Equivariant Localization in Supergravity and Holography

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based on a series of works
with Pietro Benetti Genolini, Chris Couzens,
Jerome Gauntlett, Yusheng Jiao, and James Sparks

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Motivation

$$\begin{array}{ccc} \text{SUGRA} & & \text{SCFT} \\ \mathrm{AdS}_d \times M_q & \longleftrightarrow & \mathbb{R}^{d-1} \end{array}$$

CFT observables: free energy, central charges, fluxes, ...

$$\updownarrow$$

Integrals over (subspaces of) the internal space

$$\mathcal{F} \sim \int_{M_q} \mathrm{vol}_{M_q}$$

Motivation

$$\begin{array}{ccc} \text{SUGRA} & & \text{SCFT} \\ Y_d \times \tilde{M}_q & \longleftrightarrow & W_{d-1} = \partial Y_d \end{array}$$

- Observables: on-shell action, free energy, entropy, fluxes, ...
- Typically reduce on the internal space (*truncate*)
  $\longrightarrow$ d-dim supergravity

$$I \sim \int_{Y_d} \mathrm{vol}_{Y_d}(R + \dots)$$

Motivation

$$\mathcal{F} \sim \int_M \mathrm{vol}_M\,, \qquad I \sim \int_Y \mathrm{vol}_Y(R + \dots)$$

- Looks like we need to know an explicit metric
- This would involve solving supergravity EOM $\longrightarrow$ Difficult!
- Is there a simpler way to get this result?

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$$\mathcal{F} \sim \int_M \mathrm{vol}_M\,, \qquad I \sim \int_Y \mathrm{vol}_Y (R + \dots)$$

- Looks like we need to know an explicit metric
- This would involve solving supergravity EOM $\longrightarrow$ Difficult!
- Is there a simpler way to get this result?

  $\longrightarrow$ Yes! Equivariant localization

  [Benetti Genolini, Gauntlett, Sparks '23]

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Equivariant Cohomology

- U(1) action with associated vector field ξ
- Equivariant exterior derivative

$$\mathrm{d}_\xi \equiv \underset{+1}{\mathrm{d}} - \underset{-1}{\xi \lrcorner}$$

- acts on polyforms

$$\Phi = \Phi_n + \Phi_{n-2} + \cdots + \Phi_0$$

- Φ is *equivariantly closed* if $\mathrm{d}_\xi \Phi = 0$

$$\mathrm{d}\Phi_n = 0 \qquad \xi \lrcorner \Phi_n = \mathrm{d}\Phi_{n-2} \qquad \dots \qquad \xi \lrcorner \Phi_2 = \mathrm{d}\Phi_0$$

Equivariant Localization

BVAB Theorem: [Berline, Vergne '82, Atiyah, Bott '84] The integral of an equivariantly closed form *localizes* to fixed point contributions

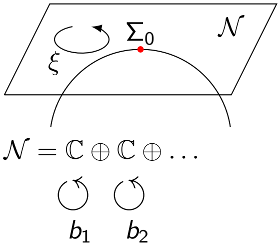
$$\int_{M_n} \Phi = \sum_{\Sigma_0} \frac{(2\pi)^k}{\prod_{i=1}^k b_i} \Phi_0\big|_{\Sigma_0} , \qquad n = 2k$$

- $\Sigma_0 \subset M_n$ fixed points set:

$$\xi|_{\Sigma_0} = 0$$

- b_i weights of $U(1)$ action:

$$\xi = \sum_{i=1}^k b_i \partial_{\varphi_i}$$



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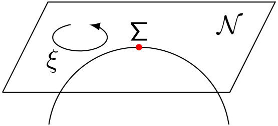
$$\xi = \sum_{i=1}^{k} b_i \partial_{\varphi_i} \, , \qquad \mathcal{N} = \underset{b_1}{L_1} \oplus \underset{b_2}{L_2} \oplus \ldots$$



Equivariant Localization

BVAB Theorem: [Berline, Vergne '82, Atiyah, Bott '84] The integral of an equivariantly closed form *localizes* to fixed point contributions

$$\int_{M_n} \Phi_n = \sum_{\Sigma_0} \text{“nut contrib.”}$$

$$+ \sum_{\Sigma_2} \text{“bolt contrib.”}$$

$\Longrightarrow$ Can be evaluated using Φ_0 and Φ_2

It circumvents the need to build an explicit solution to the EOM!

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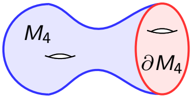
4d $\mathcal{N} = 2$ gauged supergravity

Minimal 4d gauged supergravity $\longleftrightarrow$ Einstein–Maxwell

$$I = -\frac{1}{16\pi G_4}\int_{M_4}\left(R + 6 - F^2\right)\mathrm{vol}_4$$
$$+ \cancel{\text{boundary terms}}$$

- Can add matter $\longrightarrow$ scalar & gauge fields
  Interactions specified by a *prepotential* function $\mathcal{F}$
- Dual to 3d SCFT on ∂M_4: $I = -\log Z$



Equivariant Localization

- Action

$$I = \frac{1}{8\pi G_4} \int_{M_4} \Phi_4$$

- Equivariantly closed form

$$\Phi = \Phi_4 + \Phi_2 + \Phi_0 \,, \qquad \mathrm{d}_\xi \Phi = 0$$

- Equivariant localization

$$I = \frac{\pi}{2G_4} \left\{ \sum_{\text{nuts}} \frac{\Phi_0}{b_1 b_2} + \sum_{\text{bolts}} \int_\Sigma \frac{\Phi_2}{2\pi b} - \frac{\Phi_0 c_1(L)}{b^2} \right\}$$

Minimal Result

On-shell action of minimal 4d gauged supergravity

[Benetti Genolini, Perez Ipiña, Sparks '19]

$$I = -\frac{\pi}{8G_4}\left[\sum_{\text{nuts}}\frac{(b_1-b_2)^2}{b_1 b_2} + \sum_{\text{bolts}}\left(\mathfrak{p} - \int_\Sigma c_1(L)\right)\right]$$

- Magnetic charges: $\mathfrak{p} = \frac{1}{2\pi}\int_\Sigma F$
- Adding matter straightforward: just introduces prepotential $\mathcal{F}$

Main Result

On-shell action of 4d gauged supergravity with *arbitrary* matter

[Benetti Genolini, Gauntlett, Jiao, **AL**, Sparks '24]

$$I = -\frac{\pi}{G_4}\left[\sum_{\text{nuts}} \frac{(b_1 - b_2)^2}{b_1 b_2} \mathrm{i}\mathcal{F}(u^J) + \sum_{\text{bolts}} \left(\partial_I \mathrm{i}\mathcal{F}(u^J)\mathfrak{p}^I - \mathrm{i}\mathcal{F}(u^J)\int_\Sigma c_1(L)\right)\right]$$

- $\mathcal{F}$ general prepotential (specifies the theory)
- u^J combinations of the scalar fields at the fixed points

$$\text{e.g. } \mathcal{F}_{\text{STU}}(u^I) = -2\mathrm{i}\sqrt{u^0 u^1 u^2 u^3}$$

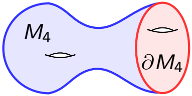
STU Result

On-shell action of 4d $\mathrm{U}(1)^4$ supergravity

[Benetti Genolini, Gauntlett, Jiao, **AL**, Sparks '24]

$$I = -\frac{2\pi}{G_4}\left[\sum_{\text{nuts}} \frac{(b_1 - b_2)^2}{b_1 b_2}\sqrt{u^0 u^1 u^2 u^3} \right. \\ \left. + \sum_{\text{bolts}} \sqrt{u^0 u^1 u^2 u^3}\left(\sum_{I=0}^{3} \frac{\mathfrak{p}^I}{u^I} - \int_\Sigma c_1(L)\right)\right]$$

- STU on $M_4 \longleftrightarrow$ dual to ABJM on ∂M_4
- Pick a topology for M_4: nuts and/or bolts



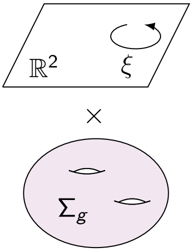
Euclidean Black Hole

$$\boxed{M_4 = \mathbb{R}^2 \times \Sigma_g}$$

Fixed pt set: Σ_g at the origin of $\mathbb{R}^2$

$$\implies I = \text{bolt contrib.}$$

$$= -\frac{\pi}{G_4}\sqrt{u^0 u^1 u^2 u^3}\sum_{I=0}^{3}\frac{\mathfrak{p}^I}{u^I}$$



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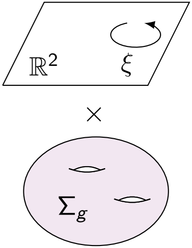
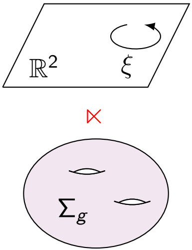


Black saddles $\longleftrightarrow$ ABJM on $S^1 \times \Sigma_g$

[Bobev, Charles, Min '20] [Benini, Hristov, Zaffaroni '15]

Rotating Black Hole

$$\boxed{M_4 = \mathcal{O}(p) \to \Sigma_g}$$



Fixed pt set: Σ_g at the origin of $\mathbb{R}^2$

$$\implies I = \text{bolt contrib. with } c_1 \neq 0$$

$$= -\frac{\pi}{G_4}\sqrt{u^0u^1u^2u^3}\left(\sum_{I=0}^{3}\frac{\mathfrak{p}^I}{u^I} - 2p\right)$$

Rotating black saddles $\longleftrightarrow$ ABJM on $S^1 \ltimes \Sigma_g$

Solution unknown! [Toldo, Willett '17]

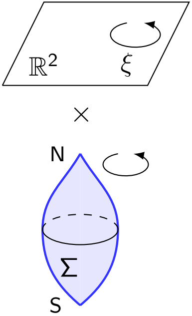
Accelerating Black Hole

$$\boxed{M_4 = \mathbb{R}^2 \times \text{spindle}}$$

Fixed pt set: 2 poles of the spindle

$$\Longrightarrow \quad I = \text{north nut} + \text{south nut}$$

$$= -\frac{2\pi}{G_4 b_0}\left[\sqrt{y_N^0 y_N^1 y_N^2 y_N^3} \pm \sqrt{y_S^0 y_S^1 y_S^2 y_S^3}\right]$$



Accelerating black saddles
[Crisafio, Fontanarossa, Martelli '24]
came after our result!

$\longleftrightarrow$

ABJM on $S^1 \times$spindle
[Colombo, Hosseini, Martelli, Pittelli, Zaffaroni '24]

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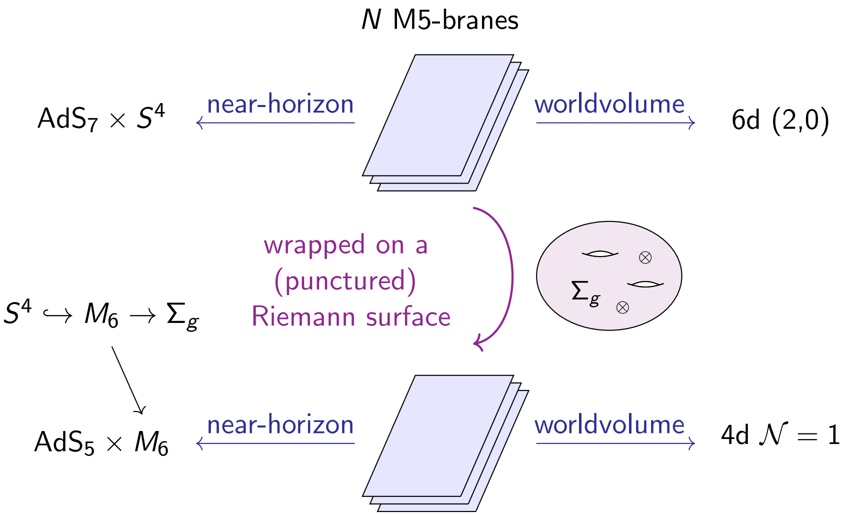
Motivation


N M5-branes
$\mathrm{AdS}_7 \times S^4$
near-horizon
worldvolume
6d (2,0)
wrapped on a
(punctured)
Riemann surface
Σ_g
$S^4 \hookrightarrow M_6 \to \Sigma_g$
$\mathrm{AdS}_5 \times M_6$
near-horizon
worldvolume
4d $\mathcal{N} = 1$

M-theory Setup

11d supergravity on $\mathrm{AdS}_5 \times M_6$ [Gauntlett, Martelli, Sparks, Waldram '04]

- Dual to 4d $\mathcal{N} = 1$ SCFT
- Central charge

$$a = \frac{1}{2(2\pi)^6 \ell_{\mathrm{p}}^9} \int_{M_6} \mathrm{e}^{9\lambda} \mathrm{vol}_{M_6}$$

Equivariant Localization

- Central charge

$$a = \frac{1}{2(2\pi)^6 \ell_{\rm p}^9} \int_{M_6} \Phi_6$$

- Equivariantly closed form

$$\Phi = \Phi_6 + \Phi_4 + \Phi_2 + \Phi_0 \,, \qquad \mathrm{d}_\xi \Phi = 0$$

- Equivariant localization

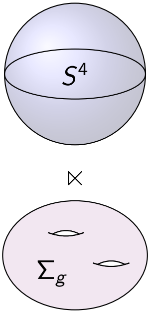
$$a = \frac{1}{2(2\pi)^3 \ell_{\rm p}^9} \Bigg\{ \sum_{\rm nuts} \frac{\Phi_0}{\epsilon_1 \epsilon_2 \epsilon_3} + \sum_{\rm bolts} \frac{1}{\epsilon_1 \epsilon_2} \int_\Sigma \frac{\Phi_2}{2\pi} - \Phi_0 \left(\frac{c_1(L_1)}{\epsilon_1} + \frac{c_1(L_2)}{\epsilon_2} \right) \Bigg\}$$

Smooth Riemann surface

[Benetti Genolini, Gauntlett, Sparks '23]

$$M_6 = S^4 \xrightarrow[\mathcal{O}(p_1)\oplus\mathcal{O}(p_2)]{} \Sigma_g \,, \qquad p_1 + p_2 = 2(g-1)$$

$$\xi = b_1 \partial_{\varphi_1} + b_2 \partial_{\varphi_2} \,, \qquad b_1 + b_2 = 1$$

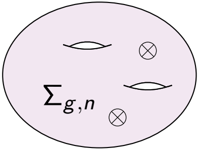


Fixed pt set: Σ_g at the poles of S^4

$$\implies a^{\Sigma_g} = \text{bolt contrib.}$$

$$= -\frac{9}{8} b_1 b_2 (b_1 p_2 + b_2 p_1) N^3$$ [Bah, Beem, Bobev, Wecht '12]

Adding Punctures



$$a^{\Sigma_{g,n}} = a^{\Sigma_g} - \sum_{I=1}^{n} \delta a^I$$

Compute puncture contrib. δa with equivariant localization

$\longrightarrow$ Completely **new** results! [Couzens, **AL**, Sparks '25]

Summary

- Lower dimensions
  - 4d [Benetti Genolini, Gauntlett, Jiao, **AL**, Sparks '24]
  - 5d [Cassani, Ruiperez, Turetta '24, Colombo, Dimitrov, Martelli, Zaffaroni '25, Benetti Genolini, Gauntlett, Jiao, Park, Sparks '25]
  - 6d [Couzens, Gregory, Muniz, Sieper, Sparks '25]
  - near-horizon [Benetti Genolini, Gauntlett, Jiao, **AL**, Sparks '24, Suh '24]
- Internal spaces
  - M-theory on $\mathrm{AdS}_5 \times M_6$ [Benetti Genolini, Gauntlett, Sparks '23]
  - Massive type IIA on $\mathrm{AdS}_4 \times M_6$ [Couzens, **AL** '24]
  - Type IIB $\mathrm{AdS}_3 \times M_7$ [Couzens, **AL**, Sparks '25]
- Orbifold geometries
  - Punctured Riemann surface [Couzens, **AL**, Sparks '25 + to appear]
  - Other punctured manifolds
  - Defects

Outlook

Take home message

Equivariant localization is a powerful tool to compute holographic observables in supergravity without needing the explicit solution.

Many interesting open questions

- Full 11d spaces
- No supersymmetry
- Higher-derivative corrections

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Thank you!
Questions?

Appendix

Supergravity Applications

- Supergravity EOM $\longleftrightarrow$ Killing spinor equations

$$(\nabla + \dots)\epsilon = 0$$

- Spinor bilinears $\Rsh$ dual to Killing vect. ξ

$$S = \bar{\epsilon}\epsilon\,, \quad P = \bar{\epsilon}\gamma_*\epsilon\,, \quad \xi^\flat = \bar{\epsilon}\gamma_\mu\gamma_*\epsilon\mathrm{d}x^\mu$$

$\longrightarrow$ follow constraints coming from KSE

- Tool to build equiv. closed forms $(\mathrm{d} - \xi)\Phi = 0$

e.g. $\mathrm{d}P = \xi \lrcorner F \quad \Longrightarrow \quad \Phi = F - P$ equiv. closed

- **Game:** build such forms whose top form part is an obs.

Supergravity Applications

Applications to supergravity [Benetti Genolini, Gauntlett, Sparks '23]

- Supersymmetry implies the existence of a U(1) symmetry
- Use *Killing spinor equations* and *spinor bilinears* to build equiv. closed forms $\longrightarrow$ once
- Localize on various topologies $\longrightarrow$ different sets of fixed points

$$\text{Obs.} = \int_{M_n} \Phi_n = \sum_{\substack{\text{fixed} \\ \text{points}}} \Phi_0$$