Quantization of the rational spin Ruijsenaars–Schneider model and the affine Yangian

DESY theory workshop

Lukas Hardi

September 25, 2025

Integrable many-body systems

Calogero–Moser–Sutherland models

The most studied class of integrable many-body systems are *Calogero–Moser–Sutherland models* with *N*-body Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \gamma^2 \sum_{i<j} v(q_i - q_j)$$

interacting via one of the following potentials:

$$v(z) = \frac{1}{z^2}, \quad v(z) = \frac{1}{\sinh^2(z)}, \quad v(z) = \frac{1}{\sin^2(z)}, \quad v(z) = \wp(z).$$

These are the *rational, hyperbolic, trigonometric,* and *elliptic* cases, which are exceptional because they give integrable models.

Integrable many-body systems

Ruijsenaars–Schneider models

Ruijsenaars–Schneider models are relativistic deformations of Calogero–Moser–Sutherland models that introduce a speed of light c with Hamiltonian

$$H = \sum_{i=1}^{N} \cosh(p_i/c) \prod_{j(\neq i)} \sqrt{\sigma^2(\gamma/c)(\wp(\gamma/c) - \wp(q_i - q_j))}$$

in the elliptic case. Taking certain limits yields the trigonometric, hyperbolic, and rational cases, as well as the periodic and non-periodic relativistic Toda chains, which are all integrable.

Integrable many-body systems with spin

Calogero–Moser–Sutherland and Ruijsenaars–Schneider models can be extended such that the ith point particle carries an ℓ-dimensional covector a_i and vector b_i that capture internal 'spin' degrees of freedom.

Calogero–Moser–Sutherland models have been extended to include spin degrees of freedom very early on (Gibbons–Hermsen). The Hamiltonian in the trigonometric case is given by

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \gamma^2 \sum_{i<j} \frac{|a_i a_j^\dagger|^2}{\sin^2(q_i - q_j)} \qquad (b_i = a_i^\dagger)$$

Ruijsenaars–Schneider models with spin were discovered by looking at doubly periodic solutions of Toda field theory (Krichever–Zabrodin).

Where do these models appear?

- Certain reductions of 2d/3d/4d gauge theories give Calogero–Moser–Sutherland and Ruijsenaars–Schneider models (Nekrasov)
- Calogero–Moser–Sutherland and Ruijsenaars–Schneider models give a low-energy effective description of 4d $\mathcal{N} = 2$ and 5d $\mathcal{N} = 1$ supersymmetric gauge theories on (Seiberg–Witten, Nekrasov, Nakajima–Yoshioka, ...)
- Spin Calogero–Moser–Sutherland models appear as models of the non-abelian quantum Hall effect and vortices in Chern–Simons theory (Bourgine–Matsuo, Hu–Li–Ye–Zhou).
- The phase space of the rational spin Calogero–Moser–Sutherland model deformds the moduli space of Yang–Mills instantons on $\mathbb{C}^2$ (a Nakajima quiver variety).
- The phase space of the rational spin Ruijsenaars–Schneider model deforms the moduli space of certain Yang–Mills instantons on $\mathbb{C} \times \mathbb{C}^\times$ (a bow variety).
- Solitons in affine Toda field theory are described by the hyperbolic spin Ruijsenaars–Schneider model (Braden–Hone).
- ...

What is missing?

The equations of motion of the spin Ruijsenaars–Schneider models were known, but their Poisson bracket and quantization remained elusive.

In the rational and trigonometric cases, the Poisson brackets are now known (Arutyunov–Frolov, Chalykh–Fairon, Arutyunov–Olivucci), but the elliptic case is missing.

The quantization is completely unknown, except for a conjectured quantization (Lamers–Pasquier–Serban, Klabbers–Lamers)

This project: Quantize the rational spin Ruijsenaars–Schneider model.

Main method

The Poisson bracket of the rational case comes about as the Hamiltonian reduction $\mu^{-1}(\gamma)/\mathrm{GL}_N$ from a natural Poisson bracket on $T^*\mathrm{GL}_N \times T^*\mathbb{C}^{N\times\ell} \cong \mathrm{GL}_N \times \mathbb{C}^{N\times N} \times \mathbb{C}^{N\times\ell} \times \mathbb{C}^{\ell\times N}$ via the moment map

$$\mu\colon T^*\mathrm{GL}_N \times T^*\mathbb{C}^{N\times\ell} \to \mathfrak{gl}_N^*, \quad (g, x, a, b) \mapsto x - gxg^{-1} + ab.$$

Letting $\mathcal{O}_\hbar(T^*M)$ denote the quantization of the Poisson bracket on T^*M, we can write down the quantum moment map as

$$\mu\colon U(\mathfrak{gl}_N) \to \mathcal{O}_\hbar(T^*\mathrm{GL}_N) \otimes \mathcal{O}_\hbar(T^*\mathbb{C}^{N\times\ell}), \quad e_{ij} \mapsto (x - gxg^{-1} + ab)_{ij}.$$

A quantum moment map gives rise to an action

$$\mathrm{ad}_x\, a := [\mu(x), a]$$

and the quantum Hamiltonian reduction is the algebra

$$\mathfrak{A}_{N,\ell} := ((\mathcal{O}_\hbar(T^*\mathrm{GL}_N) \otimes \mathcal{O}_\hbar(T^*\mathbb{C}^{N\times\ell}))/(\mu - \gamma))^{\mathfrak{gl}_N}.$$

This is the algebra of observables of the rational spin Ruijsenaars–Schneider model.

Reduction

Define the following generating function of $\mathfrak{gl}_N$-invariant elements

$$\mathbf{S}^{\alpha\beta}[n](z) := b^\alpha g^n (z-x)^{-1} a^\beta \in (\mathcal{O}_\hbar(T^*\mathrm{GL}_N) \otimes \mathcal{O}_\hbar(T^*\mathbb{C}^{N\times\ell}))^{\mathfrak{gl}_N}[[z^{-1}]].$$

The zero mode $\mathbf{T}(z) := \mathbf{S}[0](z)$ generates a quotient of the Yangian $Y(\mathfrak{gl}_\ell)$ and $\mathbf{J}[n] := \mathrm{Res}_{z=\infty} \mathbf{S}[n](z)$ generates a quotient of the loop algebra $L(\mathfrak{gl}_\ell)$. *We conjecture that $\mathfrak{A}_{N,\ell}$ is a quotient of a shifted affine Yangian.*

By the moment map equation $\mu = \gamma$, the positive modes $(n > 0)$ can be expressed in terms of

$$\mathbf{S}[1](z) = \sum_{i=1}^{N} \frac{1}{z - q_i + \hbar} \mathbf{S}_i.$$

The residues satisfy the commutation relations

$$R^{21}(q_j - q_i)\mathbf{S}_i^1 R^{12}(q_i - q_j + \gamma\hbar)\mathbf{S}_j^2 = \mathbf{S}_j^2 R^{21}(q_j - q_i + \gamma\hbar)\mathbf{S}_i^1 R^{12}(q_i - q_j), \qquad R^{12}(z) = 1 + \frac{\hbar}{z} P^{12},$$

which should be regarded as the central commutation relation of $\mathfrak{A}_{N,\ell}$.

Spectrum I

The model is integrable due to the existence of the commutative subalgebra of higher Hamiltonians given by the center of the loop algebra, which implies that the loop algebra is a symmetry of the higher Hamiltonians (⟶ superintegrability).

The lowest Hamiltonian is $H := \operatorname{Tr}\mathbf{J}[1]$, which satisfies

$$[H, \mathbf{T}(z)] = \hbar^2(\gamma + \ell)\mathbf{T}(z)\Delta\mathbf{S}(z), \qquad \Delta f(z) := \tfrac{f(z+\gamma\hbar)-f(z)}{\gamma\hbar}.$$

In particular, there is a critical level $\gamma = -\ell$, where the Yangian also becomes a symmetry of the Hamiltonian.

At non-critical level, $\bar{\mathbf{A}}^\alpha := \mathbf{T}^{\alpha\ell}(z)$ $(\alpha < \ell)$ are raising operators for a $\mathfrak{gl}_\ell$-lowest weight vector $|0\rangle$ with weight $(0, \dots, 0, N)$ satisfying

$$\mathbf{S}^{\alpha\beta}(z)|0\rangle = \delta^{\alpha\beta}\sum_{i=1}^{N}\frac{1}{z-q_i+\hbar}\prod_{j(\neq i)}\frac{q_i-q_j-\hbar}{q_i-q_j}|0\rangle, \qquad \gamma|0\rangle = |0\rangle.$$

In particular $H|0\rangle = N\ell|0\rangle$.

Spectrum II

Introducing $\tilde{H} := \frac{2}{\ell+1}H - \frac{\ell-1}{\ell+1}N$ and letting $\psi(q_1,\dots,q_N)$ be any function of the particle positions, we have

$$\begin{aligned}&\tilde{H}\psi(q_1,\dots,q_N)\bar{\mathbf{A}}^{\mu_1}(w_1)\cdots\bar{\mathbf{A}}^{\mu_k}(w_k)|0\rangle\\ &= \sum_i \psi(q_1,\dots,q_i-\hbar,\dots,q_N)\prod_{a=1}^{k}(1-2\hbar V(w_a-q_i+\hbar))\prod_{j(\neq i)}\frac{q_i-q_j-\hbar}{q_i-q_j}\bar{\mathbf{A}}^{\mu_1}(w_1)\cdots\bar{\mathbf{A}}^{\mu_k}(w_k)|0\rangle.\end{aligned}$$

where $V(z) := \frac{1}{z} - \frac{1}{z+\hbar}$ is the rational Ruijsenaars–Schneider potential. This becomes an eigenvalue equation for the Hamiltonian when ψ fulfills the difference equation

$$\sum_i \prod_{a=1}^{k}(1-2\hbar V(w_a-q_i+\hbar))\prod_{j(\neq i)}\frac{q_i-q_j-\hbar}{q_i-q_j}e^{-\hbar\partial_i}\psi = \lambda\psi.$$

When $\ell = 1$, then $k = 0$ and this difference operator reproduces exactly the spinless Ruijsenaars–Schneider Hamiltonian.

Wrap-up

- We quantized the known Poisson structure of the rational spin Ruijsenaars–Schneider model via quantum Hamiltonian reduction.
- We found quotients of the Yangian and the loop algebra inside the algebra of observables, proving integrability.
- The algebra of observables has a Fock space representation that generalizes the known rational spinless Ruijsenaars–Schneider model.
- The paper is now online at hep-th/2508.07862.

Thank you!