Generalizations of Kazhdan-Lusztig R-polynomials

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Applications of R-Polynomials

Applications

R-polynomials arise in various areas of mathematics:

- **Combinatorics:**
  - Study of Bruhat order properties.
  - Kazhdan-Luzstig-Stanley-Theory for Matroids.
- **Representation Theory:**
  - Computing Kazhdan-Luzstig Basis.
  - Counting the multiplicity of certain morphisms in the Category of Soergel Bimodules.
- **Algebraic Geometry:**
  - Schubert calculus.
  - Study of flag varieties.

Some Notation

Coxeter group

A **Coxeter group** (W, S) is a group W with generating set S and relations:

$$(st)^{m(s,t)} = 1 \quad \text{for } s, t \in S,$$

where $m(s, s) = 1$ and $m(s, t) = m(t, s) \geq 2$ for $s \neq t$.

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Length Function

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Bruhat Order

Let $x, y \in W$ and fix a reduced word $\underline{y}$ for y. We define a partial order by : $x \leq y$ if x has an expression that is a subexpression of $\underline{y}$.

R-Polynomials in Coxeter Groups

Definition

There is a unique family of Polynomials $\tilde{R}_{x,y}(v)_{x,y \in W} \subseteq \mathbb{Z}[v]$ such that

1. $\tilde{R}_{x,y}(v) = 0, \quad \text{if } x \not\leq y,$
2. $\tilde{R}_{x,y}(v) = 1, \quad \text{if } x = y,$
3. For s with $\ell(ys) < \ell(y)$,

$$\tilde{R}_{x,y}(v) = \begin{cases} \tilde{R}_{xs,ys}(v), & \text{if } \ell(xs) < \ell(x), \\ \tilde{R}_{xs,ys}(v) + v\tilde{R}_{x,ys}(v), & \text{if } \ell(xs) > \ell(x). \end{cases}$$

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Main Properties

- $\tilde{R}_{x,y}(v)$ is a polynomial in v with non-negative integer coefficients.
- The degree of $\tilde{R}_{x,y}(v)$ is $\ell(y) - \ell(x)$.
- We have $\sum_{x\leq z\leq y} \tilde{R}_{x,z}(v)(-1)^{\ell(x)+\ell(z)}\tilde{R}_{z,y}(v) = \delta_{x,y}$.

Example: S_3

- Let $S_3 =< s, t | s^2 = t^2 = e, sts = tst >$ and denote by $w_\circ = sts$.

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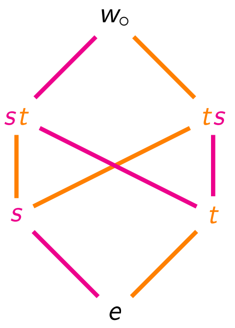

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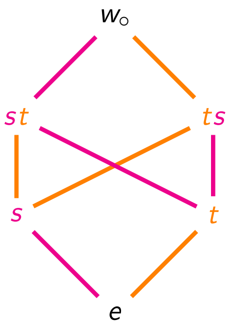

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	e	s	t	st	ts	$w_\circ$
e	1	v	v	v^2	v^2	$v^3 + v$
s	0	1	0	v	v	v^2
t	0	0	1	v	v	v^2
st	0	0	0	1	0	v
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$w_\circ$	0	0	0	0	0	1

Table: $\tilde{R}$-Polynomials of S_3.

R-Polynomials in Hecke Algebra of S_3

- Denote $v' = v - v^{-1}$. Let $\mathcal{H}$ be the $\mathbb{Z}[v, v^{-1}]$-algebra generated by $\{H_{s'}\}_{s' \in S_3}$ and relations
  1. $H_{s^2 = 1 + v' H_s}$ and $H_{t^2 = 1 + v' H_t}$
  2. $H_s H_t H_s = H_t H_s H_t$
- From the relations one gets $H_{s'}^{-1} = H_{s'} + v'$ and $H_x := \prod H_{s_i}$, where $x = \prod s_i$ is a reduced word for x, is well defined.

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- Define an involution on $\mathcal{H}$ via extending $\bar{H}_x := (H_{x^{-1}})^{-1}$, $\bar{v} = v^{-1}$ to a ring homomorphism.
- Then $\overline{H_s H_t} = (H_s + v')(H_t + v') = H_s H_t + v' H_s + v' H_t + v'^2$

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- In general for $y \in S_3$ we get $\overline{H_y} = \sum_{x \leq y} \tilde{R}(v') H_x$

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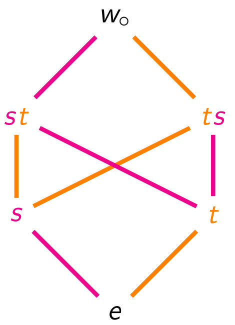
- This condition ensures that $\kappa * f = \hat{f}$ for some functions in $\mathbf{I}(P, K[v])$.

Example: S_3

- Let $\mathbf{I} = \mathbf{I}(S_3, \mathbb{Z}[v])$. We have $\delta_{x,y} \in I$.
- Define $\xi_{x,y}(v) := 1$, for $x \leq y$.



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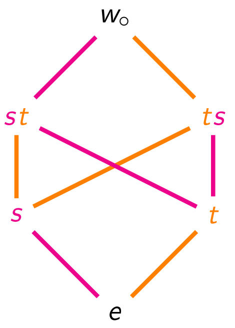
- Define $\xi_{x,y}(v) := 1$, for $x \leq y$.
- Then ξ is invertible and $\chi * \xi = \hat{\xi}$.
- Here χ is the charcteristic polymomial of the poset S_3.



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$\tilde{R}-$Polynomial as a W-Kernel

- From earlier we have $(\tilde{R}^{-1})_{x,y}(v) = (-1)^{\ell(y)-\ell(x)}\tilde{R}(v)$,
- But $\hat{\tilde{R}}_{x,y}(v) = v^{\ell(y)-\ell(x)}\hat{\tilde{R}}_{x,y}(v^{-1})$. So $\tilde{R}$ is not a W-Kernel.

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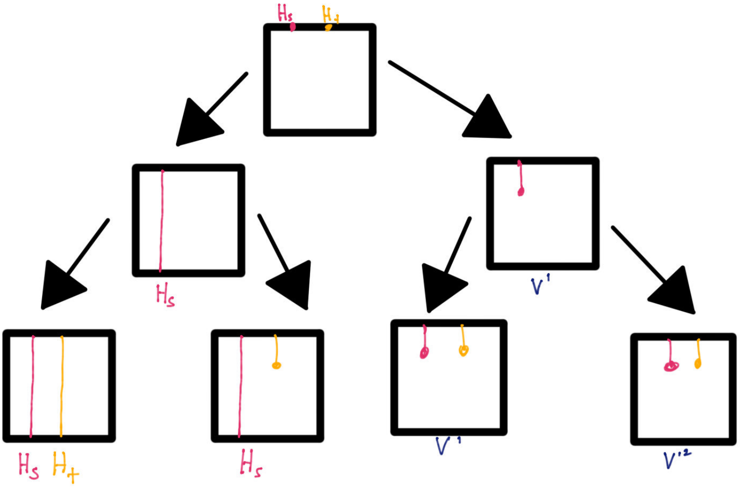
- But $\hat{\tilde{R}}_{x,y}(v) = v^{\ell(y)-\ell(x)}\hat{\tilde{R}}_{x,y}(v^{-1})$. So $\tilde{R}$ is not a W-Kernel.
- We have however $v^{\ell(y)-\ell(x)}\tilde{R}_{x,y}(v - v^{-1})$ is a W-kernel.
- Want to modify $\tilde{R}$ or the hat map to respect the substitution.

R-Polynomials and Soergel Bimodules



- Let each dot represent a factor of v' in the Product $(H_s + v')(H_t + v')$ we obtain the various $\tilde{R}_{x,st}$ for $x \leq y$.

R-Polynomials and Varities

- Related to Richardson Varities.
- Richardson Varities are indexed by a pair $x, y \in W$, such that $x \leq y$.
- These have dimension $\ell(y) - \ell(x)$.

Bibliography

[1] Anders Björner; Francesco Brenti. *Combinatorics of Coxeter Groups*. Springer Verlag, New York, 2005.

[2] James E. Humphreys. *Reflection Groups and Coxeter Groups*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1990.

[3] David Plaza. “Diagrammatics for Kazhdan–Lusztig R-polynomials”. In: *European Journal of Combinatorics* 79 (2019), pp. 193–213. ISSN: 0195-6698. DOI: https://doi.org/10.1016/j.ejc.2019.03.002. URL: https://www.sciencedirect.com/science/article/pii/S0195669819300228.

[4] David E Speyer. *Richardson varieties, projected Richardson varieties and positroid varieties*. 2024. arXiv: 2303.04831 [math.AG]. URL: https://arxiv.org/abs/2303.04831.

[5] Richard P. Stanley. “Subdivisions and Local h-Vectors”. In: *Journal of the American Mathematical Society* 5.4 (1992), pp. 805–851. ISSN: 08940347, 10886834. URL: