Deformation quantization of double Poisson brackets

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Quantum Universe Attract Workshop

Introduction

Kontsevich, 1993 & Kontsevich + Rosenberg, 1998:

A noncommutative structure of some kind on A should give an analogous "commutative" structure on all schemes $\mathrm{Rep}_N(A)$ for $N \geq 1$.

Van den Bergh, 2004:

$$A \text{ with } \{\!\{-,-\}\!\} \quad \overset{\mathrm{Rep}}{\rightsquigarrow} \quad \mathrm{Rep}_N(A) \text{ with } \{-,-\}_N$$

Calaque, 2010: How to quantize double Poisson brackets?

Double Poisson brackets

Definition 1 (Van den Bergh, 2004)

A *double Poisson bracket* on A is a linear map $\{\!\{-,-\}\!\} : A \otimes A \longrightarrow A \otimes A$ such that

- $\{\!\{a,b\}\!\} = -(12)\{\!\{b,a\}\!\}$, (cyclic antisymmetry)
- $\{\!\{a,-\}\!\} \in \mathrm{Der}(A, A \otimes A)$
- $\mathbb{J}ac(a,b,c) = 0$, (double Jacobi identity)

$$\mathrm{Rep}_N(A) = \text{algebra homomorphisms } A \longrightarrow \mathrm{Mat}_N(\Bbbk)$$

Theorem 2 (Van den Bergh, 2004)

Let $\{\!\{-,-\}\!\}$ be a double Poisson bracket on A. There is a GL_N-invariant Poisson bracket on $\mathcal{O}(\mathrm{Rep}_N(A))$.

Quantizations: commutative vs noncommutative

- $\mathcal{A}$ commutative algebra $m : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ with a Poisson bracket $\{-,-\} : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$

- A quantization of $\{-,-\}$ is an associative linear map

$$\star : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}[[\hbar]]$$

  such that $\star = m + O(\hbar)$ and $[-,-]_\star = \{-,-\}\hbar + O(\hbar^2)$.

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- A associative algebra $m : A \otimes A \longrightarrow A$ with a double Poisson bracket $\{\!\{-,-\}\!\} : A \otimes A \longrightarrow A \otimes A$
- A quantization of $\{\!\{-,-\}\!\}$ is an "associative" collection of linear maps

$$\star\star_{n,m} \colon A^{\otimes n} \otimes A^{\otimes m} \longrightarrow \Big(A^{\otimes n+m} \otimes \mathrm{S}(A_\natural) \otimes \Bbbk[S(n+m)]\Big)[[\hbar]]$$

where $A_\natural = {}^{A}\!\big/\!_{[A,A]}$ and $S(n+m)$ is the symmetric group.

Definition 3 (S., 2025)

Sketch: A quantization of a double Poisson bracket $\{\!\{-,-\}\!\}$ is a collection of linear maps

$$\star\star_{n,m} \colon A^{\otimes n} \otimes A^{\otimes m} \longrightarrow \Big(A^{\otimes n+m} \otimes \mathrm{S}(A_\natural) \otimes \Bbbk[S(n+m)]\Big)[[\hbar]]$$

such that

1. $\star\star_{n,m} = (\text{concatenation } A^{\otimes n} \otimes A^{\otimes m} \longrightarrow A^{\otimes n+m}) + O(\hbar)$
2. $\star\star_{1,1} - (12) \star\star_{1,1} (12) = \Big(\{\!\{-,-\}\!\} \otimes \mathbb{1} \otimes (12)\Big)\hbar + O(\hbar^2)$

   $+$ higher analogs for $\star\star_{n,m}$ with $n, m \geq 1$
3. "associativity":

$$\big(\alpha \star\star_{n,m} \beta\big) \star\star_{n+m,k} \gamma = \alpha \star\star_{n,m+k} \big(\beta \star\star_{n+m,k} \gamma\big)$$

4. extra conditions pairing $\star\star_{n,m}$ with the multiplication in A

The KR principle and the quantization formula

- $\{-,-\}_N$ the Poisson bracket on $\mathcal{O}(\mathrm{Rep}_N(A))$ induced by a double Poisson bracket $\{\!\{-,-\}\!\}$

Theorem 4 (S., 2025)

Any quantization $\star\star$ of the double Poisson bracket $\{\!\{-,-\}\!\}$ canonically induces a (commutative) GL_N-invariant quantization of $\{-,-\}_N$.

Quantizations of double Poisson brackets satisfy the Kontsevich-Rosenberg principle!

Kontsevich, 1997: quantization formula for $\mathbb{R}^d$

Theorem 5 (S.,2025)

Any double Poisson bracket on $A = \Bbbk\langle x_1, \dots, x_d\rangle$ admits a quantization.

Double Hochschild cochain complex

- Associative algebra $\mathcal{A}$
- Shifted Hochschild cochain complex $C^\bullet(\mathcal{A},\mathcal{A})[1]$
- Deformations of $m : \mathcal{A}\otimes\mathcal{A} \longrightarrow \mathcal{A} \iff MC\Big(C^\bullet(\mathcal{A},\mathcal{A})[1]\Big)$

Theorem 6 (S., 2025)

There is a dg Lie algebra $\mathbb{C}^\bullet(A,A)$ such that

$$\textit{double deformations}^1 \textit{ of } A \iff MC\big(\mathbb{C}^\bullet(A,A)\big)$$

and there is a homomorphism of dg Lie algebras

$$\mathbb{C}^\bullet(A,A) \longrightarrow C^\bullet\big(\mathcal{A}_N,\mathcal{A}_N\big)[1]; \qquad \mathcal{A}_N = \mathcal{O}(\mathrm{Rep}_N(A))$$

[1] Definition 3 above without 2) = without fixing the double bracket

Double Formality Theorem

Ginzburg, Schedler, 2010: Differential operators satisfying KR.

S.,2025: poly-differential operators satisfying KR+ $\mathscr{C}^{\bullet}_{\mathrm{diff}}(A,A)$.

Theorem 7 (S.,2025)

Let $A = \Bbbk\langle x_1, \ldots, x_d \rangle$. There is an L_∞-quasi-isomorphism $\mathscr{U}$ making the following diagram commute

$$\begin{array}{ccc} H\big(\mathscr{C}^{\bullet}_{\mathrm{diff}}(A,A)\big) & \overset{\mathscr{U}}{\rightsquigarrow} & \mathscr{C}^{\bullet}_{\mathrm{diff}}(A,A) \\ \downarrow & & \downarrow \\ H\big(C^{\bullet}_{\mathrm{diff}}(\mathcal{A}_N,\mathcal{A}_N)[1]\big) & \overset{\mathcal{U}}{\rightsquigarrow} & C^{\bullet}_{\mathrm{diff}}(\mathcal{A}_N,\mathcal{A}_N)[1], \end{array}$$

where $\mathcal{A}_N = \mathcal{O}(\Bbbk^{dN^2})$ and $\mathcal{U}$ is the L_∞-quasi-isomorphism constructed by M.Kontsevich in his seminal paper in 1997.