The Skein Partition Function

joint work with D. Jordan, M. Vancraeynest and M. Vazirani

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Table of Contents

Table of Contents

Introduction

Skein modules are vector spaces

Introduction

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Conjecture (Witten) : Skein modules of closed 3-manifolds are finite dimensional

Introduction

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Skein modules of tori (Gunningham, Jordan, Vazirani [3], 2024)

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Proof (Gunningham, Jordan, Safronov [2], 2023)

Skein modules of tori (Gunningham, Jordan, Vazirani [3], 2024)

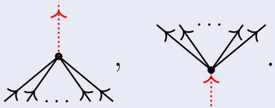
Skein modules of mapping tori (Kinnear [4], 2024)

Table of Contents

Skein modules

Definition

A **basic** GL_N**-skein** in an orientable 3-manifold M is an oriented, embedded ribbon graph, where the edges are either solid (black) or dashed (red) and where each vertex is one of the following $(N+1)$-valent vertices



The GL_N**-skein module** $\mathrm{Sk}_{\mathrm{GL}_N}(M)$, is the $\mathbb{C}(q)$-span of the isotopy classes of basic GL_N-skeins embedded in M, modulo the skein relations applied in embedded cylinders $D^2 \times I \hookrightarrow M$.

Skein relations

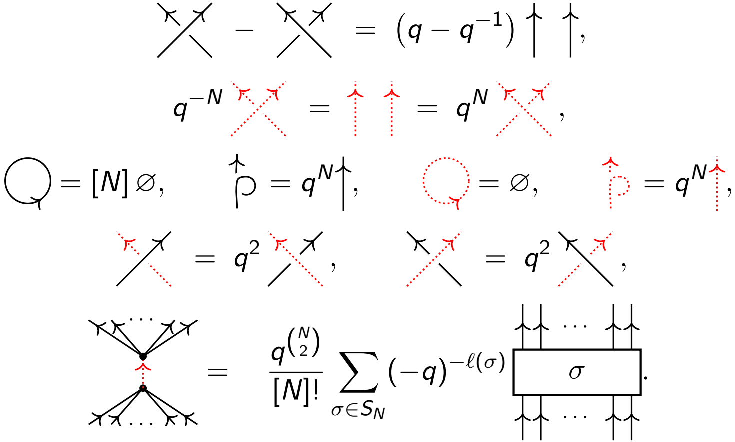


Table of Contents

Mapping Tori

Definition

The **mapping torus** of diffeomorphism $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ of the 2-torus T^2 is defined by

$$M_\gamma := T^2 \times I \Big/ (x,0) \sim (\gamma(x),1).$$

Table of Contents

The GL_N Skeins Partition Function

Definition

The *Skeins Partition Function* of a 3-manifold M is

$$\mathcal{Z}_M(t) := 1 + \sum_{N=1}^{\infty} \dim \mathrm{Sk}_{\mathrm{GL}_N}(M) \cdot t^N.$$

Theorem (B., Jordan, Vancraeynest, Vazirani 2025 [1])

For $\gamma \in \mathrm{SL}_2(\mathbb{Z})$*, the Skeins Partition Function of* M_γ *admits the following expansion:*

$$\mathcal{Z}_{M_\gamma}(t) = \prod_{k=1}^{\infty}(1 - t^k)^{-c_k},$$

$$\text{where } c_k = \frac{1}{k}\sum_{d|k}\phi\left(\frac{k}{d}\right)|\operatorname{tr}(\gamma^d) - 2| \text{ for generic } \gamma.$$

Examples

γ	$c_k(\gamma)$ for $k = 1, 2, 3, \ldots$	$\dim \mathrm{Sk}_{\mathrm{GL}_N}(M_\gamma)$ for $N = 1, 2, 3, \ldots$
$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$	2, 7, 18, 52, 146, 463, ...	2, 10, 36, 142, 520, 1980, 7344, ...
Id	1, 1, 1, 1, 1, 1, 1, 1, 1, ...	1, 2, 3, 5, 7, 11, 15, 22, 30, ...
$-$ Id	4, 1, 4, 1, 4, 1, 4, 1, 4, ...	4, 11, 28, 63, 132, 264, 504, 928, ...
S	2, 3, 2, 1, 2, 3, 2, 1, 2, ...	2, 6, 12, 25, 46, 86, 148, 255, 420, ...
TS	1, 2, 2, 2, 1, 1, 1, 2, 2, ...	1, 3, 5, 10, 15, 27, 40, 66, 97, ...
$-TS$	3, 3, 1, 3, 3, 1, 3, 3, 1, ...	3, 9, 20, 45, 90, 176, 324, 585, ...
T	1, 2, 3, 4, 5, 6, 7, 8, 9, ...	1, 3, 6, 13, 24, 48, 86, 160, 282, ...
$-T$	4, 2, 4, 3, 4, 4, 4, 5, 4, ...	4, 12, 32, 77, 172, 366, 744, 1460, ...
T^2	2, 4, 6, 8, 10, 12, 14, ...	2, 7, 18, 47, 110, 258, 568, 1237, ...

Table of Contents

Skein partition function and DT-invariants theory

Conjecture

Cohomological Donaldson-Thomas invariants $\simeq$ Skein modules

In the GL_N-Partition function of mapping tori, $c_k > 0$. Same has been proven for cohomological DT-invariants.

Conjecture

The skein partition function of any closed oriented 3-manifold admits an Euler expansion with non-negative exponents c_k.

Thank you!

References

[1] Julia Bierent et al. *The skein partition function of the mapping torus*. 2025. arXiv: 2511.19139 [math.QA]. URL: https://arxiv.org/abs/2511.19139.

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[3] Sam Gunningham, David Jordan, and Monica Vazirani. "Skeins on tori". In: (2024). arXiv: 2409.05613 [math.QA]. URL: https://arxiv.org/abs/2409.05613.

[4] Patrick Kinnear. "Skein module dimensions of mapping tori of the 2-torus". In: *Quantum Topology* (2025). DOI: 10.4171/QT/231.