Generalized Freed-Witten Anomalies

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Revisiting the ABJ anomaly

- Consider a 4d Dirac fermion on a closed spin 4-manifold X^4:

$$S = \int d^4x \, i\bar{\psi} \not{D} \psi$$

- Vector and axial U(1) symmetries:

$$\mathrm{U}(1)_V : \psi \mapsto e^{2\pi i \alpha} \psi \,, \qquad \mathrm{U}(1)_A : \psi \mapsto e^{2\pi i \beta \gamma_5} \psi$$

- Partition function vanishes if there are *zero modes*:

$$\mathcal{Z}[A_V] = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \, e^{-S} = \det(i\not{D}) = \prod_i \lambda_i$$

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- Atiyah-Singer index theorem:

$$\mathrm{ind}(i\not{D}) := n_+ - n_- = \int_{X^4} \left(\frac{1}{2} F_V^2 - \frac{1}{48} \mathrm{Tr}(R^2) \right)$$

- If the *RHS* is nontrivial, then the index is nonzero
  - $\exists$ some zero modes $\implies$ $\mathcal{Z}[A_V]$ vanishes

Mapping cylinder perspective

- There's a mixed anomaly between $\mathrm{U}(1)_V$ and $\mathrm{U}(1)_A$:

$$\mathcal{A}[\underline{A}_V, \underline{A}_A] = \int_{Y^5} \left(\frac{1}{2} \underline{F}_V^2 - \frac{1}{48} \mathrm{Tr}(\underline{R}^2) \right) \wedge \underline{A}_A$$

- It's an anomaly in the sense that

$$\mathcal{Z}[A_V + d\alpha, A_A + d\beta] = e^{2\pi i \mathcal{A}[\underline{A}_V, \underline{A}_A]} \mathcal{Z}[A_V, A_A]$$

- Place the 5d anomaly theory on the *mapping cylinder* $X^4 \times [0,1]$:
  - $\iota_0^*(\underline{A}_V) = A_V$, $\iota_1^*(\underline{A}_V) = A_V + d\alpha$, similarly for $\underline{A}_A$

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  - Let's set $\alpha = 0$ and $A_A = 0$, but allow β to be a nonzero *constant*, s.t.

$$\mathcal{Z}[A_V, 0] = e^{2\pi i \mathcal{A}[\underline{A}_V, \underline{A}_A]} \mathcal{Z}[A_V, 0]$$

  - Bulk ansatz: $\underline{A}_V = A_V$, $\underline{A}_A = \beta dt$

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Mapping cylinder perspective

- There's a mixed anomaly between $U(1)_V$ and $U(1)_A$:

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  - $\mathcal{Z}[A_V, 0]$ must *vanish* if $\left[\frac{1}{2} \underline{F}_V^2 - \frac{1}{48} \mathrm{Tr}(\underline{R}^2) \right] \neq 0 \in H^4_{\mathrm{dR}}(X^4)$

Anomaly-induced vanishing of partition functions

- A straightforward evaluation of the anomaly theory on the mapping cylinder gives the *same conclusion* from the highly nontrivial AS index theorem!
- Similar story with the vanishing of the RR partition function of a 2d Dirac fermion using the mod 2 index theorem
- Somewhat circular, but index theorems are not always easily accessible, whereas the *anomaly theory* is straightforward to determine for many QFTs

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- In our paper, we developed the second perspective in greater detail, and also in the language of differential cohomology
  - Examples: (Generalized) Maxwell theory, Dijkgraaf-Witten theory, 4d $\mathfrak{u}(N)$ Yang-Mills theory, 2d chiral boson etc.
  - The formalism can be applied even to *strongly coupled QFTs* without Lagrangian descriptions

Moduli space description

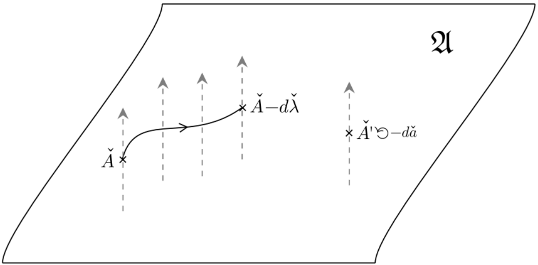

Figure 2: The partition function $\mathcal{Z}[\check{A}]$ is a section of a line bundle $\mathcal{L}$ over the space of gauge fields, $\mathfrak{A} \simeq \check{Z}^{p+2}(X^d)$. A 't Hooft anomaly is a phase $e^{2\pi i \mathcal{A}[\check{A},\check{\lambda},0]}$ acquired by $\mathcal{Z}[\check{A}]$ as we move along a non-trivial path $\check{A} \to \check{A} - d\check{\lambda}$ for some $\check{\lambda} \in \check{C}^{p+1}_{\text{flat}}(X^d)$. If the phase is trivial, then $\mathcal{L}$ can be lifted to a line bundle over $\mathfrak{A}/\mathfrak{G} \simeq \check{H}^{p+2}(X^d)$, where points along a given gauge orbit on $\mathfrak{A}$ are identified. On the other hand, a basepoint anomaly is a phase $e^{2\pi i \mathcal{A}[\check{A}',0,\check{a}]}$ acquired by $\mathcal{Z}[\check{A}']$ as we act on a fixed $\check{A}'$ with an automorphism $\check{A}' \to \check{A}' - d\check{a} = \check{A}'$ for some $\check{a} \in \check{Z}^{p+1}_{\text{flat}}(X^d)$. If such a phase is non-trivial, then $\mathcal{Z}[\check{A}']$ must vanish at this point on $\mathfrak{A}$.

Can be more naturally phrased in the language of *moduli stacks* (wip)

Application to branes

- Worldvolume theory of D3-brane in Type IIB string theory:
  - Partition function is non-vanishing only if

$$[H_3] + [W_3] = 0\,, \qquad [G_3] + [W_3] = 0$$

  - $H_3 \sim dB_2$ is the 3-form NSNS flux, $G_3 \sim dC_2$ is the 3-form RR flux, and $W_3 = \beta(w_2)$ is the third integral Stiefel-Whitney class
  - Precisely the *Freed-Witten anomaly cancellation condition*!

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- Worldvolume theory of M5-brane:
  - Partition function on spin 6-manifolds is non-vanishing only if

$$[G_4] - \frac{1}{4}[p_1] = 0$$

  - Consistent with Witten's *M-theory 4-form flux quantization*!
  - We derived the analogues for more general tangential structures

Further applications

- D3-brane can be obtained by reducing M5-brane over T^2:
  - 6d worldvolume of M5-brane: $T^2 \hookrightarrow X^6 \to X^4$
  - For trivial fibrations, reduction of the M-theory 4-form flux quantization condition reproduces the FW anomaly cancellation condition

Further applications

- D3-brane can be obtained by reducing M5-brane over T^2:
  - 6d worldvolume of M5-brane: $T^2 \hookrightarrow X^6 \to X^4$
  - For trivial fibrations, reduction of the M-theory 4-form flux quantization condition reproduces the FW anomaly cancellation condition
- We can also study D3-branes in generic F-theory backgrounds:
  - Make X^6 a nontrivial fibration, possibly with a nontrivial local system
  - Case study: 4d $\mathcal{N} = 3$ S-folds
    - *Non-perturbative* generalizations of orientifolds
- Similar considerations for Class $\mathcal{S}$ theories:
  - Take $\Sigma_g^2 \hookrightarrow X^6 \to X^4$