Quantization of symplectic singularities from vertex algebras

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Quantum Universe Attract.Workshop

Deformation quantization of symplectic manifolds

Setup (everything over $\mathbb{C}$):

- X: symplectic manifold.
- A: non-commutative, filtered associative algebra, $\mathrm{gr} A$: Poisson algebra induced from the commutator.
- A: **quantization of** X if $\mathbb{C}[X] \cong \mathrm{gr} A$ as Poisson algebras.

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Examples

- Algebra of differential operators $\mathcal{D}(\mathbb{C}^n) \overset{Vect}{=} \mathbb{C}[x_1, \cdots, x_n, \partial_1, \cdots, \partial_n]$ quantizes $T^*\mathbb{C}^n$.

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- $U(\mathfrak{g})/Z(\mathfrak{g})$ quantizes flag manifold $T^*(G/B)$ (this is the resolution of the nilpotent cone of $\mathfrak{g}$).

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- Get back quantization by modding out $\hbar$: $\mathcal{A}/\hbar\mathcal{A} \cong \mathcal{O}_{T^*(G/B)}$.

Quantization of arc spaces

How do we generalize to infinite dimension?

X: finite-dimensional symplectic manifold, $J_\infty X$: ∞-**jet scheme of** X, defined by

$$J_\infty X(R) = \mathrm{Hom}(\operatorname{Spec} R,\, J_\infty X) = \mathrm{Hom}(\operatorname{Spec} R[[t]],\, X) = X(R[[t]]).$$

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So we define **chiral quantization of** X (or **quantization of** $J_\infty X$) by a sheaf of $\hbar$-adic vertex algebras $\mathcal{A}_X^{ch}$ such that $\mathcal{A}_X^{ch}/\hbar\mathcal{A}_X^{ch} \cong \mathcal{O}_{J_\infty X}$ as sheaves of Poisson vertex algebras.

Examples

We start with an analog example of cotangent type [Beilinson and Drinfeld, 2004; Gorbounov, Malikov, and Schechtman, 2000]

- $X = T^*M$, such that the second graded piece of the Chern character $\mathrm{ch}_2(TM)$ vanishes, there exists an **algebra of chiral differential operators (CDO)** $\mathcal{D}^{ch}_M$ over M.

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- If $M = G/B$, $\mathcal{D}_M^{ch}(M) = V^{-h^\vee}(\mathfrak{g})$.
- Performing (micro)localization gives a sheaf of $\hbar$-adic vertex algebras $\mathcal{A}_{X,\hbar}^{ch}$ on T^*M which acts as the chiral quantization of X.

Other recent examples include

- X obtained from Hamiltonian reduction of Lie group G, the vertex algebra of global sections gives affine W-algebra at critical level $\mathcal{W}^{-h^\vee}(\mathfrak{g}, f)$ [Arakawa, Kuwabara, and Malikov, 2014].
- X: Hypertoric variety [Kuwabara, 2017].
- X: Hilbert scheme of n-points on $\mathbb{C}^2$ $\mathrm{Hilb}^n(\mathbb{C}^2)$ [Arakawa, Kuwabara, and Möller, 2023].

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For example, the Zhu algebra of CDO $\mathcal{D}^{ch}_M$ over M recover the algebra of differential operators $\mathcal{D}_M$ [Arakawa, Chebotarov, and Malikov, 2009].

Theorem (D. 2025)
The Zhu algebra of the simple $\mathcal{N} = 4$ Virasoro superconformal vertex algebra of central charge $c = -9$ is

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This vertex algebra comes from chiral quantization of $\mathrm{Hilb}^2(\mathbb{C}^2) \cong T^*\mathbb{P}^1$ and the Zhu algebra should recover the quantization of certain nilpotent thickening of $T^*\mathbb{P}^1$.

Arakawa, T., Kuwabara, T., & Malikov, F. (2014).**Localization of affine w-algebras.** *Communications in Mathematical Physics*, *335*(1), 143–182. https://doi.org/10.1007/s00220-014-2183-x

Arakawa, T., Chebotarov, D., & Malikov, F. (2009). **Algebras of twisted chiral differential operators and affine localization of $\mathfrak{g}$-modules.** https://arxiv.org/abs/0810.4964

Arakawa, T., Kuwabara, T., & Möller, S. (2023). **Hilbert schemes of points in the plane and quasi-lisse vertex algebras with $\mathcal{N} = 4$ symmetry.** https://arxiv.org/abs/2309.17308

Beilinson, A., & Drinfeld, V. (2004). **Chiral algebras (Vol. 51).** American Mathematical Society.

Gorbounov, V., Malikov, F., & Schechtman, V. (2000). **Gerbes of chiral differential operators. ii.** https://arxiv.org/abs/math/0003170

Kuwabara, T. (2017). **Vertex algebras associated with hypertoric varieties.** https://arxiv.org/abs/1706.02203