Stress-Tensor Correlators

Exact Consequences of the Trace Anomaly in 4d

Trace Anomaly

- Discovered by Capper & Duff in the 1970s, arises in curved space

$$\mathscr{A} = g^{\mu\nu}\langle T_{\mu\nu}\rangle$$

- Curvature scalar of dimension d ($\rightarrow$ not present for odd d)

$$\mathscr{A}^{(2d)} = \hat{c}R$$

$$\mathscr{A}^{(4d)} = a\mathbb{E}_4 + b\nabla^2 R - 3c(Weyl)^2$$

- Theorem: $a, \hat{c}$ decrease between critical points along RG-flow

Background of the thesis

- Based on a paper by Cappelli, Guida and Magnoli (2001)
  $\rightarrow$ attempt to extend „$\hat{c}$-theorem“ to 4d

- Idea: Extend and generalize approach of Cappelli et al.
  $\rightarrow$ less assumptions + restrict to CFT

- „a-theorem“ was proven by investigating on $\langle T^\mu_\mu T^\nu_\nu T^\alpha_\alpha T^\beta_\beta \rangle$ or $\langle T^\alpha_\alpha T_{\mu\nu} T^\beta_\beta \rangle$
  $\rightarrow$ no full solutions of 3- and 4-point function presented yet (!)

Symmetries and Ward identities

- Assumed space time symmetries

  1) Diffeomorphism invariance $x^\mu \to x^\mu + \epsilon^\mu(x)$

  2) Weyl invariance $g_{\mu\nu} \to e^{2\sigma} g_{\mu\nu}$

$$\Longrightarrow \quad \boxed{\begin{aligned} 0 &= \nabla^\mu \langle T_{\mu\nu} \rangle \\ \mathscr{A} &= g^{\mu\nu} \langle T_{\mu\nu} \rangle \end{aligned}}$$

Ward identities

- Combination of both symmetries for dilatations $x^\mu \to \lambda x^\mu, \ \lambda > 0$

$$\boxed{6c \int_M d^4x \sqrt{g} (Weyl)^2 = \int_M d^4x \sqrt{g} \langle T_{\mu\nu} \rangle \left(\nabla^{(\mu} x^{\nu)} - \frac{g^{\mu\nu}}{4} \nabla \cdot x \right)}$$

Dilatation identity

Stress-tensor correlators

- Matter stress-tensor from curved space action: $T_{\mu\nu}(x) := \frac{1}{\sqrt{g(x)}} \frac{\delta S[g_{\alpha\beta};\phi]}{\delta g^{\mu\nu}(x)}$
- Effective action $W[g_{\alpha\beta}] = \frac{1}{Z_0} \int \mathscr{D}\phi\, e^{-S[g_{\alpha\beta};\phi]}$ is *generating functional*

$$\langle T_{\mu_1\nu_1}(x_1)\cdots T_{\mu_N\nu_N}(x_N)\rangle := \frac{1}{\sqrt{g(x_1)\cdots g(x_N)}} \frac{\delta^N W[g_{\alpha\beta}]}{\delta g^{\mu_1\nu_1}(x_1)\cdots g^{\mu_N\nu_N}(x_N)}\Bigg|_{g_{\mu\nu}=\delta_{\mu\nu}}$$

→ invariant under translations, i.e. $p_1 + \cdots + p_N = 0$

→ invariant under permutation of stress-tensors, i.e. S_N *symmetry*

2-point function

Ward identities

1) Diff-Ward: $$p^\mu \langle T_{\mu\nu}(p) T_{\alpha\beta}(-p) \rangle = 0$$

2) Weyl-Ward: $$\delta^{\mu\nu} \langle T_{\mu\nu}(p) T_{\alpha\beta}(-p) \rangle = b(p_\alpha p_\beta - p^2 \delta_{\alpha\beta})$$

3) Dilatation identity:

$$\begin{aligned}(p \cdot \partial_p - 4) \langle T_{\mu\nu}(p) T_{\alpha\beta}(-p) \rangle \\ = -2c(p_\mu p_\nu - p^2 \delta_{\mu\nu})(p_\alpha p_\beta - p^2 \delta_{\alpha\beta}) \\ +6c\left(p_\mu p_\nu p_\alpha p_\beta - 2p^2 p_{(\mu} \delta_{\nu)(\alpha} p_{\beta)} + p^4 \delta_{\mu(\alpha} \delta_{\beta)\nu}\right)\end{aligned}$$

Ansatz and solution

- Most general Ansatz has 6 independent tensor structures
- Amplitudes are functions of p^2
- 2-point function is completely determined by Ward identities

$$\langle T_{\mu\nu}(p)T_{\alpha\beta}(-p)\rangle = \left(f_0 - f_2(p^2)\right)\left(p_\mu p_\nu - p^2\delta_{\mu\nu}\right)\left(p_\alpha p_\beta - p^2\delta_{\alpha\beta}\right)$$
$$+3f_2(p)\left(p_\mu p_\nu p_\alpha p_\beta - 2p^2 p_{(\mu}\,\delta_{\nu)(\alpha}\,p_{\beta)} + p^4\delta_{\mu(\alpha}\delta_{\beta)\nu}\right)$$

where $f_0 = b/3,\ f_2(p^2) = -\,c\log(p^2/\Lambda)$

3-point function

Ansatz of the 3-point function

- Most general Ansatz depends on 2 independent momenta

$$\langle T_{\mu\nu}(p) T_{\alpha\beta}(k) T_{\rho\sigma}(q) \rangle = \sum_{i=1}^{137} C_i(p,k) \cdot \mathbb{T}_i(p,k), \qquad p + k + q = 0$$

→ C_i are functions of p^2, k^2 and $p_\mu k^\mu$
→ $\mathbb{T}_i$ are tensor structures of p, k

- $(\mu, \nu; p) \leftrightarrow (\alpha, \beta; k)$ symmetry reduces to 77 independent structures

$$(\mu, \nu; p) \leftrightarrow (\alpha, \beta; k) \qquad\qquad i, j \in \{1,2,3\}$$

Ward identities

1) Diff-Ward → 62 conditions

$$0 = 2q_{(\rho}\delta_{\sigma)}{}^{\mu}\,\langle T_{\mu\nu}(p)T_{\alpha\beta}(-p)\rangle + k^{\nu}\langle T_{\alpha\beta}(p)T_{\rho\sigma}(-p)\rangle + (p;\mu,\nu) \leftrightarrow (k;\rho,\sigma)$$
$$-\delta_{\rho\sigma}\,k^{\mu}\langle T_{\mu\nu}(p)T_{\alpha\beta}(-p)\rangle + (p;\mu,\nu) \leftrightarrow (k;\rho,\sigma) + 2\langle T_{\mu\nu}(q)T_{\alpha\beta}(p)T_{\rho\sigma}(k)\rangle$$

2) Weyl-Ward → 21 conditions

$$2\mathscr{A}''(p;k) = \langle T_{\alpha\beta}(p)T_{\rho\sigma}(-p)\rangle + \delta_{\alpha\beta}\delta^{\mu\nu}\langle T_{\mu\nu}(p)T_{\rho\sigma}(-p)\rangle + (p;\alpha,\beta) \leftrightarrow (k;\rho,\sigma)$$
$$+2\delta^{\mu\nu}\langle T_{\mu\nu}(q)T_{\alpha\beta}(p)T_{\rho\sigma}(k)\rangle$$

3) Dilatation identity

$$\mathscr{F}\int d^4x\, 6c\left(\sqrt{g}(Weyl)^2\right)''' = (p\cdot\partial_p + k\cdot\partial_k - 4)\langle T_{\mu\nu}(q)T_{\alpha\beta}(p)T_{\rho\sigma}(k)\rangle$$

Conclusions

- Derived dilatation identity, allowing derivation for all correlation functions

$$6c \int_M d^4x \sqrt{g} (Weyl)^2 = \int_M d^4x \sqrt{g} \langle T_{\mu\nu} \rangle \left(\nabla^{(\mu} x^{\nu)} - \frac{g^{\mu\nu}}{4} \nabla \cdot x \right)$$

- Corrections to 3-point Ward identities + homogeneous solutions are trivial

$$\left.\begin{aligned} q^\mu \langle T_{\mu\nu}(q) T_{\alpha\beta}(p) T_{\rho\sigma}(k) \rangle &= 0 \\ \delta^{\mu\nu} \langle T_{\mu\nu}(q) T_{\alpha\beta}(p) T_{\rho\sigma}(k) \rangle &= 0 \end{aligned}\right\} \implies \langle T_{\mu\nu}(q) T_{\alpha\beta}(p) T_{\rho\sigma}(k) \rangle = 0$$

- Revealed complexity in finding most general 3-point function
  $\rightarrow$ more efficient algorithms / computational power

Outlook

- c-theorem imposes a strong ordering principle

- Trace Anomaly gained more interest recently
  → alternative proof for a-theorem presented
  → new bounds on a, c in free CFT found

$$\frac{1}{3} \leq \frac{a}{c} \leq \frac{31}{18}$$

- Fundament for investigations on non-conformal field theories with

$$\mathscr{A} = g^{\mu\nu}\langle T_{\mu\nu}\rangle - \langle g^{\mu\nu}T_{\mu\nu}\rangle$$