
1. Localize on $P_0 = 0$

(sole master of the subsystem)

$$\Psi_3 = \varepsilon^4 U(\vec{z})\frac{z_1^2}{P_0}\eta$$



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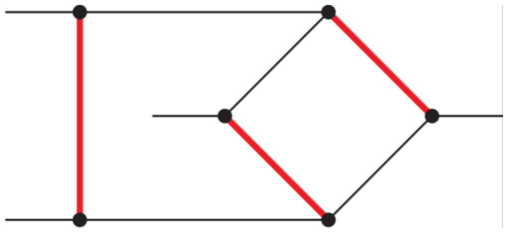

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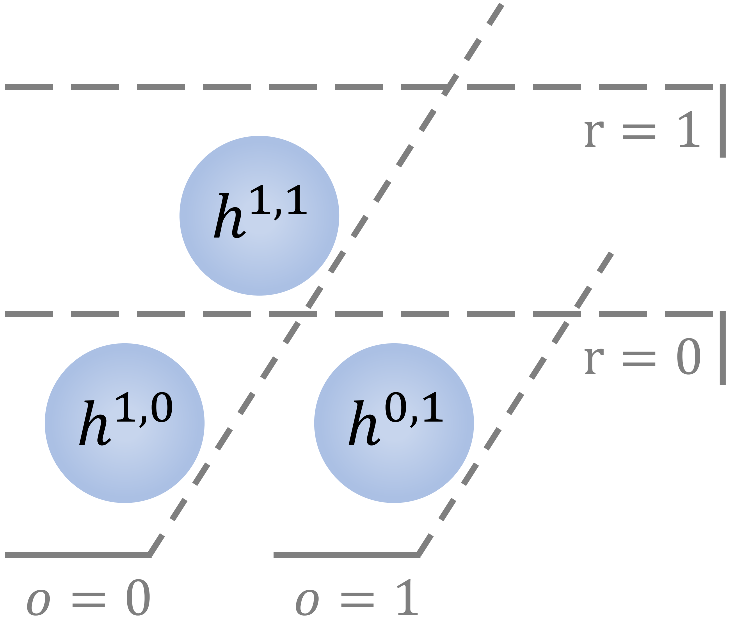
$$\Psi_3 = \varepsilon^4 U(\vec{z})\frac{z_1^2}{P_0}\eta \quad \Big\} \quad \begin{matrix} a=-1, r=1, \\ o=1, |\mu|=1 \end{matrix}$$



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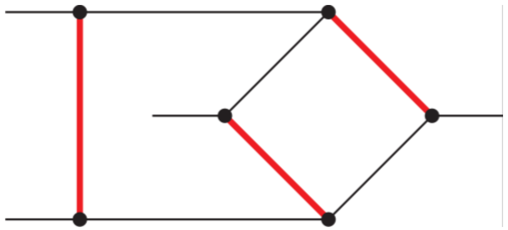

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2. Reduce the full system, preferring Ψ_3 as a master

$$\Psi_3 = \varepsilon^4 U(\vec{z})\frac{z_1^2}{P_0}\eta$$

$a = -1, r = 1,$
$o = 1, |\mu| = 1$

$$\Psi_1 = \varepsilon^4 U(\vec{z}) z_0 \eta$$

$$\Psi_2 = \varepsilon^4 U(\vec{z}) z_1 \eta$$

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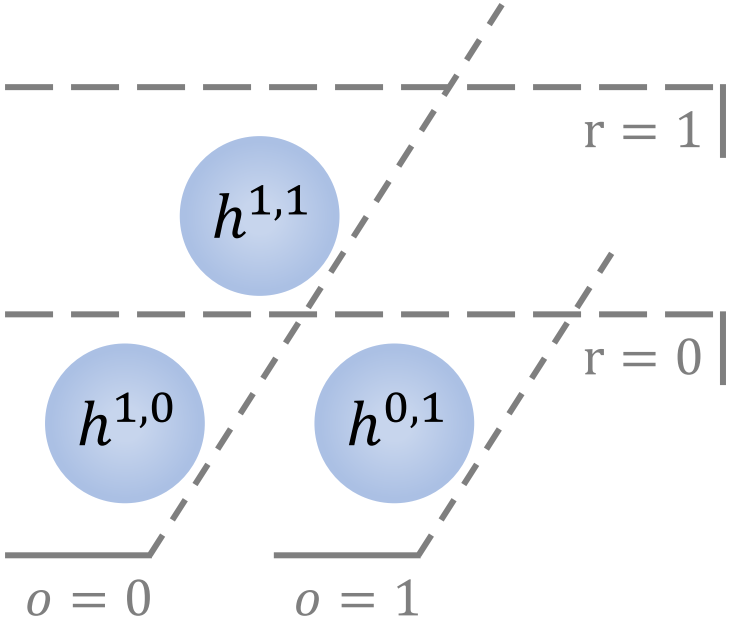
$$\Psi_5 = \varepsilon^4 U(\vec{z})\frac{z_1 Q}{P_3}\eta$$



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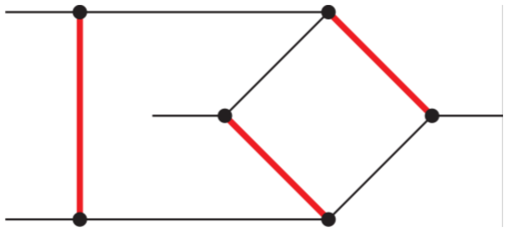

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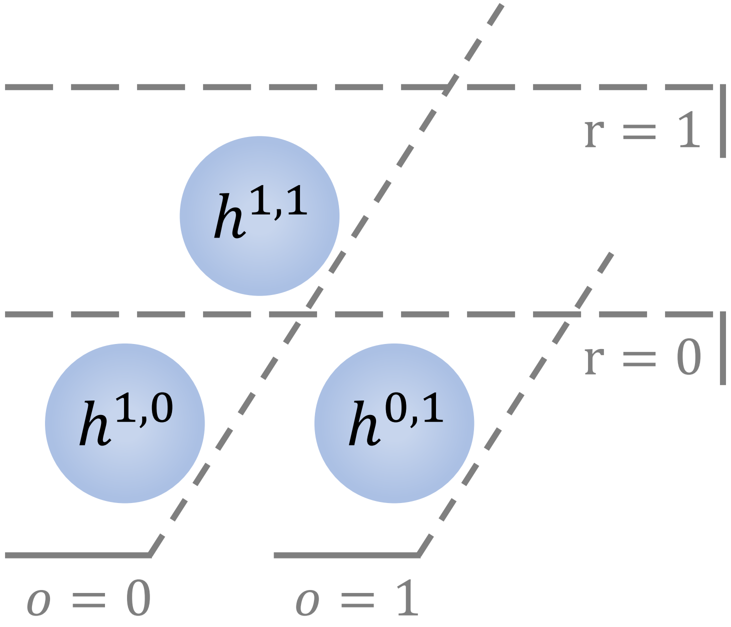

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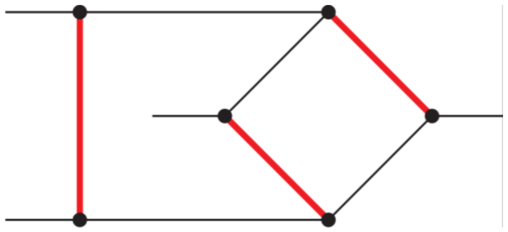

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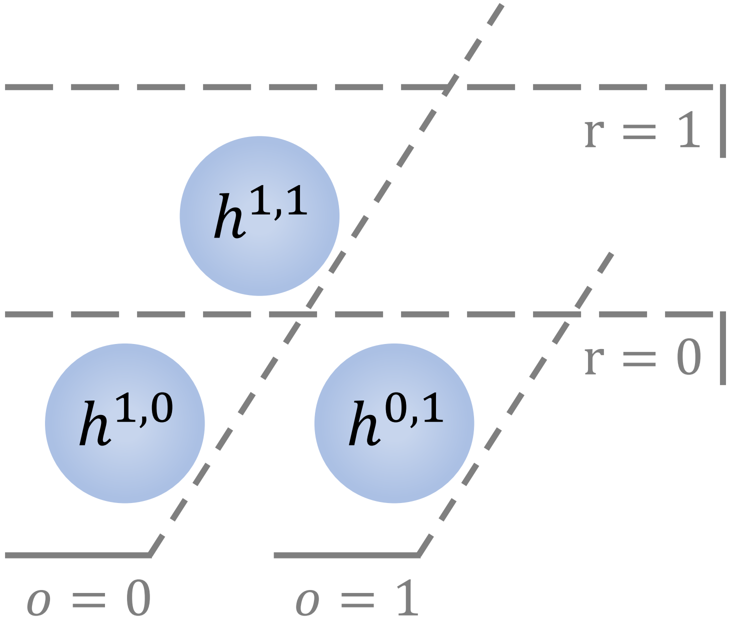

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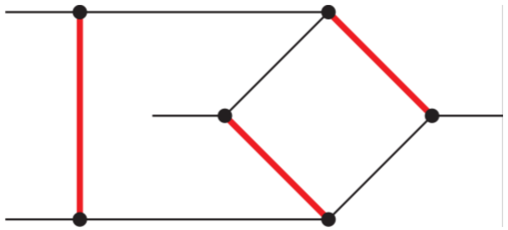

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Translate back to FI:

5 Forms $\Rightarrow$ **5 Integrals**

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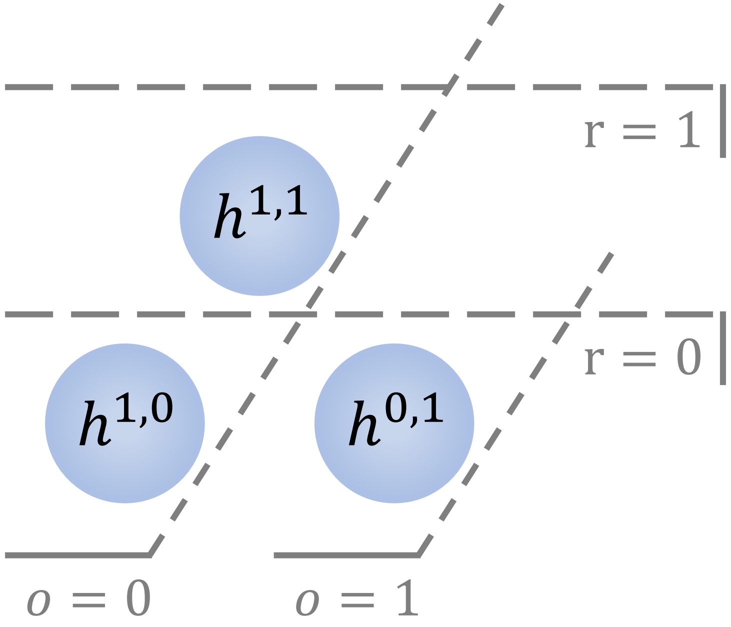

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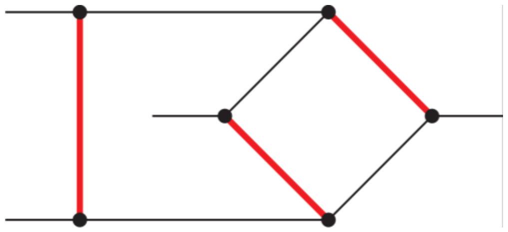
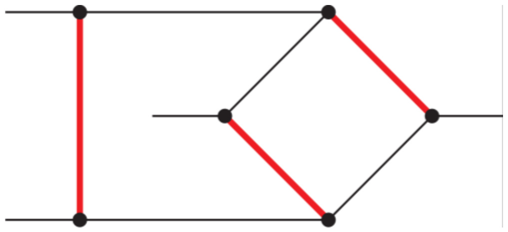


Corresponding DEQ matrix is given by Laurent series in ε:

Top Sector: Non-Planar



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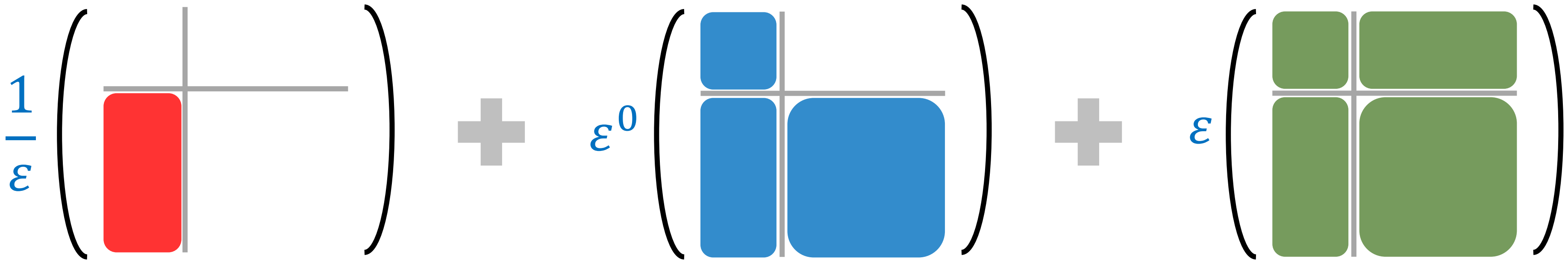

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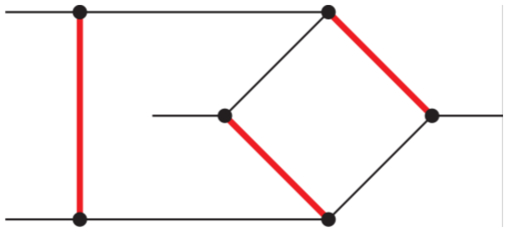

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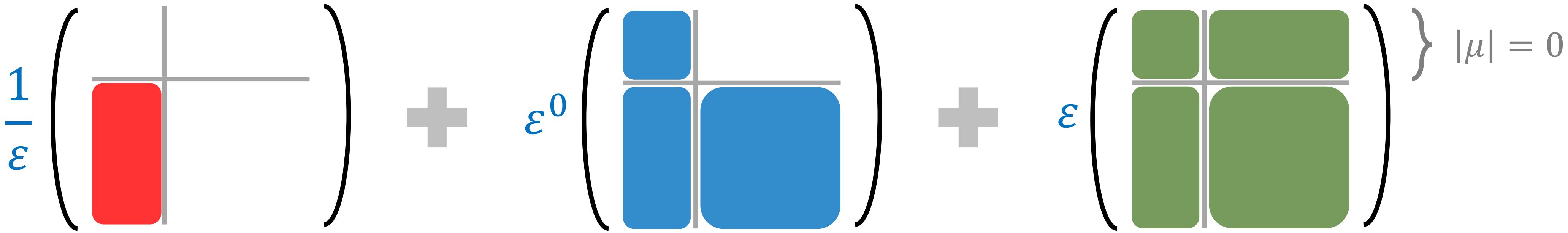

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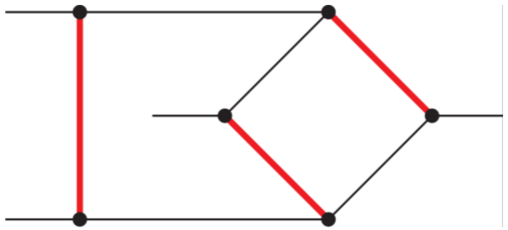

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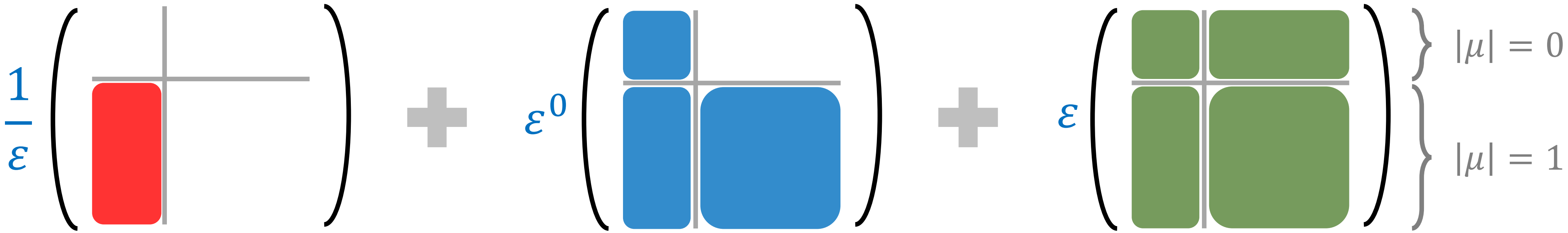

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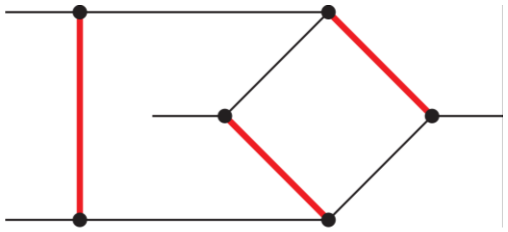

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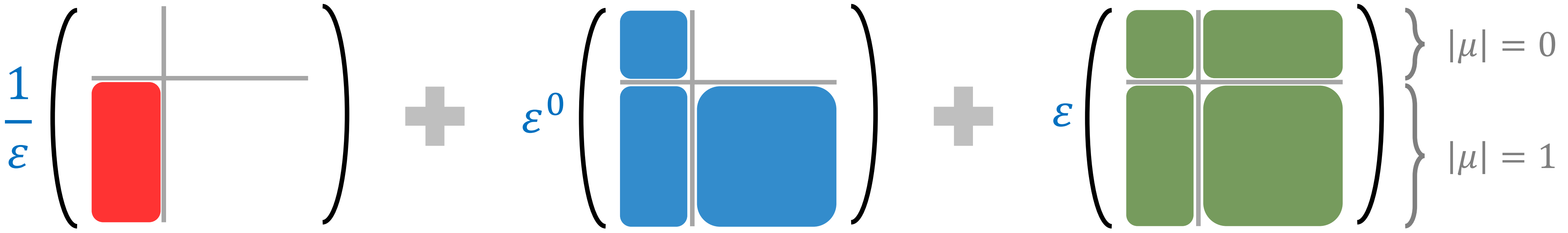
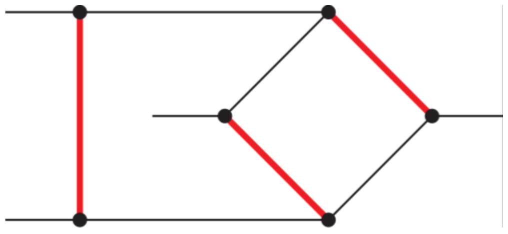


Rotation into ε-factorized basis can be constructed as:

$$\vec{K} = R^{-1}(\varepsilon, x)\vec{J} = \left[\, R^{(-1)}(\varepsilon, x) \;\; R^{(0)}(\varepsilon, x) \,\right]^{-1} \vec{J}$$

Top Sector: Non-Planar



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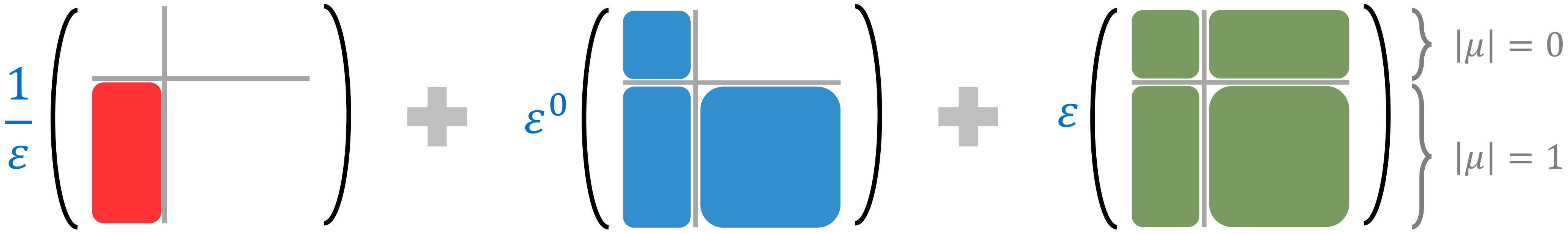

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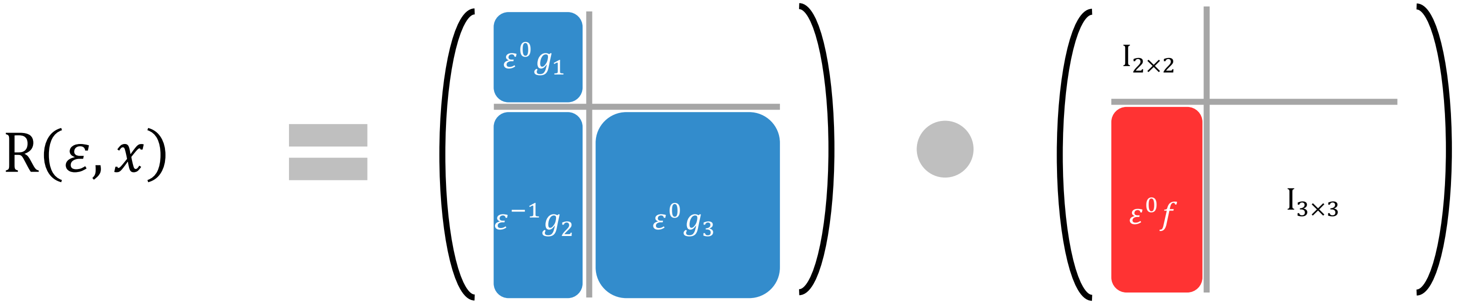

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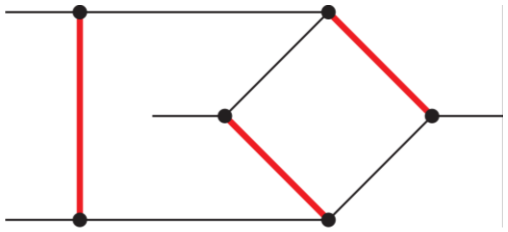

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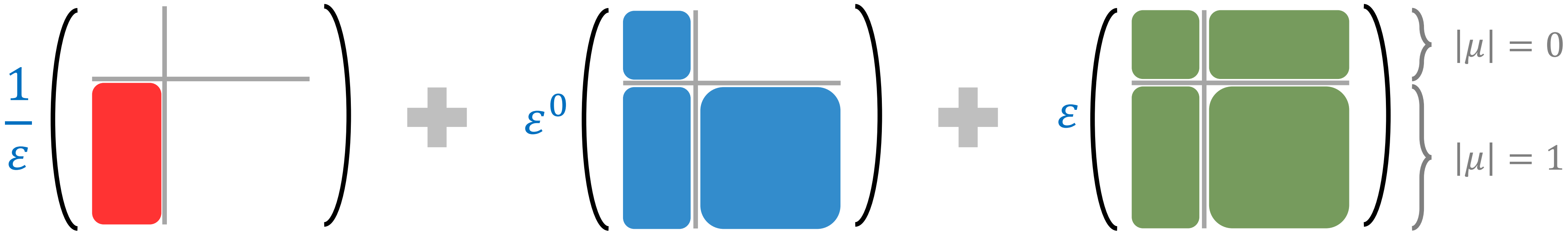

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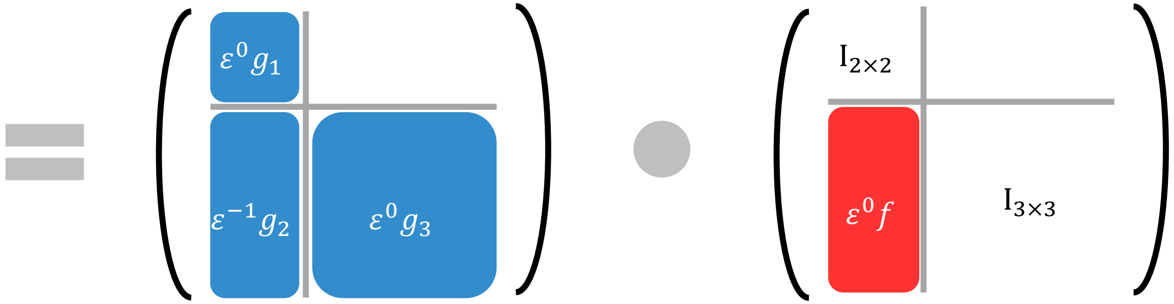
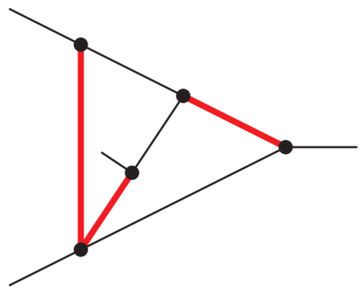
Functions g and f depend on kinematics only

$\mathrm{R}(\varepsilon, x) =$



K3 Sector: Non-Planar



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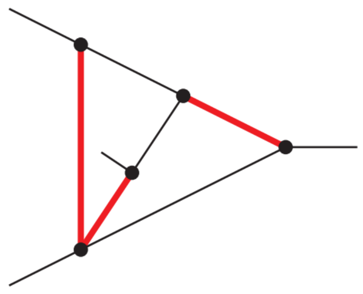
K3 Sector: Non-Planar



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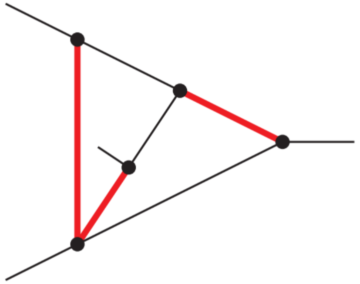
K3 Sector: Non-Planar



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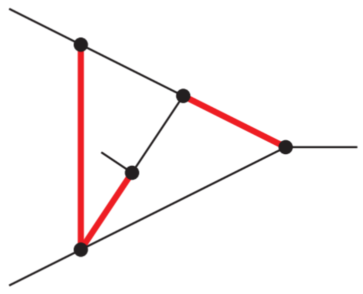
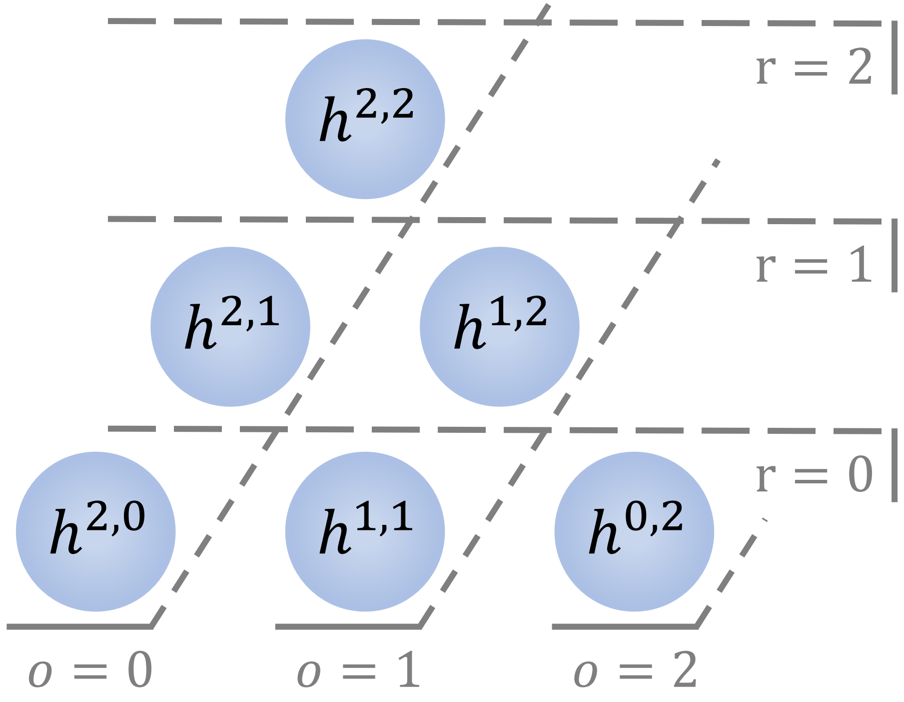
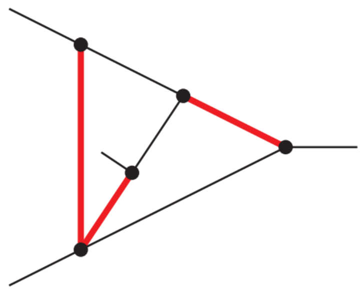




K3 Sector: Non-Planar



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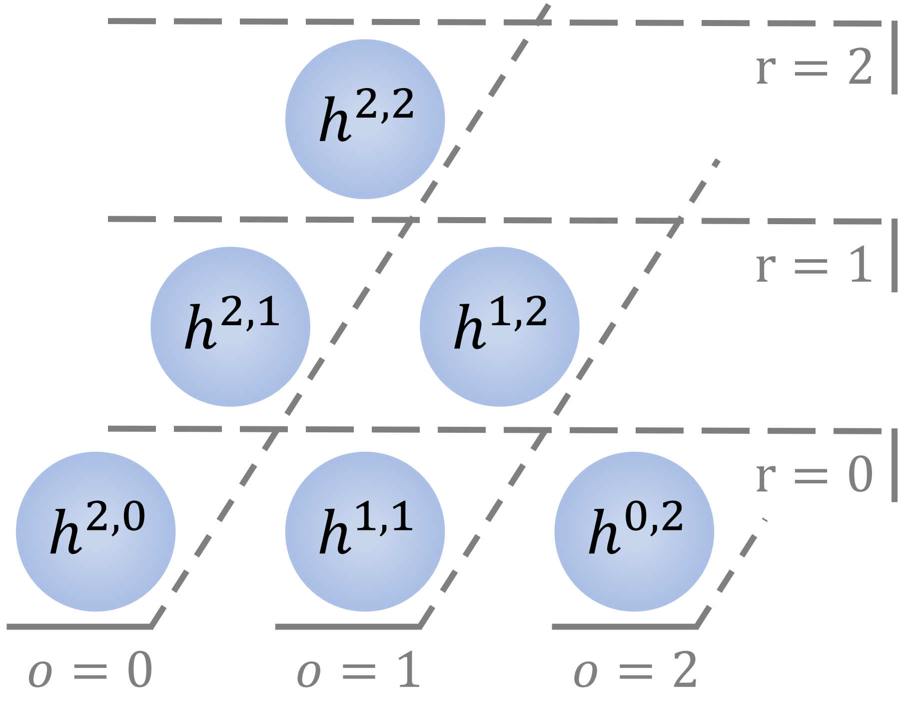
(P_{odd} become even)



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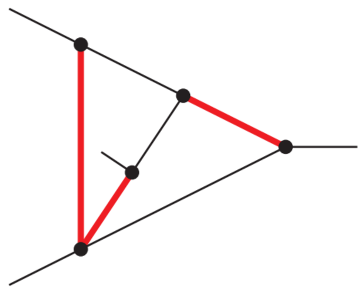

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2. Take one more residue

(we find six forms)

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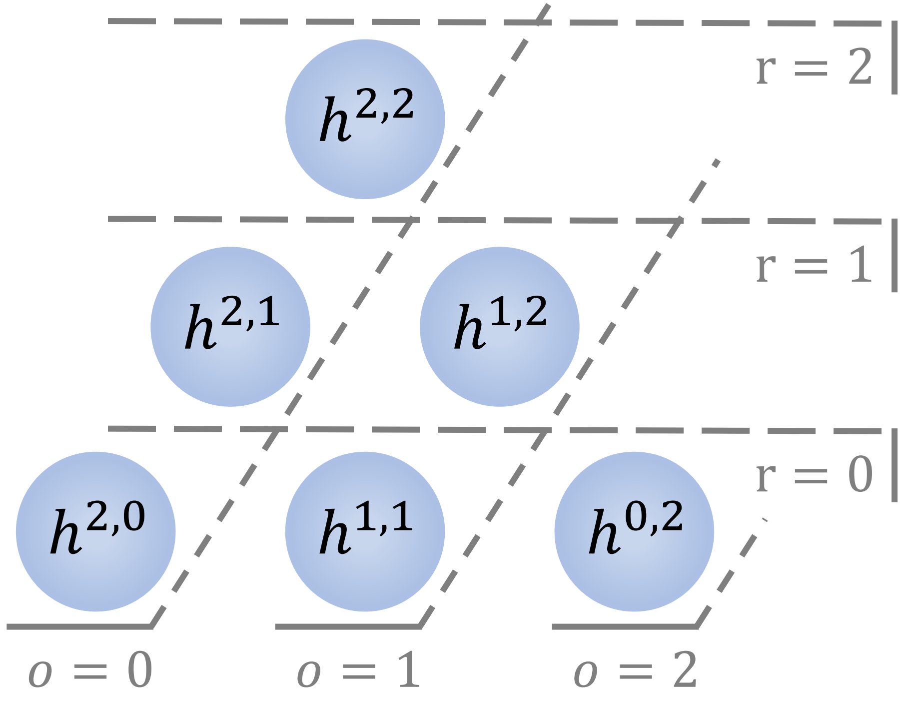

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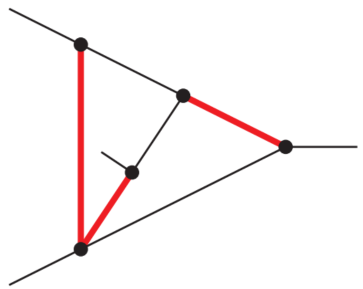

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$a = -2, r = 2,$
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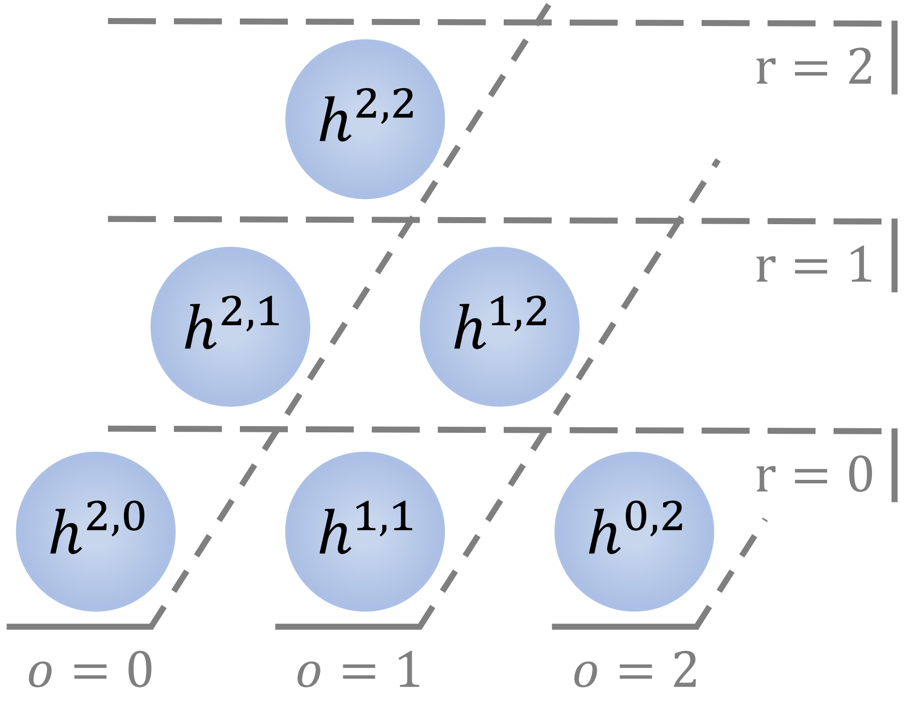

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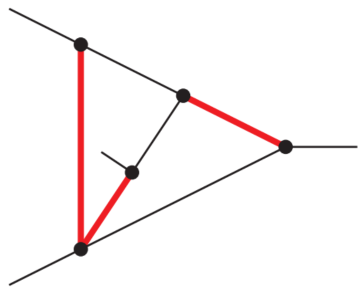

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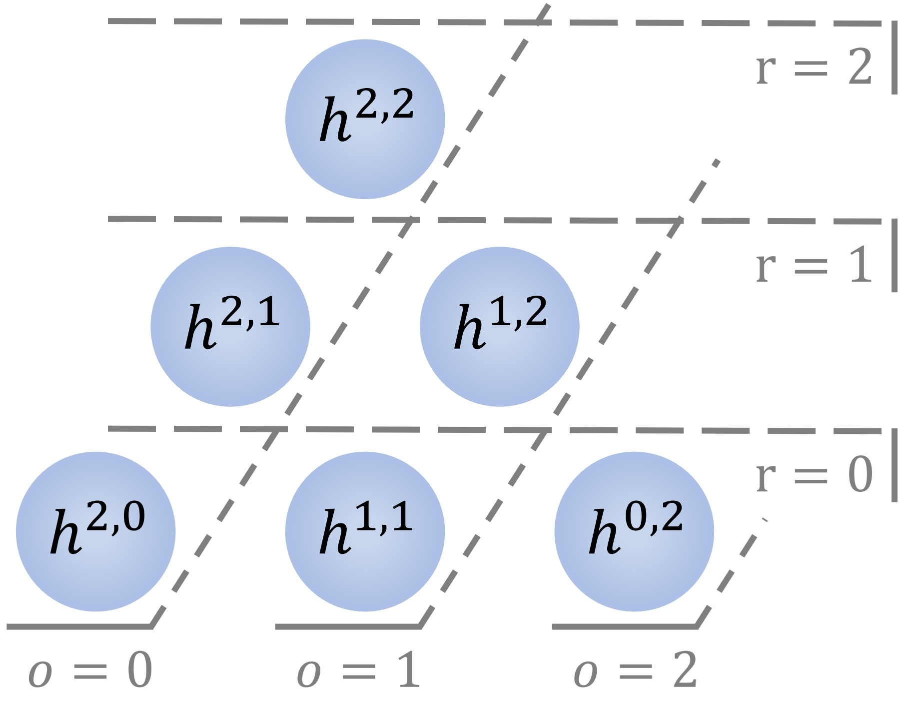

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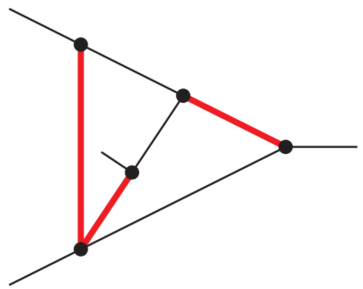

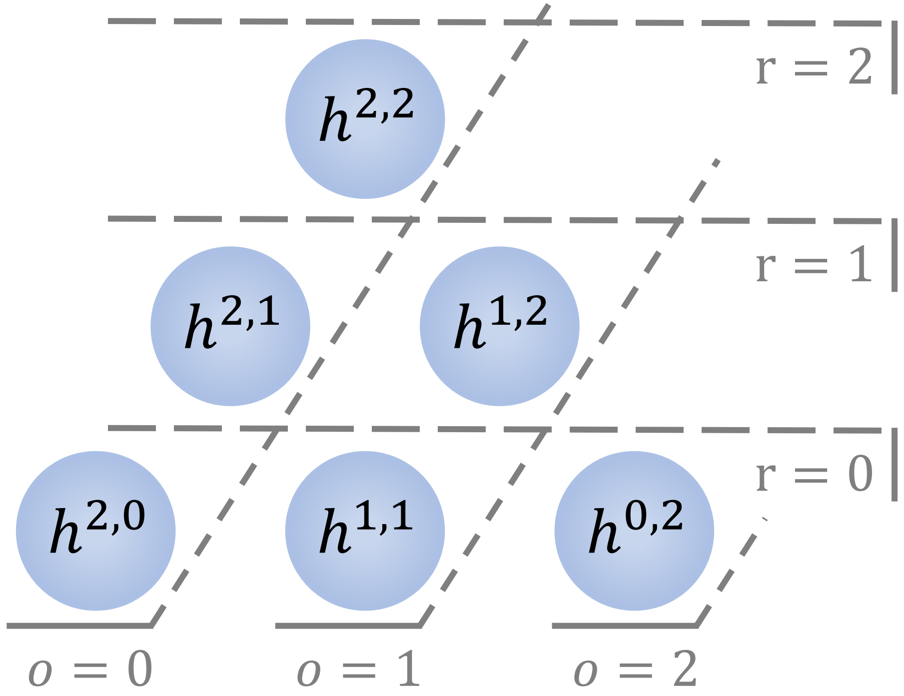

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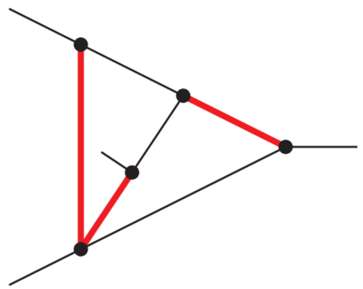

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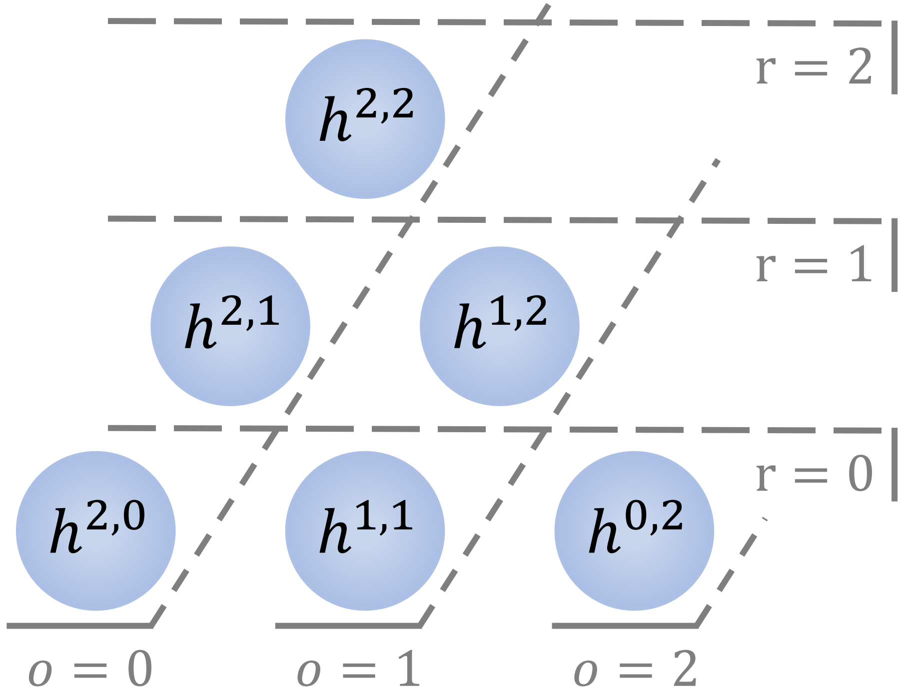

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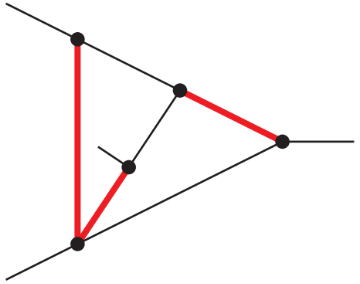

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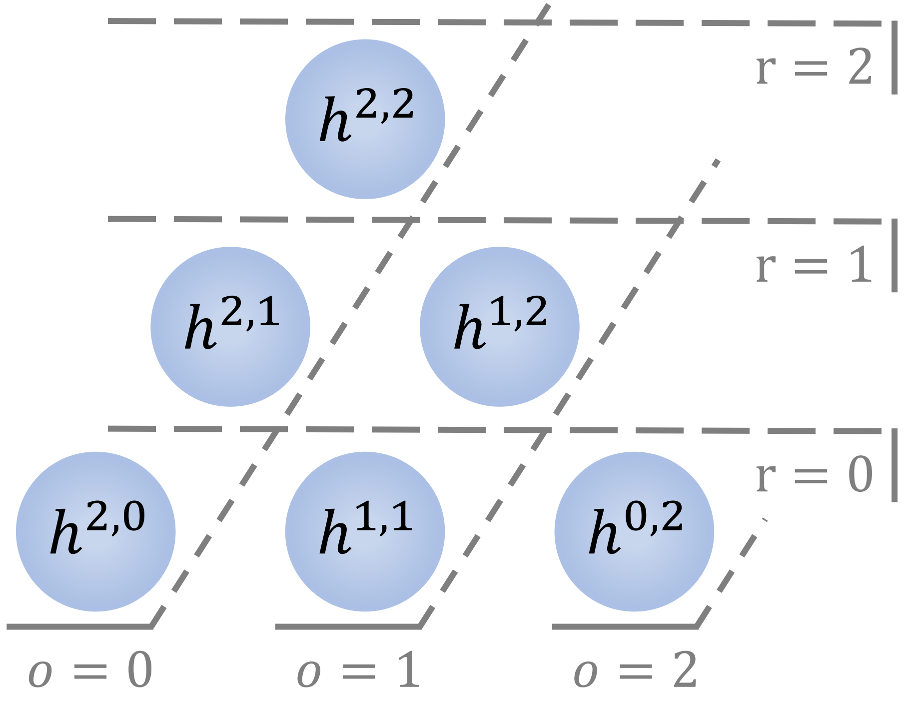

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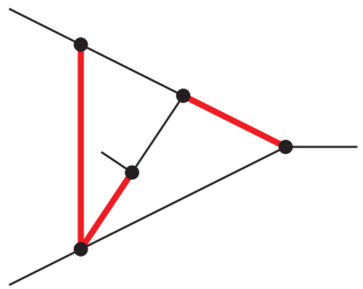

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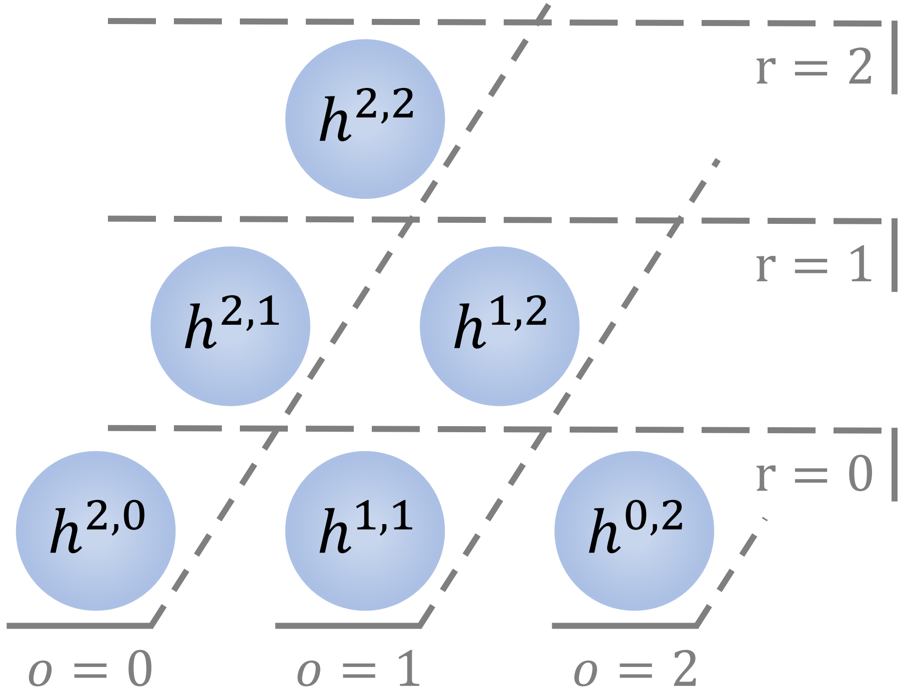

12 Forms ⇨ **8 Integrals**

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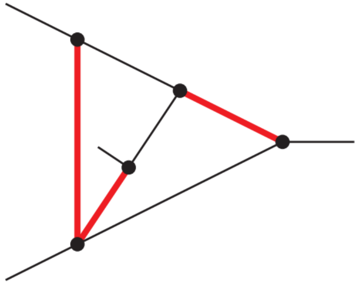
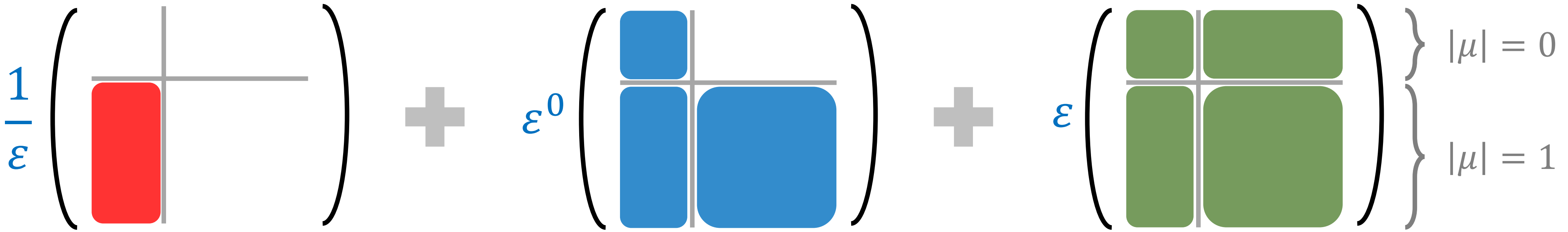


Corresponding DEQ matrix is given by Laurent series in ε:



Rotation into ε-factorized basis can be constructed as:

$$\vec{K} = R^{-1}(\varepsilon, x)\vec{J} = \left[\boxed{R^{(-1)}(\varepsilon, x)}\ \boxed{R^{(0)}(\varepsilon, x)}\right]^{-1} \vec{J}$$

$$\mathrm{R}(\varepsilon, x) = \begin{pmatrix} \varepsilon^0 g_1 & \\ \varepsilon^{-1} g_2 & \varepsilon^0 g_3 \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ \varepsilon^0 f & \mathrm{I}_{7\times 7} \end{pmatrix}$$

Summary

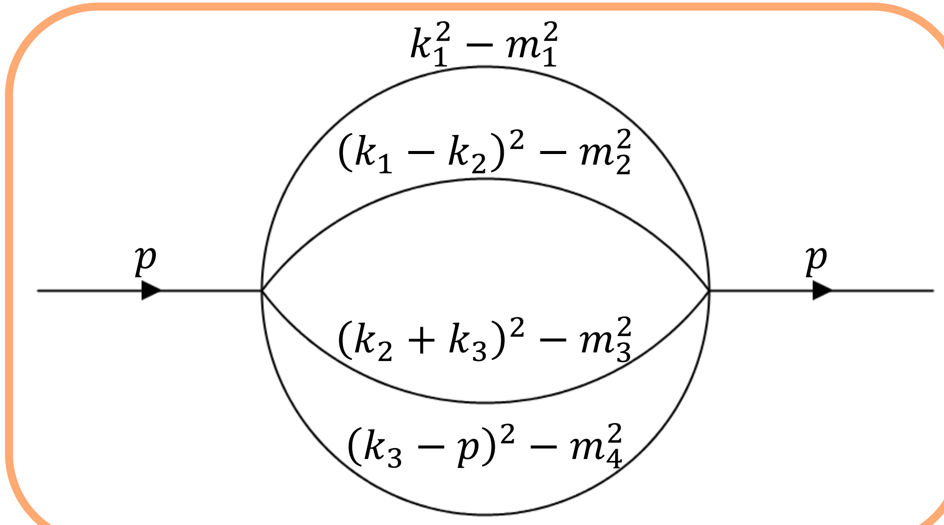
- Studied planar and non-planar double boxes with **three equal masses**
- Found multiple sectors associated with **non-trivial geometries**
- **Tested the algorithm** proposed in [arXiv:2506.09124] to solve the problem on the maximal cut
- Constructed solutions required **no prior knowledge of geometries**

Thank you for attention!

Back-Up Slides

Feynman Integrals

Feynman Diagram:



Integral Family:

$D = D_{\text{int}} - 2\varepsilon$

$\sigma_j = q_j^2 - m_j^2$

$$I_{n_1,\dots,n_N} \equiv e^{l\epsilon\gamma_E}\left(\mu^2\right)^{n-\frac{lD}{2}} \int \frac{d^D k_1}{(i\,\pi)^{D/2}} \cdots \frac{d^D k_l}{(i\,\pi)^{D/2}} \frac{1}{\sigma_1^{n_1}\cdot \ldots \cdot \sigma_N^{n_N}}$$

loop momenta

propagators

Integration-by-Parts Identities

$$I_{n_1,\dots,n_N} \propto \int \prod_{j=1}^{j=l} \frac{d^D k_j}{(i\,\pi)^{D/2}} \prod_{r=1}^{r=N} \frac{1}{\sigma_r^{n_N}}$$

∂_{k_q}

$$\int \prod_{j=1}^{j=l} \frac{d^D k_j}{(i\,\pi)^{D/2}} \frac{\partial}{\partial k_{q,\mu}} \left\{ v_\mu \prod_{r=1}^{r=N} \frac{1}{\sigma_r^{n_N}} \right\} = 0$$

loop momentum   any momentum

Laporta Algorithm

Solve ***complicated*** integrals through ***simple*** integrals

$J_1 - J_2 + 8 n^2 J_3 - 2 n J_4 - 2 n J_5 - 4 n J_6 == 0$
$-8 n J_3 + 2 J_4 - 2 J_5 + J_6 + J_7 == 0$
$J_2 - 8 n J_3 - 2 J_4 + 2 J_5 + \left(1 + \frac{1}{eps}\right) J_8 == 0$
$-2 n J_3 + J_8 - J_9 + 2 n^2 J_{10} - 2 n J_{11} - 4 n J_{12} == 0$
$-2 J_3 - 2 n J_{10} + 2 J_{11} + J_{12} + J_{13} == 0$
$2 J_3 + J_9 - 2 n J_{10} - 2 J_{11} + J_{14} == 0$
$J_2 - J_{15} + 2 n^2 J_{16} - 2 n J_{17} - 2 n J_{18} - 4 n J_{19} == 0$
$-2 n J_{16} + 2 J_{17} - 2 J_{18} + J_{19} + J_{20} == 0$

Gauss Elimination

Ordering?

Minimal set of ***unknown J***

=

Master Integrals

ε-Factorized Form

$$R^{-1}\tilde{A}(\varepsilon, x)R - R^{-1}\frac{d}{dx}R$$

"Rotate" the basis $\vec{K} = R^{-1}(\varepsilon, x)\vec{I}$:

$$\frac{d}{dx}\vec{K} = \varepsilon A(x)\vec{K}$$

Ansatz $\vec{K} = \sum_{n=0} \varepsilon^n \vec{K}^{(n)}(x)$:

LHS: $$\frac{d}{dx}\sum_{n=0} \varepsilon^n \vec{K}^{(n)}(x)$$

RHS: $$\varepsilon A(x)\sum_{n=0} \varepsilon^n \vec{K}^{(n)}(x)$$

Iterative solution:

$$O(\varepsilon^0): \frac{d}{dx}\vec{K}^{(0)} = 0$$

$$O(\varepsilon^1): \frac{d}{dx}\vec{K}^{(1)} = A(x)\vec{K}^{(0)}$$

$$\vdots$$

$$O(\varepsilon^n): \frac{d}{dx}\vec{K}^{(n)} = A(x)\vec{K}^{(n-1)}$$

Baikov Representation

$$I_{n_1,\dots,n_N} \propto \int \prod_{j=1}^{j=l} \frac{d^D k_j}{(i\,\pi)^{D/2}} \prod_{r=1}^{r=N} \frac{1}{\sigma_r^{n_N}}$$

Momentum Rep

Integrate in loop momenta k_j

$\Longleftrightarrow$

Baikov Rep

Integrate in propagators σ_j

$$I_{n_1,\dots,n_N} \propto \int_C \frac{B(\sigma)^{\beta} d^n \sigma}{\prod_j \sigma_j^{a_j}}$$

$\mathcal{B} = \mathsf{Gram}(k_1, \dots, k_L, p_1, \dots, p_E)$ $\qquad$ $\beta = (d{-}E{-}L{-}1)/2$

Scaleless Integrals

- To show the above expectation is too generous, consider an example of a *Scaleless Integral*:

$$I \equiv \int_{k_1,k_2} \frac{1}{[k_1 \cdot p_i]^m [k_2 \cdot p_i]^n \cdot \ldots} \xrightarrow[k_j \to \lambda k_j]{k_i \to \lambda k_i} \int_{k_1,k_2} \frac{1}{\left[k_1 \cdot \lambda p_j\right]^m \left[k_2 \cdot \lambda\left(p_i + p_j\right)\right]^n \cdot \ldots}$$

rescale back k_1, k_2

$$= I \qquad\qquad = \underbrace{\lambda^{-m-n+\cdots}}_{\equiv \lambda^{\alpha}} \cdot I$$

- $\alpha \neq 0$ in dimensional regularization, hence:

$$(\lambda^{\alpha} - 1) I = 0 \Rightarrow I = 0$$

scaleless integrals vanish in dimreg = **fully reducible sectors**