Analysing quantum field theories at criticality

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Background

Several recent interesting areas of study in quantum field theory centre on understanding properties of phase transitions and fixed points of the β-function

One motivation is to determine if gravity is asymptotically safe as well as if there are emergent symmetries that give hints to what lies beyond the Standard Model

To explore such ideas related and simple quantum field theories are used as testing grounds to draw out the underlying principles and properties

As a by-product of some such theories they have applications in real world physics primarily in material science

In particular one active area is the phase transition in graphene governing the change from conducting to insulator properties that is believed to be described by a continuum field theory

Specifically the model involves a 4-fermi interaction or Yukawa type interacation not dissimilar to the Standard Model

A secondary motivation for studying the test ground systems is to gauge how accurate high order perturbative results are in comparison

Will examine that specific Yukawa system at a new loop order to extract critical exponent estimates in the dimension of physical interest which is three

Other continuum methods are available to study critical phenomena such as the functional renormalization group formalism and the conformal bootstrap method

Basic critical field theory

There is a simple relation between the renormalization group functions and properties of critical phenomena

Phase transitions are defined in quantum field theory by zeroes of the β-function

$$\beta(g^*) \; = \; 0$$

where g^* is the critical coupling

Two main types of field theoretic phase transitions that depend on the spacetime dimension

Banks-Zaks fixed points, usually associated with gauge theories, are for fixed dimension and arise at two loops

In QCD the Banks-Zaks critical point is infrared stable due to asymptotic freedom

For non-asymptotically free theories then analogous non-trivial fixed point would be ultraviolet stable and termed asymptotic safe

With

$$\beta(g) \;=\; ag^2 \;+\; bg^3 \;+\; O(g^4)$$

where a and b depend on parameters of the theory, has

$$g^\star \;=\; -\,\frac{a}{b}$$

and is a *bona fide* fixed point if $a/b < 0$

For example in QCD

$$a \;=\; -\,11 \;+\; \frac{2}{3}N_f \quad , \quad b \;=\; \frac{2}{3}[19N_f - 153]$$

where N_f is the number of quarks and defines the conformal window as $8 < N_f < 16$

Another type of fixed point is Wilson-Fisher which is present in d-dimensional theory

$$\beta(g) \;=\; \epsilon g \;+\; ag^2 \;+\; bg^3 \;+\; O(g^4)$$

where $d = D_c - 2\epsilon$ and D_c is the critical dimension of the field theory

The Wilson-Fisher fixed point is at

$$g^* \ = \ - \ \frac{\epsilon}{a} \ + \ O(\epsilon^2)$$

In the $\overline{\text{MS}}$ scheme the coefficients a, b and higher ones are ϵ independent

At either fixed point the evaluation of the renormalization group functions leads to the critical exponents

For example the field anomalous dimension is $\eta = \gamma_\phi(g^*)$ and $\omega = \beta'(g^*)$ is the correction to scaling exponent

They will depend on any parameters such as colour group Casimirs and d in the Wilson-Fisher case

They define the properties of the universal quantum field theory which describes the Wilson-Fisher fixed point in *all* dimensions

Moreover exponents are renormalization group invariants and evaluate to the same value in different renormalization schemes

Applications - emergent symmetries

Scalar theory background

$O(N)$ ϕ^4 theory in four dimensions has connections with $O(N)$ ϕ^3 theory in six dimensions; Lagrangian of former is

$$L^{\phi^4} = \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i - \frac{1}{8}g^2\left(\phi^i\phi^i\right)^2$$

can be rewritten with cubic interactions called the scalar decomposition as

$$L^{\phi^4} = \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i + \frac{g}{2}\sigma\phi^i\phi^i + \frac{\sigma^2}{2}$$

By contrast $O(N)$ ϕ^3 Lagrangian is, with $D_c = 6$,

$$L^{\phi^3} = \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i + \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{g_1}{2}\sigma\phi^i\phi^i + \frac{g_2}{6}\sigma^3$$

Renormalization group functions for $O(N)$ theory available to high loop order [Klebanov et al; JAG; Kompaniets & Pikelner; Schnetz] and agree with large N exponents in the $O(N)$ scalar field theory universality class

Connection between both theories relies on the common σ and ϕ^i field interaction deriving from the quartic self-interaction in four dimensions and the introduction of the auxiliary field there

$O(N)$ ϕ^3 theory is regarded as the ultraviolet completion of $O(N)$ ϕ^4 theory

However the decomposition of $O(N)$ ϕ^4 theory into the scalar field auxiliary is not unique [Herbut, Janssen; Herbut, Roscher] as there is a tensor decomposition

$$L^{\phi^4} \;=\; \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i \;+\; \frac{g}{2}\sigma^a\phi^i\Lambda^a_{ij}\phi^j \;+\; \frac{1}{2}\sigma^a\sigma^a$$

where $1 \leq a \leq \frac{1}{2}(N-1)(N+2)$ and Λ^a are real, traceless, symmetric N-dimensional matrices satisfying

$$\Lambda^a_{ij}\Lambda^a_{kl} = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{N}\delta_{ij}\delta_{kl}$$

Six dimensional partner is

$$L^{\phi^3} = \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i + \frac{1}{2}\partial_\mu\sigma^a\partial^\mu\sigma^a + \frac{g_1}{2}\sigma^a\phi^i\Lambda^a_{ij}\phi^j + \frac{g_2}{6}d^{abc}\sigma^a\sigma^b\sigma^c$$

where $d^{abc} = \mathrm{Tr}\left(\Lambda^a\Lambda^b\Lambda^c\right)$

Low loop renormalization and FRG analyses showed richer and different fixed point structure compared with scalar decomposition [Herbut, Janssen; Herbut, Roscher] and has been extended to four loops

There were two 2-point functions to evaluate to four loops with a total of 3, 12, 108 and 1350 graphs at each respective loop order

Contributions to vertex functions can be generated automatically from 2-point function computation

Results for $O(3)$ are

$$
\begin{aligned}
\beta_1(g_i) &= -\frac{1}{2}\epsilon g_1 - \left[-17g_1^2 + 42g_1g_2 - 7g_2^2\right]\frac{g_1}{36} \\
&+ \left[-6617g_1^4 + 3591g_1^3g_2 - 1988g_1^2g_2^2 + 2625g_1g_2^3 - 791g_2^4\right]\frac{g_1}{1944} \\
&- \Big[4380480\zeta_3 g_1^6 + 2469685g_1^6 - 1959552\zeta_3 g_1^5 g_2 + 9953370g_1^5 g_2 \\
&\quad +9897552\zeta_3 g_1^4 g_2^2 - 3206105g_1^4g_2^2 + 10723104\zeta_3 g_1^3 g_2^3 \\
&\quad -3644172g_1^3g_2^3 + 2830464\zeta_3 g_1^2 g_2^4 + 4790919g_1^2g_2^4 \\
&\quad -2231712\zeta_3 g_1 g_2^5 + 4461618g_1g_2^5 + 371952\zeta_3 g_2^6 \\
&\quad -1217531g_2^6\Big]\frac{g_1}{419904}
\end{aligned}
$$

and

$$
\begin{aligned}
\beta_2(g_i) \quad = \quad & -\frac{1}{2}\epsilon g_2 - \left[6g_1^3 - 3g_1^2 g_2 - 13g_2^3\right]\frac{1}{12} \\
& + \left[549g_1^5 - 537g_1^4 g_2 + 747g_1^3 g_2^2 - 420g_1^2 g_2^3 - 2951g_2^5\right]\frac{1}{648} \\
& - \Big[-629856\zeta_3 g_1^7 + 1501722g_1^7 + 1648512\zeta_3 g_1^6 g_2 \\
& \quad -469209g_1^6 g_2 + 2822688\zeta_3 g_1^5 g_2^2 - 1281780g_1^5 g_2^2 \\
& \quad +1006992\zeta_3 g_1^4 g_2^3 + 1066437g_1^4 g_2^3 - 637632\zeta_3 g_1^3 g_2^4 \\
& \quad +1008810g_1^3 g_2^4 - 563883g_1^2 g_2^5 + 2549232\zeta_3 g_2^7 \\
& \quad -1449209g_2^7\Big]\frac{1}{139968}
\end{aligned}
$$

to three loops

Can solve $\beta_i(g_j) = 0$ and study fixed point properties; there are several non-trivial solutions for $(g_1^\star, g_2^\star)$

$O(3)$ as an example

One fixed point solution η_ϕ, and η_σ as well as g_1^* and g_2^* was

$$
\begin{aligned}
(g_1^*)^2 &= (g_2^*)^2 \\
&= \frac{3}{11}\epsilon + \frac{1301}{3993}\epsilon^2 + [1672704\zeta_3 + 3301487]\frac{\epsilon^3}{5797836} \\
&\quad + [107879214960\zeta_3 + 2732361984\zeta_4 - 76753591200\zeta_5 \\
&\quad -6440858957]\frac{\epsilon^4}{9470765106} + O(\epsilon^5) \\
\eta_\phi &= \eta_\sigma \\
&= \frac{10}{33}\epsilon + \frac{1000}{11979}\epsilon^2 + 10[104544\zeta_3 + 220057]\frac{\epsilon^3}{4348377} \\
&\quad + 10[5936914368\zeta_3 + 170772624\zeta_4 - 5025952800\zeta_5 \\
&\quad -192710239]\frac{\epsilon^4}{4735382553} + O(\epsilon^5)
\end{aligned}
$$

Exponent equality is an example of an emergent flavour symmetry where the original symmetry group is superseded by another

At this fixed point the symmetry group enlarges to $SU(3)$ as there are 3 ϕ^i fields and 5 σ^a fields

Defining

$$M = \sigma^a \Lambda^a + \phi^i S^i$$

where $(S^i)_{jk} = i\epsilon_{ijk}$ then

$$L^{\phi^3} = \frac{1}{4}\mathrm{Tr}\,(\partial_\mu M)^2 + \frac{g_2}{6}\mathrm{Tr}\left(M^3\right)$$

as $g_1 = -\, g_2$ at the fixed point

Exponents agree with those at the Wilson-Fisher fixed point of

$$L = \frac{1}{2}(\partial_\mu \phi^a)^2 + \frac{g}{6} d^{abc} \phi^a \phi^b \phi^c$$

when $N_c = 3$ where d^{abc} is the completely symmetric tensor in $SU(N_c)$

Chiral Gross-Neveu-Yukawa model

Emergence of higher symmetries is not restricted to flavour symmetries

Another example is that supersymmetry can emerge in systems with bosons and fermions

One such example is the chiral Gross-Neveu-Yukawa (GNY) system,

$$L^{\text{cGN}} = i\bar{\psi}\partial\!\!\!/\psi + |\partial_\mu\sigma|^2 + g\bar{\psi}(\sigma_1 + i\gamma^5\sigma_2)\psi + \lambda|\sigma|^4$$

where there are N fermions ψ^i, $\sigma = \sigma_1 + i\sigma_2$ and there is a continuous $U(1)$ symmetry

It is sometimes called the chiral XY Gross-Neveu model

As with most two coupling theories there are several non-trivial critical points in $d = 4 - \epsilon$ dimensions

These include the Gaussian point $(g_\star^2, \lambda_\star) = (0,0)$ and that of ϕ^4 theory with $g_\star^2 = 0$ and $\lambda_\star \neq 0$

Additionally there are two interior non-trivial fixed points with $(g_\star^2, \lambda_\star) \neq (0,0)$ with one a saddle point and the other a fully stable fixed point

The explicit values are known up to four loops [Zerf et al] as functions of N and ϵ as are the critical exponents

There is an interesting feature for $N = \frac{1}{2}$ which is that

$$g_\star^2 ~=~ \lambda_\star ~~~~,~~~~ \eta_\psi ~=~ \eta_\sigma ~=~ \frac{1}{3}\epsilon$$

which corresponds to an emergent supersymmetry

There is a caveat to this particular emergence in that to adduce it one must regularize the original chiral GNY using a regularization that respects supersymmetry even though the original Lagrangian is not supersymmetric

Otherwise using dimensional regularization the emergent supersymmetry fails at four loops

Moreover at this critical point the theory equates to the Wess-Zumino model and the exponents of that model agree with those at this fixed point

Wess-Zumino model

In superspace the Wess-Zumino model involves a cubic superfield chiral interaction

$$S = \int d^4x \left[\int d^2\theta d^2\bar{\theta} \; \bar{\hat{\Phi}}(x,\theta,\bar{\theta})\hat{\Phi}(x,\theta,\bar{\theta}) + \frac{g}{3!} \int d^2\theta \, \hat{\Phi}^3(x,\theta,\bar{\theta}) + \frac{g}{3!} \int d^2\bar{\theta} \, \bar{\hat{\Phi}}^3(x,\theta,\bar{\theta}) \right]$$

where θ and $\bar{\theta}$ are 2 component anti-commuting spinor supercoordinates

In terms of component fields the Lagrangian is

$$L = i\bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\pi)^2 + \bar{g}\bar{\psi}\left(\sigma + i\pi\gamma^5\right)\psi + \frac{1}{24}\bar{g}^2\left(\sigma^2 + \pi^2\right)^2$$

which has one coupling $\bar{g}$

There is a non-renormalization theorem that implies $\beta(g) = 3g^2\gamma_\Phi(g)$ giving $\eta = \frac{1}{3}\epsilon$ *exactly*

There is potential for this property of emergent supersymmetry to be observed in Nature in that it is believed to be present on the boundary of a three dimensional topological insulator

An obvious question is: could such a mechanism of emergent supersymmetry be a property of the Standard Model leading to or be a guide to Standard Model extensions?

Separately GNY systems have been of interest in condensed matter science as the phase transitions in this broad universality class are believed to describe transitions in graphene

Gross-Neveu-Yukawa system

Background

Graphene is a sheet of carbon which is one atom thick; it is incredibly strong and is an excellent electrical conductor

It has several interesting properties such as stretching the sheet can effect a quantum phase change from a conductor to a Mott-insulating phase

Such properties have the potential to lead graphene based electronics

Equally the Mott transition could be a mimic of spontaneous symmetry breaking in particle physics

Electrons are located at the corners of a honeycomb or hexagonal lattice and can be studied with lattice methods or spin models

There is also a connection with GNY continuum field theories whose Wilson-Fisher fixed points underlie the phase transitions in graphene

The phase transitions of interest are described by properties of the *three* dimensional quantum field theory

Gross-Neveu universality class

The original Gross-Neveu model is a renormalizable quantum field theory in two dimensions based on a 4-fermi interaction with Lagrangian

$$L^{\mathrm{GN}} = i\bar{\psi}^i \partial\!\!\!/ \psi^i - m\bar{\psi}^i\psi^i + \frac{g^2}{2}\left(\bar{\psi}^i\psi^i\right)^2$$

or introducing an auxiliary field σ

$$L = i\bar{\psi}^i \partial\!\!\!/ \psi^i + g\sigma\bar{\psi}^i\psi^i - \frac{1}{2}\sigma^2$$

and the theory is asymptotically free in two dimensions

The σ field effective potential in two dimensions shows that the true vacuum is not the perturbative one

In the true vacuum σ develops a non-zero expectation value leading to dynamical mass generation

Also via the $1/N$ expansion the σ field in the true vacuum becomes dynamical

It is the σ field and the common scalar-Yukawa connection that are at the centre of the GNY system universality class

GNY system

In GN model has critical dimension 2 and is

$$L^{\text{GN}} = i\bar{\psi}^i \partial\!\!\!/ \psi^i + g\sigma\bar{\psi}^i\psi^i - \frac{1}{2}\sigma^2$$

whereas the GNY system has critical dimension 4 and is

$$L^{\text{GNY}} = i\bar{\psi}^i \partial\!\!\!/ \psi^i + \frac{1}{2}\sigma\Box\sigma + \frac{1}{2}g\sigma\bar{\psi}^i\psi^i + \frac{1}{24}\lambda\sigma^4$$

The universality is driven by the common 3-point Yukawa vertex which is core to the large N critical point method of Vasil'ev et al with formal universal Lagrangian

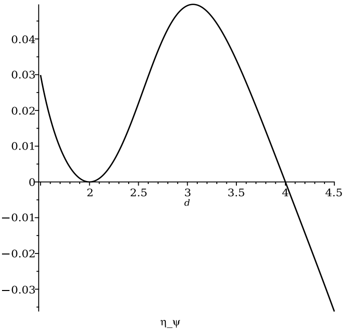
$$L^{\text{univ}} = i\bar{\psi}^i \partial\!\!\!/ \psi^i + \frac{1}{2}\sigma\Box^{\frac{1}{2}d-1}\sigma + \frac{1}{2}g\sigma\bar{\psi}^i\psi^i + f(\sigma)$$

where $f(\sigma)$ represents the set of operators built from σ and its derivatives that are relevant in the (even) critical dimensions above two dimensions

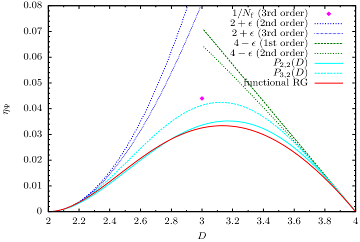
The universal theory in d-dimensions is described by the critical exponents such as $\eta(d, N)$

Their behaviour can be ascertained via a variety of methods such as the ϵ expansion, functional renormalization group, conformal bootstrap and large N critical point formalism

The $[1, 1]$ Padé for η in the GN model for a particular value of N is zero in $d = 2$ and 4



In the context of other methods Janssen and Herbut (2014) produced a plot of η_ψ summarizing the use of the ϵ expansion around two and four dimensions, as well as functional renormalization group, large N and two-sided Padé approximants



Subsequent work in this area has included conformal bootstrap estimates for this and other exponents as well as a recent five loop GNY renormalization [Maier, Marquard, Schröder, JAG]

Five loop renormalization

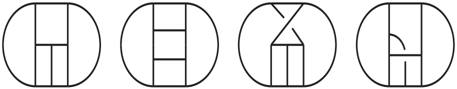
The motivation for extending the GNY renormalization to five loops is to refine exponent estimates in three dimensions using the two-sided analysis

Recent conformal bootstrap analysis of [Erramilli et al] are relatively precise

Method to renormalize GNY is to introduce an auxiliary mass for the scalar and fermion fields and then expand in external momenta

This results in the ultraviolet divergences being isolated in fully massive vacuum integrals

At five loops all integrals can be accommodated within four integral families



Using the Laporta integration by parts algorithm all the Feynman integrals are reduced to a set of 110 five loop master integrals

The ϵ expansion of the masters was evaluated by solving high precision difference equations numerically followed by reconstruction using tools like PSLQ

Only certain combinations of the top level families arise which can be expressed as rationals and zeta values

There were various checks such as the agreement of the two and higher order ϵ poles with the expectation of the renormalization group equation

A second check was the reproduction of the five loop $\overline{\text{MS}}$ $O(N)$ ϕ^4 renormalization group functions

The final check was agreement with the large N critical exponents of the underlying universality class where the fermion wave function is available at $O(1/N^3)$ and the scalar, mass and the two β-function eigen-exponents are known at $O(1/N^2)$ [Vasil'ev et al; Manashov & Strohmaier; JAG]

The large N results give independent access to five loop coefficients in the renormalization group functions

Recall there are two coupling constants in the GNY theory in four dimensions

For the application to graphene and other systems need to construct the critical exponents which are derived from the renormalization group functions evaluated at the Wilson-Fisher fixed point in $d = 4 - \epsilon$ dimensions

The fixed points $\lambda_\star$ and $y_\star$ where $y = g^2$ are given by

$$\beta_y(y_\star, \lambda_\star) \; = \; 0 \quad , \quad \beta_\lambda(y_\star, \lambda_\star) \; = \; 0$$

So the exponents of interest are

$$\eta_\psi \; = \; \gamma_\psi(y_\star, \lambda_\star) \;\; , \;\; \eta_\sigma \; = \; \gamma_\sigma(y_\star, \lambda_\star) \;\; , \;\; \eta_{\sigma^2} \; = \; \gamma_{\sigma^2}(y_\star, \lambda_\star)$$

Additionally there are correction to scaling exponents $\omega_\pm$ that are derived from the critical slope of the β-functions via the eigen-anomalous dimensions of

$$\beta_{ij} \; = \; \left(\begin{array}{cc} \frac{\partial \beta_y}{\partial y} & \frac{\partial \beta_y}{\partial \lambda} \\ \frac{\partial \beta_\lambda}{\partial y} & \frac{\partial \beta_\lambda}{\partial \lambda} \end{array} \right) \; \equiv \; \left(\begin{array}{cc} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{array} \right)$$

with

$$\omega_\pm(y, \lambda) \; = \; \frac{1}{2} \left(\beta_{11} + \beta_{22} \right) \; \mp \; \frac{1}{2} \sqrt{[(\beta_{11} - \beta_{22})^2 + 4 \beta_{12} \beta_{21}]}$$

As an example of the results the fermion exponent for $N = 2$ is

$$\begin{aligned}
\eta_\psi \quad = \quad & \frac{1}{14}\epsilon - \frac{71}{10584}\epsilon^2 + \left[-\frac{2432695}{158696496} - \frac{18}{2401}\zeta_3 \right] \epsilon^3 \\
& + \left[\frac{150}{16807}\zeta_5 - \frac{27}{4802}\zeta_4 + \frac{11109323}{555437736}\zeta_3 - \frac{111266497289}{11557548410688} \right] \epsilon^4 \\
& + \left[-\frac{89279362217932877}{117839838991501992 96} - \frac{1905116200933}{377546581415808}\zeta_3 \right. \\
& \quad - \frac{136650473}{2744515872}\zeta_5 - \frac{5643}{537824}\zeta_7 - \frac{2004}{823543}\zeta_3^2 + \frac{375}{33614}\zeta_6 \\
& \quad \left. + \frac{11109323}{740583648}\zeta_4 \right] \epsilon^5 \; + \; O(\epsilon^6)
\end{aligned}$$

Or numerically

$$\eta_\psi \; = \; 0.071429\epsilon - 0.006708\epsilon^2 - 0.024341\epsilon^3 + 0.017584\epsilon^4 - 0.051782\epsilon^5 + O(\epsilon^6)$$

Resummation

Numerical values of the ϵ expansion of the exponents do not appear to be useable when $\epsilon = 1$

Two resummation methods employed: two-sided Padé approximants and interpolating polynomial

For the former the $[m/n]$ approximant is defined by

$$\mathcal{P}_{[m/n]}(d) \;=\; \frac{\sum_{p=0}^{m} a_p d^p}{1 + \sum_{q=1}^{n} b_q d^q}$$

and the interpolating polynomial by

$$\mathcal{I}_{[i,j]}(d) \;=\; \sum_{m=0}^{i} \eta^{(2)}_m (d-2)^m \;+\; \sum_{n=i+1}^{i+j+1} c_n (d-2)^n$$

The unknown coefficients are determined from the ϵ expansion of the exponent close to two and four dimensions

For a generic exponent this translates to

$$\eta(2+\epsilon) \ = \ \sum_{n=0}^{\infty} \eta_n^{(2)} \epsilon^n \quad , \quad \eta(4-\epsilon) \ = \ \sum_{n=0}^{\infty} \eta_n^{(4)} \epsilon^n$$

with two dimensional renormalization group functions available at four loops

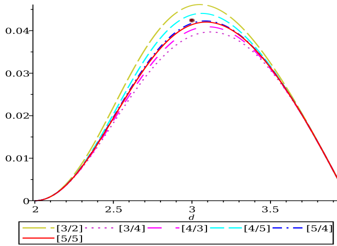
Expressing $\mathcal{I}_{[i,j]}(d)$ as a polynomial in $(d-4)$ produces the same d dependent result as the $(d-2)$ definition

One benefit of the d-dimensional expressions is that the approximations can be plotted in $2 \leq d \leq 4$

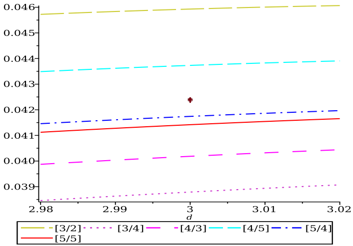
We only constructed Padés using the same loop order from both boundary dimensions

For the interpolating polynomial approach we constructed the $[L, L]$ and $[L, L+1]$ polynomials at each loop order L

Method	η_ψ	η_ϕ	$1/\nu$
Large *N*	0.044	0.743	0.952
Monte Carlo	——	0.754(8)	1.00(4)
Monte Carlo	0.38(1)	0.62(1)	1.20(1)
FRG	0.032	0.760	0.982
FRG	0.0276	0.7765	0.994(2)
4 loop $d = 2$ naive Padé	0.082	0.745	0.931
3 loop $d = 4$ naive Padé	0.0740	0.672	1.048
Conformal bootstrap	0.044	0.742	0.880
Monte Carlo	——	0.65(3)	1.2(1)
Monte Carlo	——	0.54(6)	1.14(2)
4 loop $d = 4$ naive Padé	0.0539	0.7079	0.931
4 loop two-sided Padé	0.042	0.735	1.004
4 loop interp poly	0.043	0.731	0.982
Monte Carlo	0.05(2)	0.59(2)	1.0(1)
Conformal bootstrap	0.04238(11)	0.7329(27)	0.998(12)
Monte Carlo	0.043(12)	0.72(6)	1.07(12)
5 loop two-sided Padé	0.04142	0.73453	1.02129
5 loop interp poly	0.04111	0.73092	1.01014



N = 2 η_ψ



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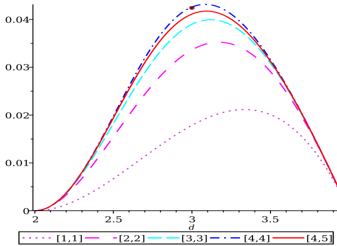
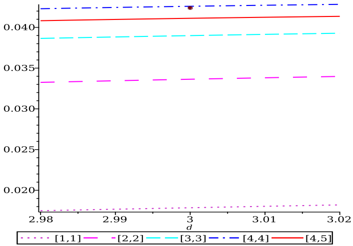
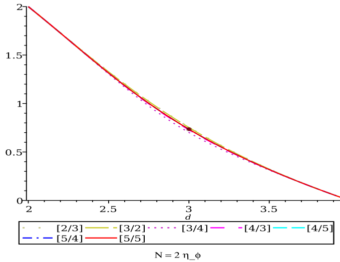


N = 2 η_ψ



N = 2 η_ψ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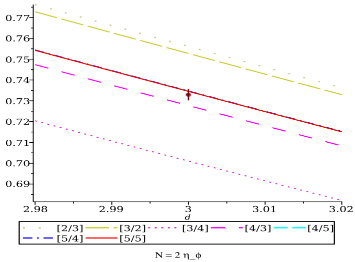
d
N = 2 η_ϕ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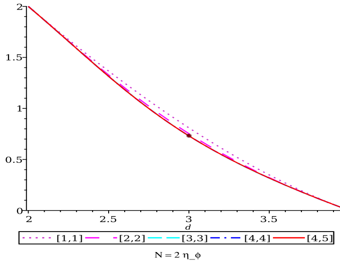
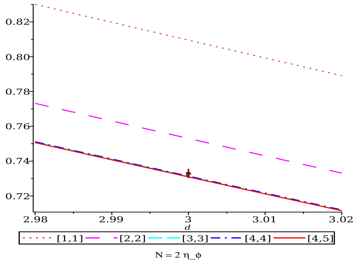
d
N = 2 η_ϕ


[1,1]
[2,2]
[3,3]
[4,4]
[4,5]
d
N = 2 η_ϕ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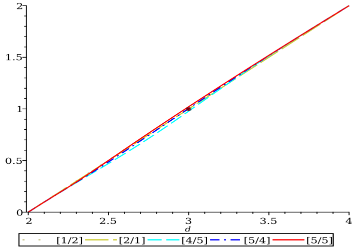
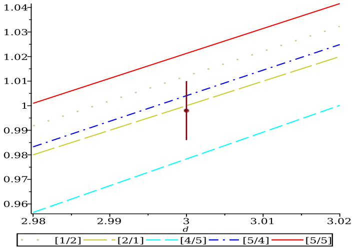
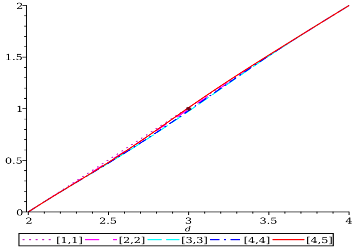
N = 2 η_ϕ



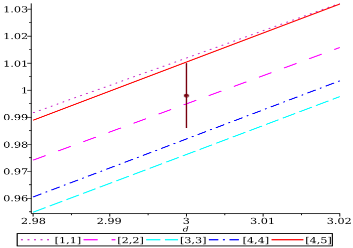
N = 2 1/ν



N = 2 1/ν



N = 2 1/ν



N = 2 1/ν

The plots of η_ψ are qualitatively similar to those of the earlier large *N*, Padé and functional renormalization group ones

The progression of the successive loop order two-sided constructions show a gradual convergence towards the latest conformal bootstrap value

A similar picture emerges for the other exponents η_ϕ and $1/\nu$

For the latter two the canonical dimensions are not equal in the two critical dimensions meaning the *d*-dimensional behaviour is roughly linear

This means that convergence is quicker and for both cases is on the edge of one standard deviation

Analysis for other values of *N* was carried out and agreement with exponents from other methods is closer as *N* increases

Scalar ϕ^3 theory

Background

There are other critical point quantum field theories with connections to applications to real world problems

Equally perturbative methods can contribute to the debate and complement conformal bootstrap studies

For instance ϕ^3 theory is the continuum field theory underlying certain percolation problems as well as the Lee-Yang edge singularity problem

The critical exponent σ of Lee-Yang theory in three dimensions plays a fundamental role in lattice studies of the QCD equation of state as well as baryon number studies

Having a precise value for σ connects with the volume scaling of Lee-Yang zeroes in the situation where there is crossover behaviour

Lee-Yang theory

The core Lagrangian for the Lee-Yang situation is

$$L = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{ig}{6}\phi^3$$

which is a non-unitary theory [Fisher] involving one scalar field

In terms of renormalization the anomalous dimension of the mass operator $\frac{1}{2}\phi^2$ is not independent and proportional to the β-function

The renormalization group functions are available to five loops [Macfarlane & Woo; McKane et al; JAG; Kompaniets & Pikelner; Borinsky, Kompanients, Schnetz & JAG]

More recently this has been extended to *six* loops by Schnetz

So the critical exponents are available to $O(\epsilon^6)$

While the main dimensions of interest are three, four and five ϕ^3 theory is renormalizable in six dimensions leading to a similar convergence problem GNY

For example, we deduced

$$\begin{aligned}\sigma \quad = \quad & \frac{1}{2} - \frac{1}{12}\epsilon - \frac{79}{3888}\epsilon^2 + \left[\frac{1}{81}\zeta_3 - \frac{10445}{1259712}\right]\epsilon^3 \\ & + \left[\frac{1}{108}\zeta_4 - \frac{4047533}{408146688} - \frac{161}{26244}\zeta_3 - \frac{5}{729}\zeta_5\right]\epsilon^4 \\ & + \left[\frac{17}{1458}\zeta_3^2 + \frac{20101}{78732}\zeta_5 + \frac{112399}{944784}\zeta_3 - \frac{1601178731}{132239526912} - \frac{1043}{2916}\zeta_7 \right. \\ & \quad \left. - \frac{161}{34992}\zeta_4 - \frac{25}{2916}\zeta_6\right]\epsilon^5 \\ & + \left[\frac{17}{972}\zeta_3\zeta_4 + \frac{224}{2187}\zeta_3^3 + \frac{289}{1215}\zeta_{5,3} + \frac{32669}{236196}\zeta_3^2 + \frac{49645}{157464}\zeta_6 \right. \\ & \quad + \frac{112399}{1259712}\zeta_4 + \frac{216839}{39366}\zeta_9 - \frac{158574097133}{14281868906496} - \frac{156752701}{153055008}\zeta_5 \\ & \quad \left. - \frac{107952203}{344373768}\zeta_3 - \frac{7029955}{1889568}\zeta_7 - \frac{7781}{6480}\zeta_8 - \frac{976}{19683}\zeta_3\zeta_5\right]\epsilon^6 \quad O(\epsilon^7)\end{aligned}$$

$$\sigma \ = \ 0.5000 - 0.0833\epsilon - 0.0203\epsilon^2 + 0.0065\epsilon^3 - 0.0144\epsilon^4 + 0.0381\epsilon^5 - 0.1227\epsilon^6 + O(\epsilon^7)$$

There is a rise in the magnitude of the $O(\epsilon^6)$ coefficients across all the main exponents which means convergence for $\epsilon = 3$ could be problematic

A two-sided Padé analysis would require the perturbative renormalization group functions of the field theory of the universality class that is relevant *near* two dimensions

What this field theory is has yet to be determined

Instead the purely two dimensional conformal theory is known which is the $M(2,5)$ minimal model

The exponents of all two dimensional conformal field theories are known [Friedan, Qiu, Shenker]

Additionally the *one* dimensional exponents are also available giving two low dimension boundary conditions for a two-sided Padé analysis

An interpolating polynomial approach relies on the ϵ expansion up from the lower dimensional boundary

As the six loop exponents have seven known coefficients including the ϵ independent term plus two low dimension boundary conditions a large number of possible approximants can be constructed

This means a statistical analysis can be carried out in principle

However this is qualified by the fact that a rational polynomial approximation to a continuous function of d in $2 \leq d \leq 6$ may not in itself be continuous or monotonic

Discontinuous approximants and non-monotonic ones are discarded but this still leaves a reasonable number for an analysis

For the valid ones, $\mathcal{P}$, we define a weighted average

$$\mu_{\rm wt} ~=~ \frac{\sum_m w_m \mathcal{P}_m}{\sum_n w_n}$$

and a standard deviation $\varsigma_{\rm wt}$ where the weights w_n are the loop order L

Method	d	η	σ	ν_c
CB	3	$-$ 0.530(5)	0.085(1)	0.3617(4)
	4	$-$ 0.3067(5)	0.2685(1)	0.3171(3)
	5	$-$ 0.090(3)	0.4105(5)	0.2821(1)
FRG	3	$-$ 0.586(29)	0.0742(56)	0.3581(19)
	4	$-$ 0.316(16)	0.2667(32)	0.3167(8)
	5	$-$ 0.126(6)	0.4033(12)	0.2807(2)
CB	3	$-$ 0.651	0.062	0.354
	4	$-$ 0.353	0.259	0.315
	5	$-$ 0.124	0.404	0.2801
ϵ	3	$-$ 0.580(7)	0.078(2)	0.359(1)
	4	$-$ 0.356(6)	0.2604(14)	0.3151(3)
	5	$-$ 0.1521(13)	0.3984(2)	0.2797(1)
CB	3	$-$ 0.564	0.078	0.359
	4	$-$ 0.346	0.261	0.315
	5	$-$ 0.140	0.399	0.280
6 loop	3	$-$ 0.578(5)	0.078(8)	0.3584(12)
	4	$-$ 0.354(6)	0.260(5)	0.3147(6)
	5	$-$ 0.152(1)	0.3985(8)	0.27965(11)

Two separate conformal bootstrap exponent measurements in three dimensions were made using fuzzy spheres that included ω values [2505.06369(A), 2505.06342(B)]

Work	η	σ	ν_c	θ	ω
A	$-$ 0.572(4)	0.0768(8)	0.3589(3)	0.579(3)	1.613(6)
	$-$ 0.5790(32)	0.0774(6)	0.3591(2)	———	———
	$-$ 0.5698(16)	0.0772(3)	0.3591(2)	———	———
B	$-$ 0.42	0.11	0.37	0.63	1.71
6 loop	$-$ 0.578(5)	0.078(8)	0.3584(12)	0.590(20)	1.615(1)

There appears to be good agreement of the six loop exponent estimates using the one and two dimensional boundary conditions with A especially with the two correction to scaling exponents θ and ω

Percolation theory

Again the critical properties of percolation theory are given by a decoration of ϕ^3 theory that gives the $(N+1)$-state Potts model with Lagrangian

$$L = \frac{1}{2}\left(\partial_\mu \phi^i\right)^2 + \frac{g}{6} d^{ijk} \phi^i \phi^j \phi^k$$

where d^{ijk} is fully symmetric, for example,

$$d^{ipq} d^{jpq} = (N+1)^2 (N-1) \delta^{ij} \quad , \quad d^{ipq} d^{jpr} d^{kqr} = (N+1)^2 (N-2) d^{ijk}$$

The first has parallels with T_F in Yang-Mills and the second would equate to $\frac{1}{2} C_A$

The $N \to 0$ limit of the six loop renormalization group functions calculated by [Schnetz] gives the percolation theory results

Followed a similar two-sided Padé approximant approach as for Lee-Yang theory except that only the values of the two dimensional exponents are known

For example from the mass renormalization we derive

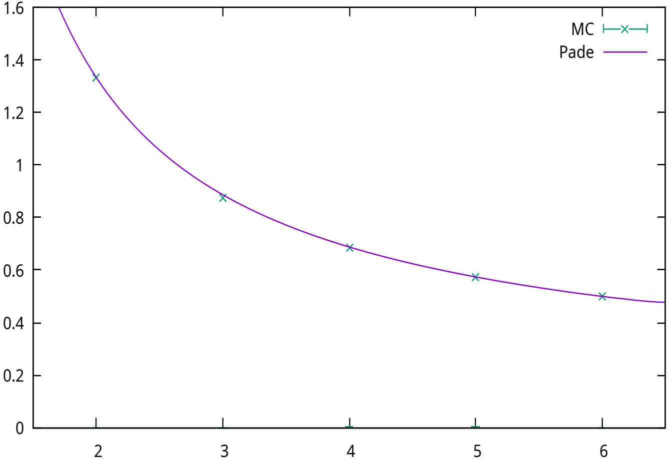
$$
\begin{aligned}
\nu \;=\;& \frac{1}{2} + \frac{5}{84}\epsilon + \frac{589}{37044}\epsilon^2 + \left[\frac{716519}{130691232} - \frac{89}{7203}\zeta_3\right]\epsilon^3 \\
&+ \left[\frac{344397667}{230539333248} - \frac{222359}{38118276}\zeta_3 - \frac{89}{9604}\zeta_4 + \frac{940}{21609}\zeta_5\right]\epsilon^4 \\
&+ \left[\frac{33500}{7411887}\zeta_3^2 + \frac{141995802917}{58095911978496} - \frac{313903867}{9605805552}\zeta_3 - \frac{711901}{9529569}\zeta_5 \right. \\
&\quad \left. - \frac{222359}{50824368}\zeta_4 - \frac{738}{16807}\zeta_7 + \frac{1175}{21609}\zeta_6\right]\epsilon^5 \\
&+ \left[\frac{1893082324019}{16944640993728}\zeta_3 + \frac{29757051275785}{34160396243355648} - \frac{313903867}{12807740736}\zeta_4 \right. \\
&\quad - \frac{13649285}{152473104}\zeta_6 - \frac{6170698}{3176523}\zeta_9 - \frac{361511}{8235430}\zeta_8 \\
&\quad - \frac{56136}{4117715}\zeta_{5,3} - \frac{11744}{352947}\zeta_3^3 + \frac{16750}{2470629}\zeta_3\zeta_4 + \frac{975664}{7411887}\zeta_3\zeta_5 \\
&\quad \left. + \frac{3793208}{466948881}\zeta_3^2 + \frac{17842757}{8470728}\zeta_7 + \frac{83802155}{600362847}\zeta_5\right]\epsilon^6 \; + \; O(\epsilon^7)
\end{aligned}
$$

$$
\nu \;=\; 0.5000 + 0.0595\epsilon + 0.0159\epsilon^2 - 0.0094\epsilon^3 + 0.0296\epsilon^4 - 0.1015\epsilon^5 + 0.4220\epsilon^6 + O(\epsilon^7)
$$

ν estimates

Year	$d = 3$	$d = 4$	$d = 5$
1976	0.80(5)	———	———
1985	———	———	0.51(5)
1990	0.872(7)	0.6782(50)	0.571(3)
1997	———	0.689(10)	———
1998	0.875(1)	———	———
2000	0.8765(18)	———	———
2005	———	———	0.569(5)
2013	0.8764(12)	———	———
2014	0.8751(11)	———	———
2014	0.8762(12)	———	———
2015	**0.8960**	**0.6920**	**0.5746**
2016	0.8774(13)	0.6852(28)	0.5723(18)
2018	———	0.693	———
2020	———	0.6845(6)	0.5757(7)
2021	**0.88(2)**	**0.686(2)**	**0.5739(1)**
2021	———	0.6845(23)	0.5737(33)
2022	0.8762(7)	0.6842(16)	0.5720(43)
6 **loop**	**0.895(20)**	**0.691(8)**	**0.5745(10)**

ν



Data points are for ν from Brzeski and Kondrat using overlapping discrete hyperspheres on hypercubic lattices in three, four and five dimensions

For ν majority of previous results are extracted from high-temperature expansions or Monte Carlo simulations of discrete spin models

The numerical investigations are in fixed dimensions and cannot access the dependence when the dimension is continuous

Similar broad agreement for all the other exponents across the three different dimensions but most measurements are made for η, ν, τ and the fractal dimension d_f

There are fewer measurements of the correction to scaling exponents, ω and Ω leading a less clear picture of the convergence of different methods to a consensual value

Conclusions

Have examined a variety or relatively simple theories where critical exponents can be extracted to very high loop order

Exponents can reveal interesting properties of a theory such as the emergence of symmetries at certain points

The underlying structural similarities of the GNY system, and its extension to include gauge fields, with the Standard Model indicate there may be hidden clues as to SM extensions

The immediate application of GNY to the particular cases that relate to condensed matter systems suggest that with high enough loop order exponent estimates are commensurate with those from other methods such as the conformal bootstrap

Equally exponents for Lee-Yang theory are also consistent with recent bootstrap studies in three dimensions

In the case of percolation theory the main comparison at the moment is with Monte Carlo studies rather than any other continuum technique