Reduction of two-loop integrals using tadpole expansion

Sara Rösler

Institut für Physik
Humboldt-Universität zu Berlin

November 25, 2025

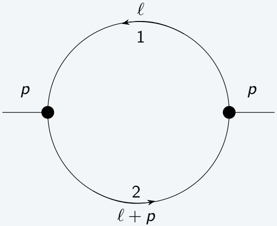
Motivation

Theory precision $\Leftrightarrow$ Experimental precision

$\Rightarrow$ Higher order pertubation theory

$\Rightarrow$ **Calculation of L loop integrals**:

$$\boxed{\mathrm{I}(d; n_1, ..., n_N) := \prod_{i=1}^{L} \int_{-\infty}^{\infty} \frac{d^d \ell_i}{i\pi^{\frac{d}{2}}} \prod_{j=1}^{N} \frac{1}{(D_j)^{n_j}}}$$



Propagator:

$$\boxed{D_j := q_j^2 - m_j^2}$$

$$D_1 = \ell^2 - m_1^2\,,\; D_2 = (\ell + p)^2 - m_2^2$$

q_j : Linear combination of ℓ_j and **external** momenta p_j

Reduction to Master Integrals

- **IBP-Relations**

$$\int\int d^d\ell_1 d^d\ell_2 \, \frac{\partial}{\partial \ell_i^\nu}\left(k_i^\nu \prod_{j=1}^{N} \frac{1}{(D_j)^{n_j}}\right) = 0$$

  - **Reduction to independent Master Integrals (MIs)**
  - **Finite number** of MIs
  - **Rational** coefficients of kinematic variables $\vec{s}$ and d
  - Has to reduce many integrals to compute one integral

$\rightarrow$ **Auxiliary Mass Flow** Liu and Ma [2019]

$$D_i \rightarrow D_i + \eta^2$$

  - Number of MIs **increase**
    - Example: $L = 1$ and one external momentum p
      MI : {I(d; 1, 1)} for $m_1 = m_2 = 0 \implies$ MI : $\{\mathrm{I}(d;1,0),\mathrm{I}(d;1,1)\$$ for $m_1 = m_2 = M$

Series expansion

$$\frac{1}{((\ell_j + p_j)^2 - m_j^2 + \eta^2)^{n_j}} = \frac{1}{\left(\ell_j^2 + \eta^2\right)^{n_j}} \frac{1}{\left(1 + \frac{2\ell_j p_j + p_j^2 - m_j^2}{\ell_j^2 + \eta^2}\right)^{n_j}}$$

$$= \frac{1}{\left(\ell_j^2 + \eta^2\right)^{n_j}} \sum_{k=0}^{\infty} \binom{n-1+k}{k} \left(-\frac{2\ell_j p_j + p_j^2 - m_j^2}{\ell_j^2 + \eta^2}\right)^k \qquad \mid \tilde{\ell}_i = \frac{\ell_i}{\eta}$$

$$= \frac{\eta^{-2n_j}}{\left(\tilde{\ell}_j^{\,2} + 1\right)^{n_j}} \sum_{k=0}^{\infty} \eta^{-2k} \binom{n-1+k}{k} \left(-\frac{2\tilde{\ell}_j p_j \eta + p_j^2 - m_j^2}{\tilde{\ell}_j^{\,2} + 1}\right)^k$$

Elimination of p_i and m_l from the denominator

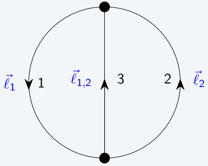
Series expansion for $L = 2$

$$\int \frac{d^d\ell_2}{i\pi^{\frac{d}{2}}} \int \frac{d^d\ell_1}{i\pi^{\frac{d}{2}}} \prod_{j=1}^{N} \frac{1}{(D_j)^{n_j}} = \sum_i c_i(d,\vec{s},\eta) \cdot \int\int d^d\tilde{\ell}_2 d^d\tilde{\ell}_1 \frac{(\tilde{\ell}_1\cdot\tilde{\ell}_2)^{N_{i,1}}(\tilde{\ell}_1\cdot\tilde{\ell}_1)^{N_{i,2}}(\tilde{\ell}_2\cdot\tilde{\ell}_2)^{N_{i,3}}(\tilde{\ell}_1\cdot p_j)^{N_{i,4}}(\tilde{\ell}_2\cdot p_j)^{N_{i,5}}\ldots}{(\tilde{\ell}_1^2+1)^{n'_{i,1}}(\tilde{\ell}_2^2+1)^{n'_{i,2}}((\tilde{\ell}_1-\tilde{\ell}_2)^2+1)^{n'_{i,3}}}$$

→ Depending on known **Tadpole Integrals**

→ MIs of the tadpole integrals:

$$MI^{L=2}(d) = \{I(d,1,1,1), I(d,1,1,0)\}$$



$$\boxed{\mathrm{I}(d;n_1,\ldots,n_N) = \eta^{Ld-\sum_{j=1}^{n_N} 2n_j} \cdot \sum_i C_i(d,\vec{s},\eta)\cdot(MI^L)_i(d)}$$

Expansion of the coefficients

- IBP: **Polynomial coefficients Q_i**

$$\boxed{Q_0(d,\vec{s},\eta)\mathrm{I}(d;n_1,...,n_N)=\sum_{i=1}^{N_{\mathrm{MI}}}Q_i(d,\vec{s},\eta)(\mathrm{MI})_i(d,\vec{s},\eta)}$$

- **Mass dimensions:**

$$\dim(\mathrm{I})+\dim(Q_0)=\dim((\mathrm{MI})_1)+\dim(Q_1)=...=\dim((\mathrm{MI})_{N_{\mathrm{MI}}})+\dim(Q_{N_{\mathrm{MI}}})$$

$\rightarrow$ $\dim(Q_i)$ fixed up to **one degree of freedom** d_{It}

Expand the coefficients:

$$\boxed{Q_i(d,\vec{s},\eta)=\sum_{j=0,\vec{\lambda}\in\Omega_{\mathrm{d}_i}}^{n_{max}}\tilde{C}_{i,j,\lambda_1,...,\lambda_N}(d)\eta^{2j}s_1^{\lambda_1}\cdot...\cdot s_N^{\lambda_N}\,|\,\Omega_{\mathrm{d}_i}=\{\vec{\lambda}\in\mathbb{N}^N|\lambda_1+...+\lambda_N\leq\mathrm{d}_i\geq 0\}}$$

Fix coefficients by expanding MIs and I for large η

Relations of Expansions

Finite number of MI :

$$Q_0(d,\vec{s},\eta)\mathrm{I}(d,n_1,\ldots,n_N)=\sum_{i=1}^{N_{\mathrm{MI}}}Q_i(d,\vec{s},\eta)(\mathrm{MI})_i(d,\vec{s},\eta)$$

Rational coefficients :

$$Q_i(d,\vec{s},\eta)=\sum_{j=0,\vec{\lambda}\in\Omega_{\mathrm{d}_i}}^{n_{max}}\tilde{C}_{i,j,\lambda_1,\ldots,\lambda_N}(d)\eta^{2j}s_1^{\lambda_1}\cdot\ldots\cdot s_N^{\lambda_N}$$

Series expansion approach

$$\mathrm{I}(d;n_1,\ldots,n_N)=\eta^{Ld-\sum_{j=1}^{n_N}2n_j}\cdot\sum_i C_i(d,\vec{s},\eta)\cdot(MI^L)_i(d)$$

$$0=\sum_j^{B_L}\sum_{\vec{i}}f_j(\tilde{C}_{i,j,\lambda_1,\ldots,\lambda_N}(d),d)\cdot\eta^{i_0}s_1^{i_1}\cdot\ldots\cdot s_N^{i_N}\cdot(MI^L)_j(d)$$

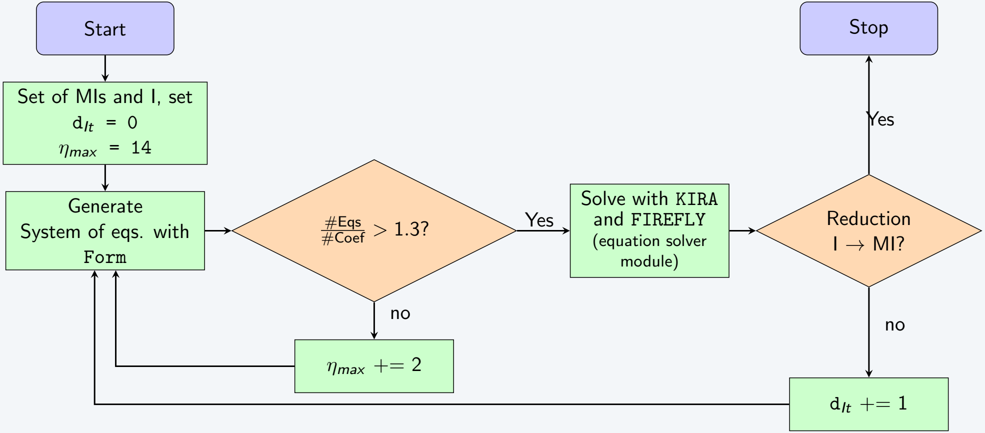
Number of f_j determind by order of η of the series expansion

System of linear equations

$$0=f_j(\tilde{C}_{i,j,\lambda_1,\ldots,\lambda_N}(d),d)$$

Algorithm and Implementation

Alternative Implementation of the Algorithm of Liu and Ma [2019]



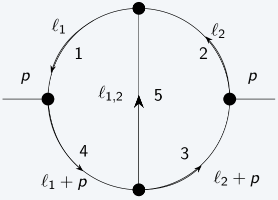
Example Self-Energy-Two-Loop Integrals

The Program can reduce arbitrary two loop integrals

Tested for the example

- $\mathrm{I}(d; n_1, n_2, n_3, n_4, n_5)$:

$$
\begin{aligned}
D_1 &= \ell_1^2 - m_1^2 \ ; \\
D_2 &= \ell_2^2 - m_2^2 \ ; \\
D_3 &= (\ell_2 + p)^2 - m_3^2; \\
D_4 &= (\ell_1 + p)^2 - m_4^2 \\
D_5 &= (\ell_1 - \ell_2)^2 - m_5^2
\end{aligned}
$$



$$\mathrm{MI}_\eta = \{\mathrm{I}_\eta(d; 1, 1, 0, 0, 0)\, ;\ \mathrm{I}_\eta(d; 1, 0, 0, 0, 1)\, ;\ \mathrm{I}_\eta(d; 0, 1, 0, 0, 1)\, ;\ \mathrm{I}_\eta(d; 1, 1, 0, 0, 1)\}$$

Example Self-Energy-Two-Loop Integrals

Example of output:

$$\mathrm{I}(\mathrm{d},1,1,-1,-1,1)\cdot \mathrm{d} =$$
$$\mathrm{I}(\mathrm{d,0,1,0,0,1})\cdot(2\cdot s + \mathrm{d}\cdot s - \mathrm{d}\cdot m_3^2 + \mathrm{d}\cdot m_2^2)$$
$$+\ \mathrm{I}(\mathrm{d,1,0,0,0,1})\cdot(2\cdot s + \mathrm{d}\cdot s - \mathrm{d}\cdot m_4^2 + \mathrm{d}\cdot m_1^2)$$
$$+\ \mathrm{I}(\mathrm{d,1,1,0,0,0})\cdot(-2\cdot s)$$
$$+\ \mathrm{I}(\mathrm{d,1,1,0,0,1})\cdot(-2\cdot m_5^2\cdot s + 2\cdot m_2^2\cdot s + 2\cdot m_1^2\cdot s - 2\cdot \eta^2\cdot s + \mathrm{d}\cdot s^2 - \mathrm{d}\cdot m_4^2\cdot s$$
$$-\ \mathrm{d}\cdot m_3^2\cdot s + \mathrm{d}\cdot m_3^2\cdot m_4^2 + \mathrm{d}\cdot m_2^2\cdot s - \mathrm{d}\cdot m_2^2\cdot m_4^2 + \mathrm{d}\cdot m_1^2\cdot s - \mathrm{d}\cdot m_1^2\cdot m_3^2 + \mathrm{d}\cdot m_1^2\cdot m_2^2)$$

Reduced a sample of 840 integrals
$\rightarrow$ All reduction formulae agree with KIRA

d_{It}	0	1	2	3	4	5	6	7	8	9	trivial	total
Integrale	122	117	6	42	12	36	64	25	37	63	315	840

Summary

- **Tadpole Reduction** Liu and Ma [2019]
    - Extension and expanding by the **auxiliary mass** η^2
      $\Rightarrow$ Reduction to master vacuum integrals
    - Expansion of the coefficients
      $\Rightarrow$ Generating and solving a system of linear equations
  - **Advantage**
    - Computation of single integrals
    - MIs independent of propagator power
- **Alternative implementation**
  - Using the Algorithm from Liu and Ma [2019]
  - Program which can reduce **arbitary** two loop integrals for given MIs
    - Generation of the equation system with **FORM**
    - Solve with the linear solver module from **KIRA** and **FIREFLY**

Outlook

- **Optimisation**
  - Extension by auxiliary masses only for **some** propagators, Guan, Liu, and Ma [2020]

    $\rightarrow$ fewer MIs

    $\rightarrow$ fewer coefficients
- **Outlook**
  - Searching for and using **trigonal block matrices** , Guan, Liu, and Ma [2020]
  - Application to more advanced examples
  - **Calculation of the MIs**
    - Creating the differential equation
    - Boundry condition for large η

Thank you for your attention.

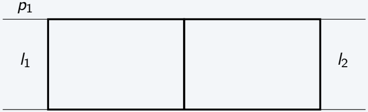
Sara Rösler

Institut für Physik
Humboldt-Universität zu Berlin

November 25, 2025

Multi Loop Integrals

- **Passarino Veltman Reduction** :
  Write products of ℓ_i with D_j
- Works for $L = 1$
- The set of D_j is **not** complete for $L > 1$
- Passarino Veltman **not** possible



$$D_1 = \ell_1^2 \,,\; D_2 = (\ell_1 + p_1)^2 \,,\; D_3 = (\ell_1 - p_2)^2 \,,$$
$$D_4 = (\ell_2 - p_3)^2 \,,\; D_5 = \ell_2^2 \,,\; D_6 = (\ell_2 + p_4)^2 \,,$$
$$D_7 = (\ell_1 + \ell_2 + p_1 - p_3)^2$$

$$\frac{(p_1 + \ell_2)^2 - \ell_2^2 - p_1^2}{(\ell_1^2)^{\nu_1}((\ell_1 + p_1)^2)^{\nu_2}((\ell_1 - p_2)^2)^{\nu_3}((\ell_2 - p_3)^2)^{\nu_4}(\ell_2^2)^{\nu_5}((\ell_2 + p_4)^2)^{\nu_6}((\ell_1 + \ell_2 + p_1 - p_3)^2)^{\nu_7}}$$

Tensor Reduction

1 **Schwinger Parametrization**

$$\frac{1}{D_1^{n_1}\cdot\ldots\cdot D_n^{n_n}} = \int Dx\; exp[\sum_i^n x_i D_i]$$

- $\sum_i^n x_i D_i = a\ell_1^2 + b\ell_2^2 + 2c\cdot(\ell_1\cdot\ell_2) + 2(d\cdot\ell_2) + 2(e\cdot k_2) + f$
- $Dx = \prod_i^n \frac{(-1)^{n_i}}{\Gamma(n_i)} x_i^{n_i-1} dx_i$

2 **Diagonalization**

- **Substitution** :
  $\ell_1^\nu = K_1^\nu - \frac{c}{a}K_2^\nu + X^\nu$, $\ell_2^\nu = K_2^\nu + Y^\nu$

  $\sum_i^n x_i D_i = aK_1^2 + \frac{a_0}{a}K_2^2 + \frac{Q}{a_0}$,

$$X^\nu = \frac{ce^\nu - bd^\nu}{a_0}\;,\; Y^\nu = \frac{cd^\nu - ae^\nu}{a_0}$$
$$Q = -ae^2 - bd^2 + 2c\,(e\cdot d) + fa_0$$
$$a_0 := ab - c^2$$

3 **Separable** gaussian integrals over K_1, K_2:

$$\int Dx\;\left(\int \frac{d^dK_1}{i\pi^{\frac{d}{2}}}(K_1^{\nu_1}\cdot\ldots\cdot K_1^{\nu_{N_1}})e^{aK_1}\right)\cdot\left(\int \frac{d^dK_2}{i\pi^{\frac{d}{2}}}(K_2^{\nu_1}\cdot\ldots\cdot K_2^{\nu_{N_2}})e^{\frac{a0}{a}K_2}\right)$$

Tensor Reduction

4 Lowering and raising operators

- **Evaluation of** Dx
- Independence of x_i $\Rightarrow$Looking at one x_i:

$$\frac{(-1)^{n_i}}{\Gamma(n_i)}\int_0^\infty dx_i\ x_i^{n_i-1}x_i = -n_i\int_0^\infty dx_i\ \frac{(-1)^{n_i+1}x_i^{n_i}}{\Gamma(n_i+1)} := -n_i\mathbf{i}^+\int_0^\infty dx_i\ \frac{(-1)^{n_i}}{\Gamma(n_i)}x_i^{n_i-1}$$

  - **Raising operator** : $\mathbf{i}^+ I^L(d; ..., n_i, ...) = I^L(d; ..., n_i+1, ...)$

- **Effect of** a_0

$$\int\int \frac{d^{d\pm2}K_1}{i\pi^{\frac{d\pm2}{2}}}\frac{d^{d\pm2}dK_2}{i\pi^{\frac{d\pm2}{2}}}\ e^{aK_1^2+\frac{a_0}{a}K_2^2} = a_0^{-\frac{d\pm2}{2}} = a_0^{\mp1}\cdot a_0^{-\frac{d}{2}} := \mathbf{d}^\pm\int\int\frac{d^dK_1}{i\pi^{\frac{d}{2}}}\frac{d^dK_2}{i\pi^{\frac{d}{2}}}\ e^{aK_1^2+\frac{a_0}{a}K_2^2}$$

  - **Dim. lowering-/raising operators** : $\mathbf{d}^\pm I^L(d; n_1, n_2, ...) = I^L(d\pm2; n_1, n_2, ...)$
  - $1 = \frac{ab-c^2}{a_0}$ change of dimension

IBP-Relations

Deriving relations between integrals by solving:

$$\int\int d^d\ell_1 d^d\ell_2 \, \frac{\partial}{\partial \ell_i^\nu}\left(k_i^\nu \prod_{j=1}^{N} \frac{1}{(D_j)^{n_j}}\right) = 0$$

- **Example:** two loop vacuum integral with $m_i^2 = -1$

  For $n_1 \neq 1$ and $n_3 = 0$:

  $$\mathbf{E} = \left(1 - \frac{d}{2(n-1)}\right)\mathbf{1}^-$$

  For $n_1 \neq 1$:

  $$\mathbf{E} = -\frac{(d-3(n_1-1))\mathbf{1}^- + 2n_3\mathbf{3}^+(\mathbf{2}^- - \mathbf{1}^-)\mathbf{1}^- + (n_1-1)(\mathbf{3}^- - \mathbf{2}^-)}{3(n_1-1)}$$