Quantisation of Character Varieties

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Character Varieties

Definition

Let S be an oriented surface, G a reductive group,

$$\mathrm{Ch}_G(S) := \{G\text{-local systems on } S\}/\sim \;\cong\; \hom(\pi_1(S), G)/G.$$

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Quantise:

- Skein theory [Turaev, Przytycki, . . .]
- Cluster theory [Fock-Goncharov, G-Shen]
- Factorisation homology [Ben-Zvi-Brochier-Jordan,J-Le-Schrader-Shapiro]

Skein partition function

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Theorem: Skein partition function [Bierent, Jordan, V, Vazirani '25]

$$\mathcal{Z}_{M_\gamma}(t) := 1 + \sum_{N=1}^{\infty} \dim \mathrm{Sk}_{\mathrm{GL_N}}(T^2 \times_\gamma S^1) \cdot t^N = \prod_{k=1}^{\infty}(1 - t^k)^{-c_k},$$

$$\text{where } c_k = \frac{1}{k}\sum_{d|k} \phi(\frac{k}{d})|(\gamma^d) - 2|, \text{ for generic } \gamma.$$

Decorated Character Varieties

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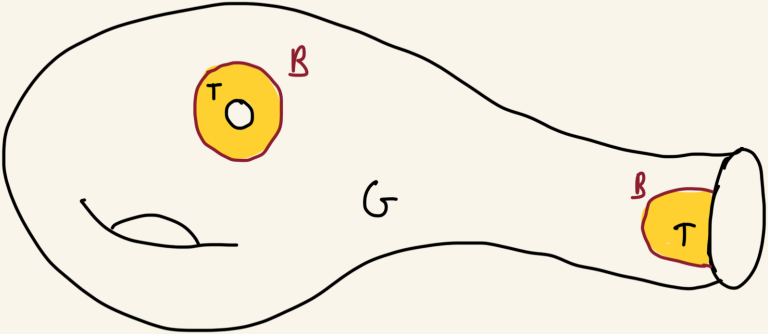

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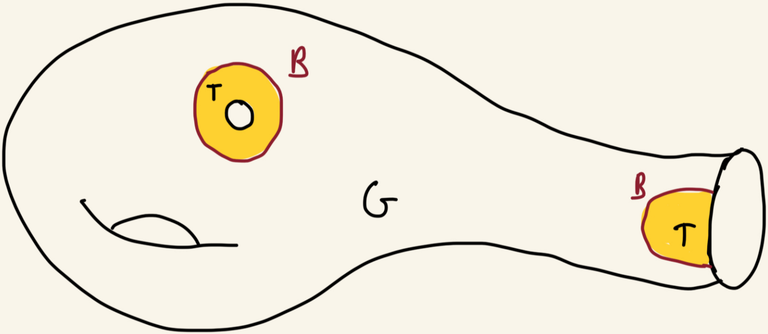

$$\mathrm{Ch}_G(\mathbb{S}) := \left\{ \begin{array}{c} G\text{-local systems on } \mathbb{S}_G, \\ T\text{-local systems on } \mathbb{S}_T, \\ \text{Borel reduction on } \mathbb{S}_B. \end{array} \right\} / \sim$$

Cluster Structure by Fock-Goncharov and G-Shen

$$\mathrm{Ch}_G(\mathbb{S}) \dashrightarrow \chi_\Delta = (\mathbb{C}^\times)^d = \mathrm{Spec}(\mathbb{C}[x_1^\pm, \ldots, x_d^\pm])$$

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Easy Poisson structure: $\{x_i, x_j\} = a_{ij} x_i x_j$

Easy quantisation: $\chi_\Delta^q = \mathbb{C}[q^{\pm}][x_1^{\pm}, \ldots, x_d^{\pm}] / < x_i x_j = q^{a_{ij}} x_j x_i >$

Clusters from Factorisation Homology

Factorisation homology:

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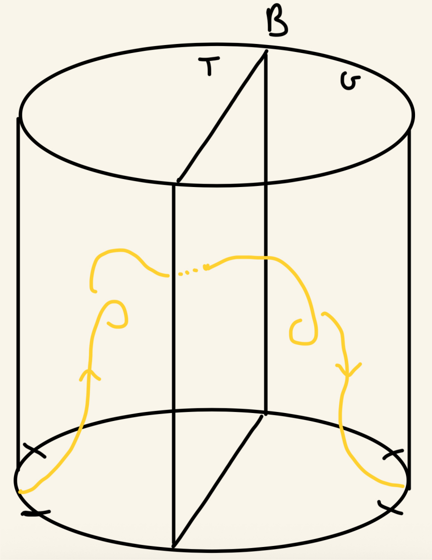
Moreover: χ_Δ^q can be understood in terms of skein theory.

Internal skein algebra



Thank you for your attention.