# A new framework for massive neutrino perturbations using moment hierarchies

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# About me

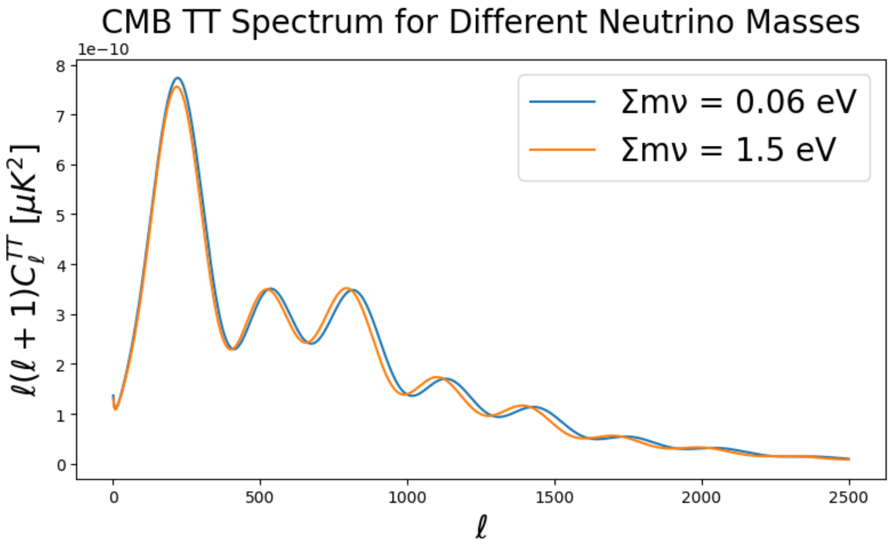
Interest in cosmology, particle physics and interplay between the two

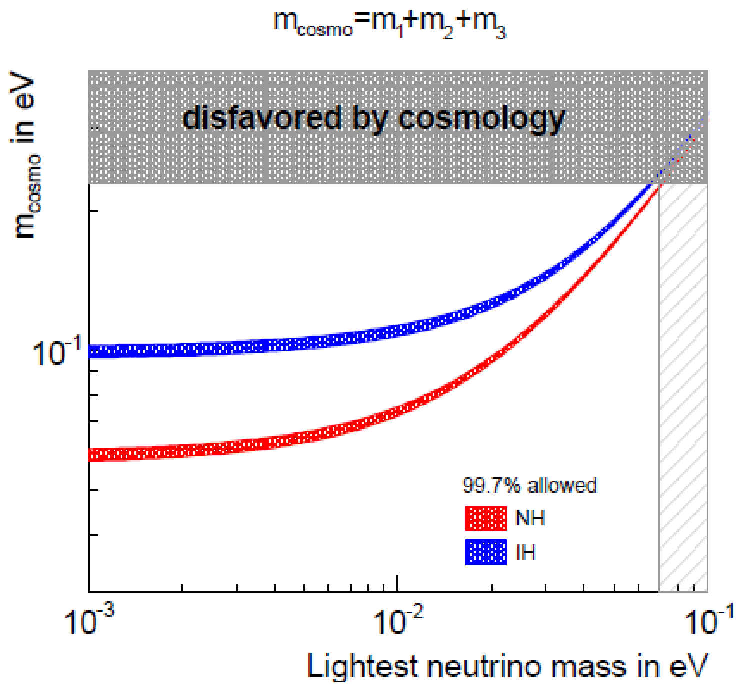
DESY Summer school

Building a satellite in a student project

# Importance of neutrinos in cosmology

- Neutrino masses affect structure formation
- Affect the CMB acoustic peaks
- Can be used to probe neutrino physics and BSM physics

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# Massive neutrino perturbations

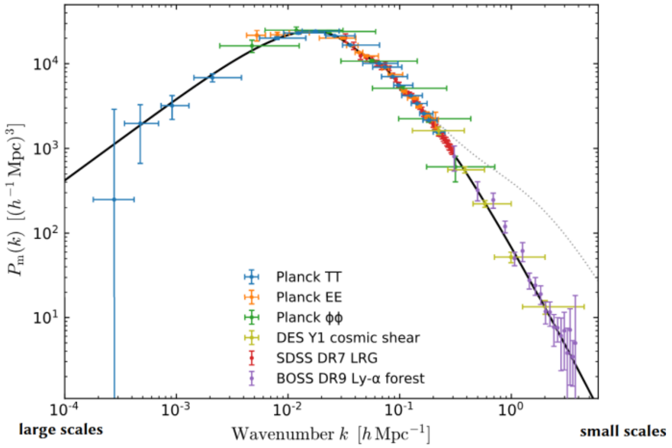
- Current framework based on Ma & Bertschinger [3]
  - Boltzmann equation expanded in Legendre modes
  - Linear perturbation framework
- Boltzmann-Einstein codes compute observables
- Boltzmann hierarchy of equations in momentum space
  - Momentum dependence cannot be integrated out

# Massive neutrino perturbations in Boltzmann codes

- First, perturbations $\Psi_\ell$, are computed
  - Coupled Hierarchy. Truncate at some $\ell_{\max}$
  - Momentum dependent. Dimension $N_q(\ell_{\max}+1)$
- Momentum integrals performed using $\Psi_\ell$
  - Another momentum dependent calculation

$$\begin{aligned}
\dot{\Psi}_0 &= -\frac{qk}{\epsilon}\Psi_1 + \frac{1}{6}\dot{h}\frac{d\ln f_0}{d\ln q}\,,\\
\dot{\Psi}_1 &= \frac{qk}{3\epsilon}\left(\Psi_0 - 2\Psi_2\right)\,,\\
\dot{\Psi}_2 &= \frac{qk}{5\epsilon}\left(2\Psi_1 - 3\Psi_3\right) - \left(\frac{1}{15}\dot{h} + \frac{2}{5}\dot{\eta}\right)\frac{d\ln f_0}{d\ln q}\,,\\
\dot{\Psi}_l &= \frac{qk}{(2l+1)\epsilon}\left[l\Psi_{l-1} - (l+1)\Psi_{l+1}\right]\,, \quad l \geq 3\,.
\end{aligned}$$

$$\begin{aligned}
\delta\rho_h &= 4\pi a^{-4}\int q^2 dq\,\epsilon f_0(q)\Psi_0\,,\\
\delta P_h &= \frac{4\pi}{3}a^{-4}\int q^2 dq\,\frac{q^2}{\epsilon} f_0(q)\Psi_0\,,\\
(\bar{\rho}_h + \bar{P}_h)\theta_h &= 4\pi k a^{-4}\int q^2 dq\, q f_0(q)\Psi_1\,,\\
(\bar{\rho}_h + \bar{P}_h)\sigma_h &= \frac{8\pi}{3}a^{-4}\int q^2 dq\,\frac{q^2}{\epsilon} f_0(q)\Psi_2\,.
\end{aligned}$$

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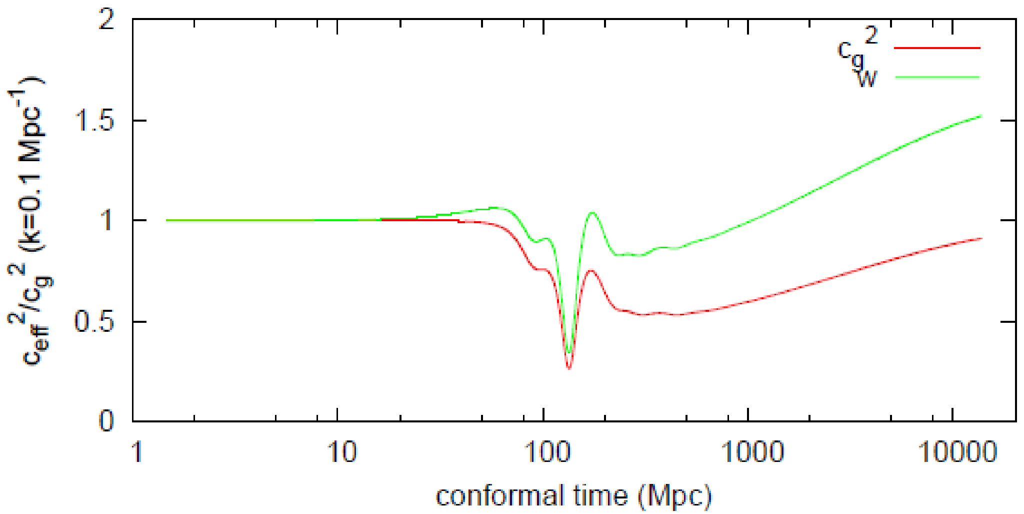
# Approximations in current approach

- Fluid approximations
  - UR-Fluid approximation
    - Reduce $\ell_{max}$ to 2
  - NRA: Pressureless fluid approximation
    - $\ell > 1$ moments decay. Larger error
- Less reliable at semi-relativistic scales



Deviations at semi relativistic scales[1]

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# Issues with the standard momentum-dependent hierarchy

- Must evolve $\Psi_\ell(k, q)$ across many momentum bins
- Must compute repeated momentum integrals for $\delta, \theta, \sigma, \ldots$
- Cost grows quickly for semi-relativistic species
- NCDM species dominate runtime in Boltzmann codes

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# Purpose of the moment hierarchy

- Replace $q$-dependent hierarchy with equivalent hierarchy of moments
- Obtain fully closed hierarchy
- Reduce dimensionality of differential equation system
- Direct computation of fluid properties

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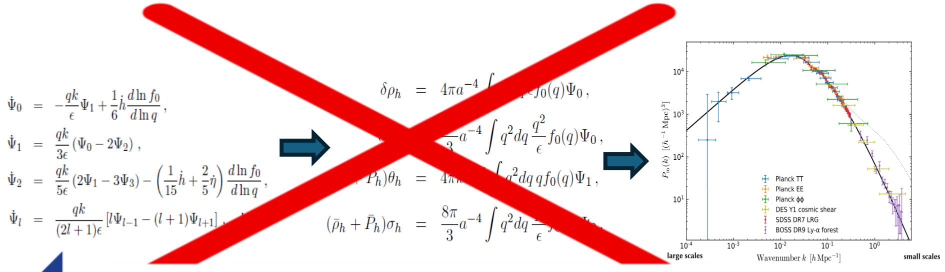
# Introducing the moment hierarchy

- $\Psi_\ell(\vec{k}, \hat{n}, q, \tau)$ ($q$-dependent) replaced by moments:
  - $\delta_n, \theta_n, \sigma_n, f_{l,n}$ ($q$-independent)
- Momentum dependence exactly integrated out
- Fully closed coupled hierarchy in $\ell, n$
  - New velocity weight parameter $n$: $\left(\frac{q}{\epsilon}\right)^{2n}$




Direct computation of perturbative fluid properties

| | | | |
|---|---|---|---|
| $f_{0,0}$ | $f_{0,1}$ | … | $f_{0,n_{\max}}$ |
| $f_{1,0}$ | $f_{1,1}$ | … | $f_{1,n_{\max}}$ |
| … | … | … | … |
| $f_{\ell_{\max},0}$ | $f_{\ell_{\max},1}$ | … | $f_{\ell_{\max},n_{\max}}$ |

| | |
|---|---|
| $f_{0,0}$ | $1/3\delta\rho$ |
| $f_{0,1}$ | $\delta P$ |
| $f_{1,0}$ | $\theta$ |
| $f_{2,0}$ | $\sigma$ |

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# Deriving the moment hierarchy

- Define higher velocity weight background

$$P_n = \rho\omega_n := \frac{4\pi}{3}a^{-4}\int_0^\infty dq q^2 f_0(q)\epsilon\left(\frac{q}{\epsilon}\right)^{2n}$$

| Velocity moment | Corresponding quantity |
|---|---|
| $\boldsymbol{P_0}$ | $\mathbf{1/3\ \rho}$ **(density)** |
| $P_1$ | $P$ (pressure) |
| $P_{n+2}$ | Higher velocity weight pressures |

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# Deriving the moment hierarchy

- Define higher velocity weight background
- Hierarchy for evolution of equation of state

$$\omega_n' = -(2n+3)\mathcal{H}\omega_n + (2n-1)\mathcal{H}\omega_{n+1} - \frac{\rho'}{\rho}\omega_n \;\forall n \geq 0$$

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# Deriving the moment hierarchy

- Define higher velocity weight background
- Hierarchy for evolution of equation of state
- Define moment integrals

$$\delta P_n = \delta_n \rho = \frac{4\pi}{3} a^{-4} \int_0^\infty dq q^2 f_0(q) \epsilon \left(\frac{q}{\epsilon}\right)^{2n} \Psi_0$$

$$(\rho + P)\theta_n = 4\pi k a^{-4} \int_0^\infty dq q^2 f_0(q) \epsilon \left(\frac{q}{\epsilon}\right)^{2n+1} \Psi_1$$

$$(\rho + P)\, f_{l,n} = 4\pi \frac{l!}{(2l-1)!!} a^{-4} \int_0^\infty dq q^2 f_0(q) \epsilon \left(\frac{q}{\epsilon}\right)^{2n+l} \Psi_l$$

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# Deriving the moment hierarchy

$$\dot{\Psi}_0 = -\frac{qk}{\epsilon}\Psi_1 + \frac{1}{6}\dot{h}\frac{d\ln f_0}{d\ln q},$$
$$\dot{\Psi}_1 = \frac{qk}{3\epsilon}(\Psi_0 - 2\Psi_2),$$
$$\dot{\Psi}_2 = \frac{qk}{5\epsilon}(2\Psi_1 - 3\Psi_3) - \left(\frac{1}{15}\dot{h} + \frac{2}{5}\dot{\eta}\right)\frac{d\ln f_0}{d\ln q},$$
$$\dot{\Psi}_l = \frac{qk}{(2l+1)\epsilon}\left[l\Psi_{l-1} - (l+1)\Psi_{l+1}\right], \quad l \geq 3.$$

Multipole evolution eqs.

- Define higher velocity weight background
- Hierarchy for evolution of equation of state
- Define moment integrals
- Derivative of moment integrals & use multipole evolution [3](Ma & Bertschinger)

$$f'_{l,n} = -\left[(2n + l - 3w)\mathcal{H} + \frac{w'}{1+w}\right]f_{l,n} + (2n + l - 1)\mathcal{H}\underline{f_{l,n+1}} + \frac{l^2}{4l^2-1}k\underline{f_{l-1,n+1}} - k\underline{f_{l+1,n}} \;\; \forall l \geq 2$$

Fully closed coupled hierarchy in $n, \ell$ with no momentum dependence

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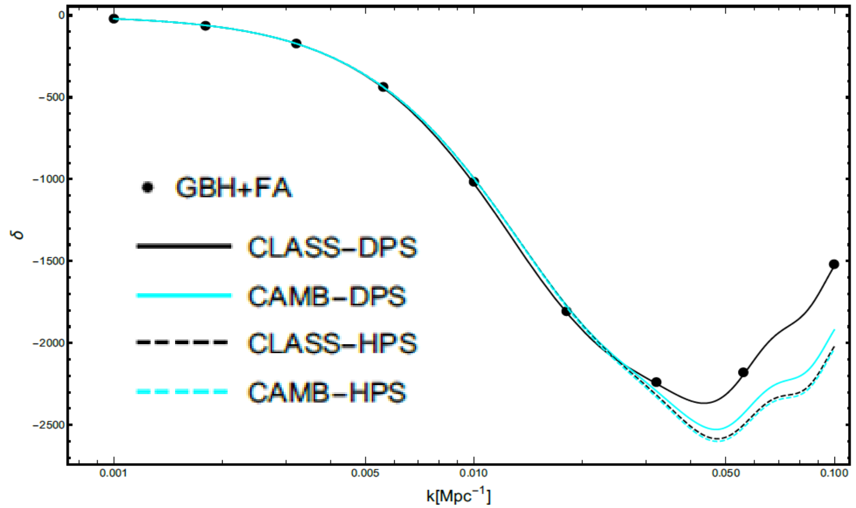
# Advantages & comparison

- Reduction in equation count
  - From $(l_{\max}+1)\times N_q$ to $(l_{\max}+1)\times(n_{\max}+1)$
- Direct calculation of perturbed quantities
  - From $\Psi_l(\vec{k},\hat{n},q,\tau)\rightarrow\delta,\theta,\sigma$
  - To $f_{l,n}\rightarrow f_{0,0}=\delta, f_{1,0}=\theta,\ldots$
- Captures semi-relativistic regime accurately
- Nice behavior in UR, NR regimes
  - $\left(\frac{q}{\epsilon}\right)\rightarrow 1,\ \left(\frac{q}{\epsilon}\right)\rightarrow 0$
- Difficulties at small scales




Differences between moment hierarchy and standard Boltzmann codes [2]

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# Current status

- Accuracy has been tested with toy code[2]
  - Agreement with CLASS within $\lessapprox 0.5\%$
    - Worse at smaller scales
- Implement in CLASS
- Improvement of approximation + truncation schemes

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# Outlook

- Accelerate neutrino calculations
  - Neutrino mass scale and hierarchy
- Potential for inverse decays (neutrino lasing)
  - Extended neutrino/dark sector models
- Applications for neutrino cosmology & beyond $\Lambda$CDM
  - SI$\nu$, SIDM, …

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# Thank you:)

Questions?

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# Sources

[1] CLASS IV: Efficient implementation of non-cold relics. arXiv:1104.2935

[2] Generalized Boltzmann hierarchy for massive neutrinos in cosmology. arXiv:2104.00703

[3] Cosmological Perturbation Theory in the Synchronous and Conformal Newtonian Gauges. arXiv:astro-ph/9506072

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