Symmetric Poisson geometry,
totally geodesic foliations
and Jacobi-Jordan algebras

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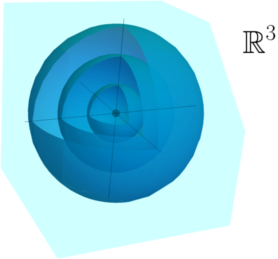

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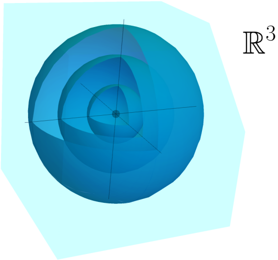

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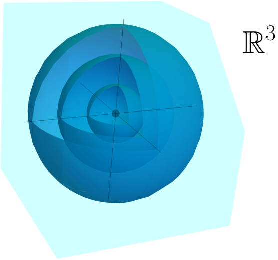

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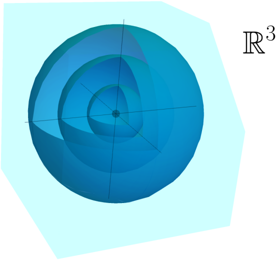

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If a symmetric bivector field $\vartheta \in \mathfrak{X}^2_{\mathsf{sym}}(M) := \Gamma(\mathrm{Sym}^2\, TM)$ is non-degenerate,

$$g := \vartheta^{-1}$$

is a (pseudo-)Riemannian metric.

Integrability condition for a symmetric bivector field ϑ

Following the Poisson geometry approach, every $\vartheta \in \mathfrak{X}^2_{\mathsf{sym}}(M)$ gives

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We resort to the symmetric Cartan calculus!

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$$[X,Y]_s := \nabla_X Y + \nabla_Y X.$$

Compare with: $[X,Y] := X\circ Y - Y\circ X = \nabla_X Y - \nabla_Y X$

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It is actually determined by ∇^s:

$$\iota_{[X,Y]_s} = [[\iota_X, \nabla^s], \iota_Y].$$

Compare with: $[X,Y] := X \circ Y - Y \circ X = \nabla_X Y - \nabla_Y X$ and $\iota_{[X,Y]} = [[\iota_X, \mathrm{d}]_{\mathrm{g}}, \iota_Y]_{\mathrm{g}}$.

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$\rightsquigarrow$ A natural extension to $\mathfrak{X}^\bullet_{\mathsf{sym}}(M)$
the **symmetric Schouten bracket** $[\ ,\ ]_s$.

Compare with: $[X,Y] := X \circ Y - Y \circ X = \nabla_X Y - \nabla_Y X$ $\rightsquigarrow$ the Schouten bracket $[\ ,\ ]$ on $\mathfrak{X}^\bullet(M)$.

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This motivates an a priori intermediate class

$$\left\{\begin{array}{c}\text{strong symmetric}\\ \text{Poisson structures}\end{array}\right\} \subseteq \left\{\begin{array}{c}\textbf{involutive symmetric}\\ \textbf{Poisson structures}\\ [\mathcal{F}_\vartheta, \mathcal{F}_\vartheta] \subseteq \mathcal{F}_\vartheta\end{array}\right\} \subseteq \left\{\begin{array}{c}\text{symmetric}\\ \text{Poisson structures}\\ [\vartheta, \vartheta]_s = 0\end{array}\right\}.$$

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but it is preserved by the symmetric bracket

$$[\mathcal{F}_\vartheta, \mathcal{F}_\vartheta]_s \subseteq \mathcal{F}_\vartheta.$$

(ϑ, ∇) is strong symmetric Poisson

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$$[\mathcal{F}_\vartheta, \mathcal{F}_\vartheta] \subseteq \mathcal{F}_\vartheta,$$

$\Rightarrow$ it yields a singular partition.

This motivates an a priori intermediate class

$$\left\{ \begin{array}{c} \text{strong symmetric} \\ \text{Poisson structures} \\ \nabla_{\operatorname{im}\vartheta}\vartheta = 0 \end{array} \right\} \subseteq \left\{ \begin{array}{c} \textbf{involutive symmetric} \\ \textbf{Poisson structures} \\ [\mathcal{F}_\vartheta, \mathcal{F}_\vartheta] \subseteq \mathcal{F}_\vartheta \end{array} \right\} \subseteq \left\{ \begin{array}{c} \text{symmetric} \\ \text{Poisson structures} \\ [\vartheta, \vartheta]_s = 0 \end{array} \right\}.$$

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Moreover, ϑ-admissible geodesics have constant square of the speed $g_\vartheta(\dot{\gamma}, \dot{\gamma})$.

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In addition, if (ϑ, ∇) is strong, ∇^N is the Levi-Civita connection of g_N.

Examples: Linear symmetric Poisson structures

In classical Poisson geometry,

$$\left\{ \begin{array}{c} \text{linear Poisson} \\ \text{structures on } V^* \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Lie algebra} \\ \text{structures on } V \end{array} \right\}.$$

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In particular, Jacobi-Jordan algebras are Jordan algebras.

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3	$z\,\partial_x \otimes \partial_x$		
	$z\,(\partial_x \otimes \partial_x + \partial_y \otimes \partial_y)$		
4	$t\,\partial_x \otimes \partial_x$		
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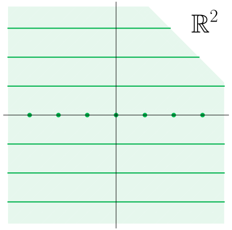

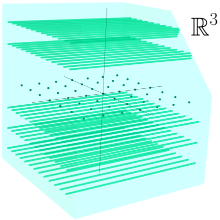

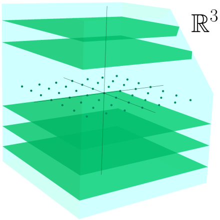

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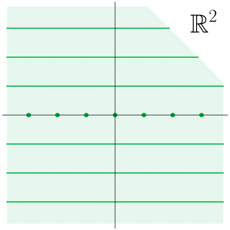

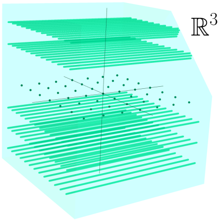

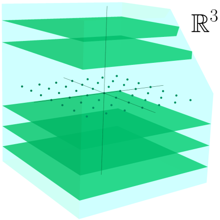

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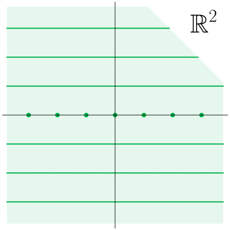

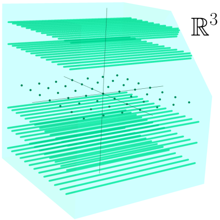

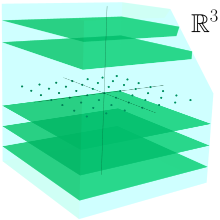

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$$\vartheta = x_2\,\partial_{x_1} \otimes \partial_{x_1} + x_5\,\partial_{x_1} \odot \partial_{x_4} - \frac{1}{2} x_3\,\partial_{x_1} \odot \partial_{x_5} + x_3\,\partial_{x_2} \odot \partial_{x_4}.$$

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⇒ The normal form of a linear **non-strong** symmetric Poisson structure for $\dim V = 5$:

$$\vartheta = x_2\,\partial_{x_1} \otimes \partial_{x_1} + x_5\,\partial_{x_1} \odot \partial_{x_4} - \frac{1}{2} x_3\,\partial_{x_1} \odot \partial_{x_5} + x_3\,\partial_{x_2} \odot \partial_{x_4}.$$

Its characteristic module $\mathcal{F}_\vartheta$ is generated by

$$X_1 := x_2\,\partial_{x_1} + x_5\,\partial_{x_4} - \frac{1}{2} x_3\,\partial_{x_5}, \qquad X_2 := x_3\,\partial_{x_4},$$

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The only non-trivial Lie bracket is $[X_1, X_3] = \frac{1}{2} X_4$, hence it is **involutive**

Dimensions of the leaves are $0, 1, 2, 4$ with the signatures: $(1,0), (0,1), (1,1), (2,2)$.

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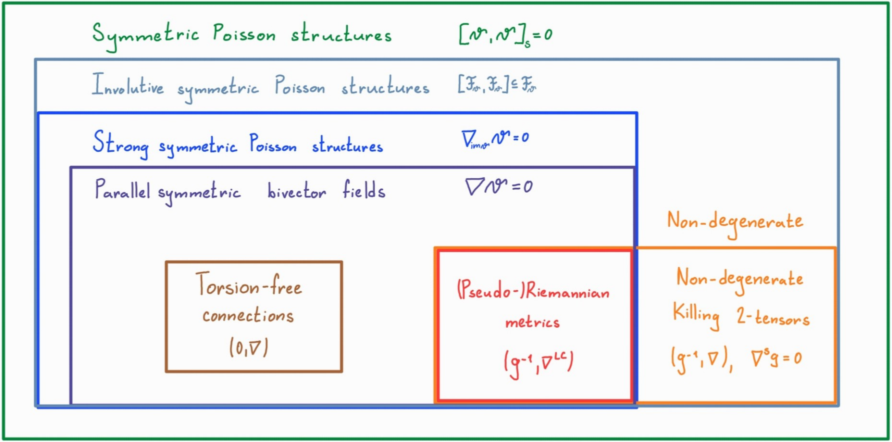
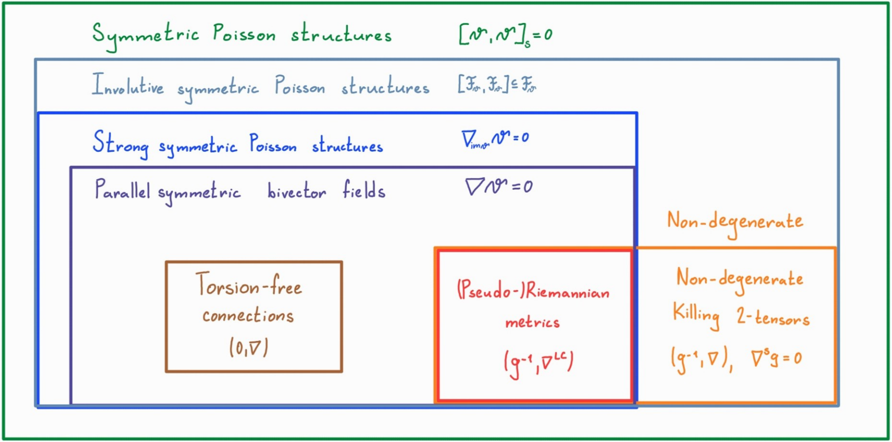
The leaf metric on a 4-dimensional leaf N_c given by $x_3 = c$, $c \neq 0$:

$$g_{N_c} = -\frac{2}{c} \mathrm{d}x_1 \odot \mathrm{d}x_5 + \frac{1}{c} \mathrm{d}x_2 \odot \mathrm{d}x_4 + \frac{2x_5}{c^2} \mathrm{d}x_2 \odot \mathrm{d}x_5 - \frac{4x_2}{c^2} \mathrm{d}x_5 \otimes \mathrm{d}x_5.$$


Symmetric Poisson structures $[\vartheta,\vartheta]_s=0$
Involutive symmetric Poisson structures $[\mathcal{F}_\vartheta,\mathcal{F}_\vartheta]\subseteq\mathcal{F}_\vartheta$
Strong symmetric Poisson structures $\nabla_{\mathrm{im}\,\vartheta}\vartheta=0$
Parallel symmetric bivector fields $\nabla\vartheta=0$
Non-degenerate
Torsion-free connections
$(0,\nabla)$
(Pseudo-)Riemannian metrics
(g^{-1},∇^{LC})
Non-degenerate Killing 2-tensors
(g^{-1},∇), $\nabla^s g=0$



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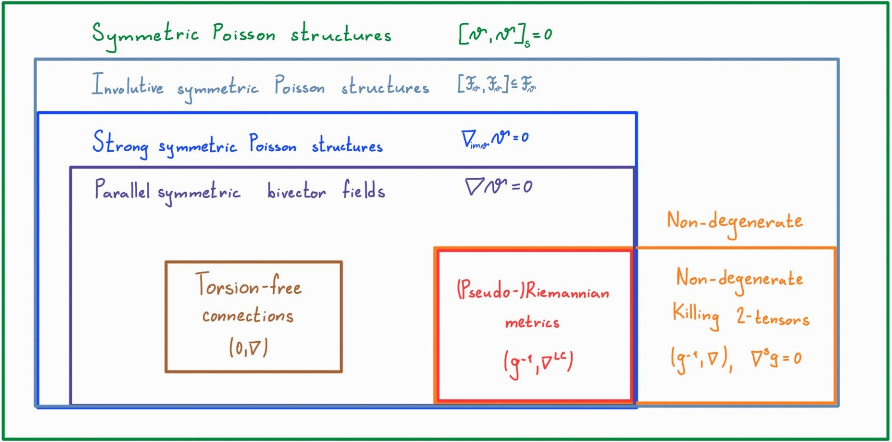

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Thank you for your attention!