Spectral Networks

& their role in 2d conformal field theory

Subrabalan Murugesan
Heriot-Watt University, Edinburgh

November 25, 2025
Quantum Universe Attract Workshop

Background

Set up & conventions

Let C be a genus g Riemann surface with $n \geq 1$ punctures, and Σ be a branched double cover,

$$\pi : \Sigma \xrightarrow{2:1} C,$$

such that it can be embedded into the cotangent bundle,

$$\iota : \Sigma \hookrightarrow T^*C.$$

Set up & conventions

Let C be a genus g Riemann surface with $n \geq 1$ punctures, and Σ be a branched double cover,

$$\pi : \Sigma \xrightarrow{2:1} C,$$

such that it can be embedded into the cotangent bundle,

$$\iota : \Sigma \hookrightarrow T^*C.$$

We fix a trivialisation of the cover for convenience. We may then label the sheets of Σ. If $\gamma \subset C$ is a (real) curve, then we denote by γ_i the lift of γ to the i^{th} sheet of Σ.

Definition

Fix a phase $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.

Definition

An ij trajectory is a curve $w \subset C$ such that

$$e^{-i\theta} \int_{w_{ij}} \iota^* \lambda \in \mathbb{R}_{\geq 0} \tag{1}$$

where $w_{ij} := w_i - w_j$ and $\lambda \in \Omega^1(T^*C)$ is the canonical 1-form.

Definition

Fix a phase $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.

Definition

An ij trajectory is a curve $w \subset C$ such that

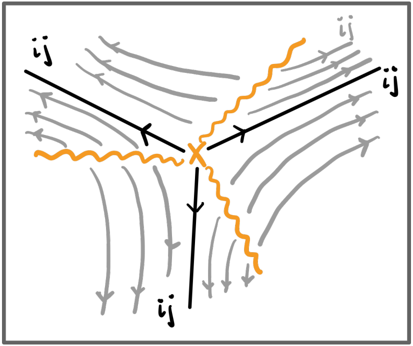
$$e^{-i\theta} \int_{w_{ij}} \iota^* \lambda \in \mathbb{R}_{\geq 0} \tag{1}$$

where $w_{ij} := w_i - w_j$ and $\lambda \in \Omega^1(T^*C)$ is the canonical 1-form.

This defines a foliation of C (known as the **horizontal foliation** or WKB foliation).

Definition

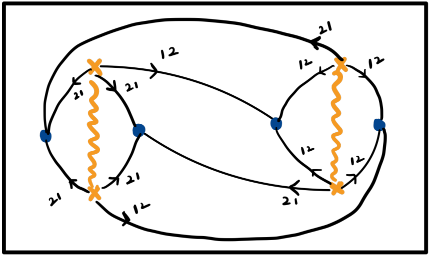
At the branch points of Σ, the foliation has a three-pronged structure. These are known as the critical ij trajectories.



We orient all the critical ij trajectories away from the branch point.

Definition

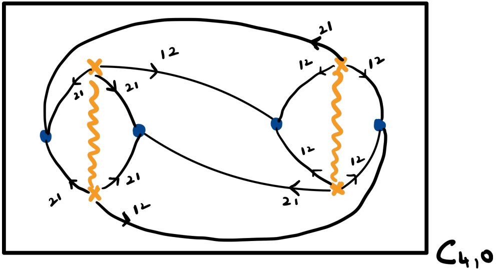
Then, the collection of all the oriented critical ij trajectories, the branch points, and the ij labels is known as the **WKB spectral network** $\mathcal{W}_\theta(\Sigma)$.



$C_{4,0}$

Definition

Then, the collection of all the oriented critical ij trajectories, the branch points, and the ij labels is known as the **WKB spectral network** $\mathcal{W}_\theta(\Sigma)$.



Working definition

A spectral network $\mathcal{W}$ is a directed graph on C with a trivalent vertex at each branch point of Σ, and all the edges terminate on either punctures or branch points.

The 4d-2d connection

BPS Spectra

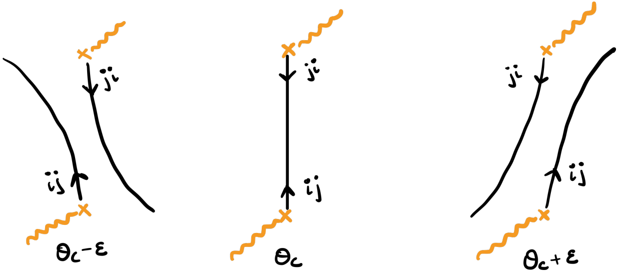
Spectral networks were originally conceived to capture the various BPS spectra, and the wall-crossing phenomena of 4d $\mathcal{N} = 2$ supersymmetric theories of type class S.

The surface C is the moduli space of (canonical) surface defects in the 4d theory, and Σ is the space of classical vacua of the coupled 2d-4d system.

The topology of the spectral network encodes the instanton effects, i.e. the BPS spectrum of the coupled 2d-4d system. [GMN13], [BHM25]

BPS Spectra

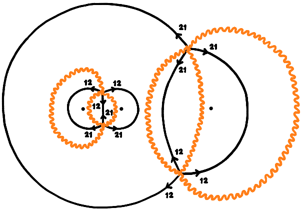
The degenerate topologies of the network encode the BPS spectrum of the 4d theory.



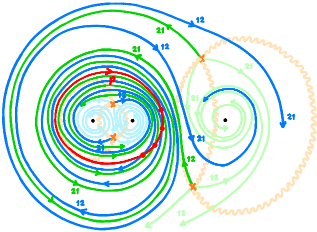
At certain phases, two critical trajectories may coincide.

BPS Spectra

Our interest lies in the maximally degenerate network called the *Fenchel-Nielsen network*.



(a) A Fenchel-Nielsen network



(b) When the phase is slightly perturbed

This is dual to a pants decomposition of C, and is most natural from the perspective of the 4d theory.

AGT Correspondence

The AGT correspondence [AGT10] states that 4d $\mathcal{N} = 2$ $SU(2)$ gauge theories are dual to Liouville theory on C.

AGT Correspondence

The AGT correspondence [AGT10] states that 4d $\mathcal{N} = 2$ $SU(2)$ gauge theories are dual to Liouville theory on C.

Definition

Liouville theory is a 2d conformal field theory whose operator spectrum is given by

$$\mathcal{H} = \int d\alpha \, \mathcal{V}_\alpha \otimes \overline{\mathcal{V}}_\alpha, \quad \Delta(\alpha) = \alpha(Q - \alpha)$$

where $\mathcal{V}_\alpha$ and $\overline{\mathcal{V}}_\alpha$ are (chiral) Virasoro Verma modules with conformal dimension $\Delta(\alpha)$, and the central charge of the theory is $c = 1 + 6Q^2$.

Conformal blocks

Fix points $z_1, \ldots, z_n \in \mathbb{P}^1$, and a representation $\mathcal{V}_{\alpha_i}$ attached to each z_i.

Definition

A genus zero (holomorphic) **conformal block** is a linear map

$$\mathcal{Z} : \mathcal{V}_{\alpha_1} \otimes \ldots \otimes \mathcal{V}_{\alpha_n} \to \mathbb{C}$$

that satisfies all the holomorphic Ward identities.

The space of all conformal blocks defines a **projectively flat vector bundle** on the moduli space of $\mathbb{P}^1$ with n marked points.

Conformal blocks

Fix points $z_1, \ldots, z_n \in \mathbb{P}^1$, and a representation $\mathcal{V}_{\alpha_i}$ attached to each z_i.

Definition

A genus zero (holomorphic) **conformal block** is a linear map

$$\mathcal{Z} : \mathcal{V}_{\alpha_1} \otimes \ldots \otimes \mathcal{V}_{\alpha_n} \to \mathbb{C}$$

that satisfies all the holomorphic Ward identities.

The space of all conformal blocks defines a **projectively flat vector bundle** on the moduli space of $\mathbb{P}^1$ with n marked points.

It is possible to extend this definition to arbitrary surfaces.

Conformal blocks

Conformal blocks only depend on the representation theory of the modules $\mathcal{V}_{\alpha_i}$, and not on any physical data.

Liouville conformal blocks *quantise* the moduli space of flat $SL(2,\mathbb{C})$ connections on C.

We can write down integral representations for conformal blocks given a suitable choice of contours on C. [DF84, CPT19]

Role of spectral networks

A matrix model - inspired argument shows that spectral networks can supply the required contours for these integral representations. [CDV11, HN24]

Role of spectral networks

A matrix model - inspired argument shows that spectral networks can supply the required contours for these integral representations. [CDV11, HN24]

In particular, the integral representations $\mathcal{Z}_{\mathcal{W}_{\rm FN}}$ defined using Fenchel-Nielsen networks reproduce the *standard* Liouville conformal blocks. [HM*]

Other spectral networks define conformal blocks in non-standard bases. [GS19]

Role of spectral networks

A matrix model - inspired argument shows that spectral networks can supply the required contours for these integral representations. [CDV11, HN24]

In particular, the integral representations $\mathcal{Z}_{\mathcal{W}_{\mathrm{FN}}}$ defined using Fenchel-Nielsen networks reproduce the *standard* Liouville conformal blocks. [HM*]

Other spectral networks define conformal blocks in non-standard bases. [GS19]

These integral representations can also be interpreted as the Borel sum of the asymptotic expansion in the direction θ,

$$\mathcal{Z}_{\mathcal{W}_\theta} = \mathcal{B}_\theta \left(\mathcal{Z}^{\mathrm{as}} \right).$$

Quantum invariants

Hitchin systems

The relation between Hitchin systems and 4d $\mathcal{N} = 2$ theories lets us define the $\mathcal{W}-$abelianisation and $\mathcal{W}-$nonabelianisation maps,

$$\mathcal{M}^{\mathcal{W}}(C, SL(2,\mathbb{C})) \underset{\text{nab}}{\overset{\text{ab}}{\rightleftharpoons}} \mathcal{M}^{\text{al}}(\Sigma, GL(1,\mathbb{C})),$$

between the moduli spaces of flat connections on the base and the cover. The spectral network plays the role of a (generalised) triangulation of C. [GMN13, HN16]

Hitchin systems

The relation between Hitchin systems and 4d $\mathcal{N} = 2$ theories lets us define the $\mathcal{W}-$abelianisation and $\mathcal{W}-$nonabelianisation maps,

$$\mathcal{M}^{\mathcal{W}}(C, SL(2,\mathbb{C})) \underset{\text{nab}}{\overset{\text{ab}}{\rightleftharpoons}} \mathcal{M}^{\text{al}}(\Sigma, GL(1,\mathbb{C})),$$

between the moduli spaces of flat connections on the base and the cover. The spectral network plays the role of a (generalised) triangulation of C. [GMN13, HN16]

The canonical quantisation of the moduli space of flat $G-$connections on a surface S is the skein algebra $\text{SkAlg}(C, G)$.

Knot invariants

For non-degenerate networks, the $\mathcal{W}-$nonabelianisation map can be refined into a $q-$nonabelianisation map [NY20, Gab17]

$$\mathrm{SkAlg}(C, SL(2,\mathbb{C})) \to \mathrm{SkAlg}(\Sigma, GL(1,\mathbb{C})).$$

Knot invariants

For non-degenerate networks, the $\mathcal{W}-$nonabelianisation map can be refined into a $q-$nonabelianisation map [NY20, Gab17]

$$\mathrm{SkAlg}(C, SL(2,\mathbb{C})) \to \mathrm{SkAlg}(\Sigma, GL(1,\mathbb{C})).$$

Under certain restrictions, such a homomorphism also exists for degenerate networks. [HM*]

For Fenchel-Nielsen networks, this can be explicitly verified via the connection to Liouville theory.

Outlook

Other keywords I could/should have used:

- Chern-Simons theory
- Toda theories
- Quantum groups
- Wall-crossing phenomena
- 3d-3d duality

Thank you!

Luis F. Alday, Davide Gaiotto, and Yuji Tachikawa.
Liouville Correlation Functions from Four-dimensional Gauge Theories.
Lett. Math. Phys., 91:167–197, 2010.

Loïc Bramley, Lotte Hollands, and Subrabalan Murugesan.
Les Houches lectures on non-perturbative Seiberg-Witten geometry.
3 2025.

Miranda C. N. Cheng, Robbert Dijkgraaf, and Cumrun Vafa.
Non-Perturbative Topological Strings And Conformal Blocks.
JHEP, 09:022, 2011.

Ioana Coman, Elli Pomoni, and Jörg Teschner.
Toda conformal blocks, quantum groups, and flat connections.
Commun. Math. Phys., 375(2):1117–1158, 2019.

V. S. Dotsenko and V. A. Fateev.
Conformal Algebra and Multipoint Correlation Functions in Two-Dimensional Statistical Models.
Nucl. Phys. B, 240:312, 1984.

Maxime Gabella.
Quantum Holonomies from Spectral Networks and Framed BPS States.
Commun. Math. Phys., 351(2):563–598, 2017.

Davide Gaiotto, Gregory W. Moore, and Andrew Neitzke.
Wall-crossing, Hitchin systems, and the WKB approximation.
Adv. Math., 234:239–403, 2013.

Alexander Goncharov and Linhui Shen.
Quantum geometry of moduli spaces of local systems and representation theory.
arXiv preprint arXiv:1904.10491, 2019.

Lotte Hollands and Andrew Neitzke.
Spectral Networks and Fenchel–Nielsen Coordinates.
Lett. Math. Phys., 106(6):811–877, 2016.

Qianyu Hao and Andrew Neitzke.
A new construction of $c = 1$ Virasoro blocks.
7 2024.

Andrew Neitzke and Fei Yan.
q-nonabelianization for line defects.
JHEP, 09:153, 2020.