Complete 3 Loop unpolarized and polarized heavy flavor corrections to DIS for $Q^2 \gg m^2$

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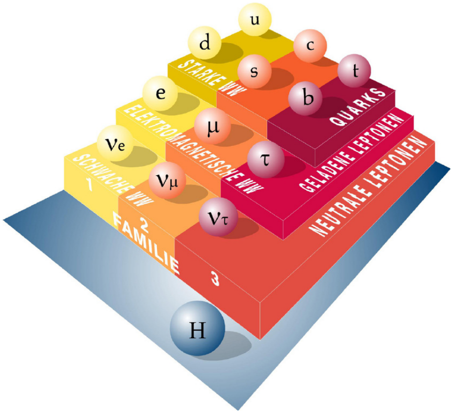
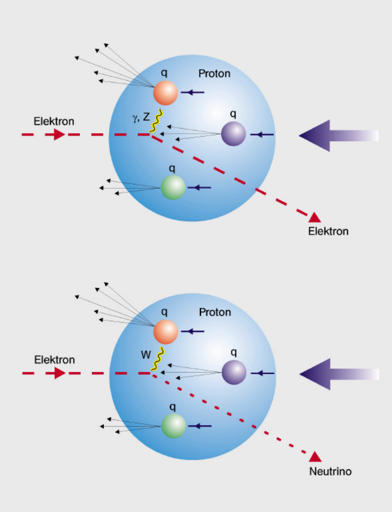
The Mission of Elementary Particle Physics

- verify: $SU_{3,c} \otimes SU_{2L} \otimes U_{1Y}$ and its breaking at high precision
- find all fundamental particles
- resolve the complete interior of hadrons:
  parton distribution functions, composition of nucleon spin
- find unexpected fundamental particles
- find new substructures
- determine the masses of all fundamental particles at high precision: heavy quarks
- determine the couplings of the fundamental forces at high precision: $\alpha_s(Q^2)$

The Mission of Elementary Particle Physics



$+\gamma, Z^0, W^{\pm}, 8$ gluons



Hadron Microscopy

The Mission of Elementary Particle Physics

$$\frac{\delta\alpha_{\rm em}(0)}{\alpha_{\rm em}(0)} = 3 \cdot 10^{-11}, \qquad \frac{\delta\alpha_{\rm weak}}{\alpha_{\rm weak}} = 7 \cdot 10^{-4}, \qquad \frac{\delta\alpha_s(M_Z^2)}{\alpha_s(M_Z^2)} > 3 \cdot 10^{-2}$$

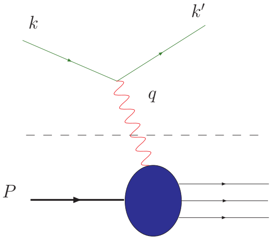
- more precision for $\alpha_s(M_Z^2)$ needed
- higher precision for m_c
- higher theoretical precision
- new high luminosity data: EIC at BNL [after 2030]

- Four loop evolution needed
- Three loop massive corrections needed

Theory uncertainties must be lower than the experimental resolution.

$$\frac{\delta_{TH}\alpha_s(M_Z^2)}{\alpha_s(M_Z^2)} < 0.005, \quad \delta_{TH} m_c < 10~\mathrm{MeV} \quad \text{currently: 70 MeV}$$

Deep–Inelastic Scattering (DIS):



$\longrightarrow \quad L_{\mu\nu}$

$\longrightarrow \quad W_{\mu\nu}$

$$Q^2 := -q^2, \quad x := \frac{Q^2}{2P.q} \quad \text{Bjorken–x}$$

$$\frac{d\sigma}{dQ^2\, dx} \sim W_{\mu\nu} L^{\mu\nu}$$

$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi}\int d^4\xi \exp(iq\xi)\langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s\rangle =$$

$$\frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x,Q^2)$$

$$+i\varepsilon_{\mu\nu\lambda\sigma}\frac{q^\lambda S^\sigma}{P.q} g_1(x,Q^2) + i\varepsilon_{\mu\nu\lambda\sigma}\frac{q^\lambda (P.qS^\sigma - S.qP^\sigma)}{(P.q)^2} g_2(x,Q^2)\,.$$

The structure functions $F_{2,L}$ and $g_{1,2}$ contain light and heavy quark contributions.
At 3-loop order also graphs with two heavy quarks of different mass contribute.
$\Longrightarrow$ Single and 2-mass contributions: c and b quarks in one graph.

Factorization of the Structure Functions

At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x,Q^2) = \sum_j \underbrace{\mathbb{C}_{j,(2,L)}\left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right)}_{perturbative} \otimes \underbrace{f_j(x,\mu^2)}_{nonpert.}$$

into (pert.) Wilson coefficients and (nonpert.) parton distribution functions (PDFs).
$\otimes$ denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz \; \delta(x - yz) f(y) g(z) \, .$$

Many of the subsequent calculations are performed in Mellin space, where $\otimes$ reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) = \int_0^1 dx \; x^{N-1} f(x) \, .$$

The PDF evolution and massless Wilson coefficients

$$Q^2 \frac{d}{dQ^2} \begin{pmatrix} \Sigma(N) \\ G(N) \end{pmatrix} = \begin{pmatrix} P_{qq}(a_s, N) & P_{qg}(a_s, N) \\ P_{gq}(a_s, N) & P_{gg}(a_s, K) \end{pmatrix} \cdot \begin{pmatrix} \Sigma(N) \\ G(N) \end{pmatrix}$$

$$P_{ij} = a_s P_{ij}^{(0)}(N) + a_s^2 P_{ij}^{(1)}(N) + a_s^3 P_{ij}^{(2)}(N)$$

[Moch, Vermaseren, Vogt, 2004, 2014]

3-loop massless Wilson coefficients:

$$C_k(a_s, N) = \delta_{kq} + a_s C_k^{(1)}(N) + a_s^2 C_k^{(2)}(N) + a_s^3 C_k^{(3)}(N)$$

Unpolarized: [Moch, Vermaseren, Vogt, 2005]

Polarized: [JB, Marquard, Schneider, Schönwald, 2022]

Wilson coefficients:

$$\mathbb{C}_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) = C_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right) + H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) .$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) = \sum_i C_{i,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right) A_{ij}\left(\frac{m^2}{\mu^2},N\right)$$

[Buza, Matiounine, Smith, van Neerven 1996]

factorizes into the light flavor Wilson coefficients C and the massive operator matrix elements (OMEs) of local operators O_i between partonic states j

$$A_{ij}\left(\frac{m^2}{\mu^2},N\right) = \langle j \mid O_i \mid j\rangle .$$

$\to$ additional Feynman rules with local operator insertions for partonic matrix elements.

The unpolarized light flavor Wilson coefficients are known up to NNLO [Vermaseren, Moch, Vogt, 2005; JB, Marquard, Schneider, Schönwald, 2022].

For $F_2(x,Q^2)$: at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

1- and 2-loops

- 1976-1982; 1991: Analytic 1-loop results
  E. Witten; J. Babcock, D. W. Sivers, S. Wolfram; M.A. Shifman, A.I. Vainshtein, V.I. Zakharov; J.P. Leveille, T.J. Weiler; M. Glück, E. Hoffmann, E. Reya; C. Watson, W. Vogelsang
- 1992-1995: Numeric 2-loop results
  E. Laenen, W. van Neerven, S. Riemersma, J. Smith
  1995-1998: Analytic 2-loop results
  M. Buza, Y. Matiounine, R. Migneron, W. van Neerven, J. Smith
- 2007, 2009: Correcting earlier 2-loop results
  I. Bierenbaum, JB, S. Klein

$$1976 \underset{19\text{ yr}}{\Longrightarrow} 1995 \underset{\substack{30\text{ yr}\\ [18\text{ yr}]}}{\Longrightarrow} 2025$$

The main time-line for the 3-loop corrections

- 2005 F_L [no massive 3-loop OMEs needed]
- 2010 All unpolarized N_F terms and $A_{qg,Q}^{(3)}$, $A_{qq,Q}^{(3).PS}$
- 2014 unpolarized logarithmic 3-loop contributions and $A_{gq,Q}^{(3)}$,$(\Delta)A_{qq,Q}^{(3),NS}$, $A_{Qq}^{(3),PS}$
- 2017 two-mass corrections $A_{gq,Q}^{(3)}$, $(\Delta)A_{qq,Q}^{(3),NS}$, $A_{Qq}^{(3),PS}$
- 2018 two-mass corrections $A_{gg,Q}^{(3)}$
- 2019 2-loop correction: $(\Delta)A_{Qq}^{(2),PS}$ whole kinematic region and $\Delta A_{Qq}^{(3),PS}$
- 2019 two-mass corrections $\Delta A_{Qq}^{(3),PS}$
- 2020 two-mass corrections $\Delta A_{gg,Q}^{(3)}$
- 2021 polarized logarithmic 3-loop contributions and $\Delta A_{qg,Q}^{(3)}$, $\Delta A_{qq,Q}^{(3).PS}$, $\Delta A_{gq}^{(3)}$
- 2022 3-loop polarized massless Wilson coefficients [JB, Marquard, Schneider, Schönwald]
- 2022 corrected the polarized 2-loop contributions [Buza, van Neerven et al.]
- 2022 $(\Delta)A_{gg,Q}^{(3)}$
- 2023 $(\Delta)A_{Qg}^{(3)}$: 1st order factorizing parts
- 2024 $(\Delta)A_{Qg}^{(3)}$, [single-mass corrections]
- 2025 $(\Delta)A_{Qg}^{(3)}$, [two-mass corrections]

- 49 physics papers (journals)
- 26 mathematical papers
  - 1998 Harmonic sums [Vermaseren; JB]
  - 2000,2005 Analytic continuations of harmonic sums to $N \in \mathbb{C}$ [JB; JB, S. Moch]
  - 2003 Concrete shuffle algebras [JB]
  - 2009 Guessing large recurrences [JB, M. Kauers, S. Klein, C. Schneider]
  - 2009 Structural relations of harmonic sums [JB]
  - 2009 MZV Data mine [JB, D. Broadhurst, J. Vermaseren]
  - 2011 Cyclotomic harmonic sums and harmonic polylogarithms [Ablinger, JB, Schneider]
  - 2013 Generalized harmonic sums and harmonic polylogarithms [Ablinger, JB, Schneider]; 2001 [Moch, Uwer, Weinzierl]
  - 2014 Finite binomial sums and root-valued iterated integrals [Ablinger, JB, Raab, Schneider]
  - 2017 ${}_2F_1$ solutions (iterated non-iterative integrals) [J. Ablinger, JB, A. De Freitas, M. van Hoeij, E. Imamoglu, C. Raab, C.S. Radu, C. Schneider]
  - 2017 Methods of arbitrary high moments [JB, Schneider]
  - 2018 Automated solution of first-order factorizing differential equation systems in an arbitrary basis [J. Ablinger, JB, P. Marquard, N. Rana, C. Schneider]
  - 2023 Analytic continuation form t to x-space [JB, Behring, Schönwald]

Important Computer-Algebra Packages

C. Schneider: `Sigma`, `EvaluateMultiSums`, `SumProduction`, `SolveCoupledSystem`
J. Ablinger: `HarmonicSums`

Why take large projects rather long ?

Examples:

- unpolarized anomalous dimensions and massless DIS Wilson coefficients
  [Vermaseren, Larin, Nogueira, van Ritbergen, Moch, Vogt] 1990-2005: 15 years
  function space: harmonic sums
- unpolarized and polarized massive OMEs and asymptotic Wilson coefficients
  [DESY-Linz collaboration] 2009 - 2025: 16 years
  function spaces: harmonic sums, generalized harmonic sums, finite cyclotomic sums, finite binomial sums, elliptic integrals, higher transcendental ${}_pF_q$ structures

- Initially the function spaces contributing were unknown.
- How to solve the systems of difference equations for the contributing topologies ?
- How to process the differential equations of the master integrals to provide large numbers of moments ?
- How to deal with all first order factorizing difference equations ?
- How to solve the elliptic-affected part ?
- How to tackle 2-mass problems analytically ?
- Are N-space solutions providing the right framework ? [Non-first order factorizable recurrences.]
- Do the computer resources suffice [in space and time] to establish all contributing recurrences ?

Why take large projects rather long ?

- At present, massless 3-loop problems are no problem anymore.
  - Typical computation times $O(1 year)$; pole-terms: $O(month)$.
  - Basically all technologies needed are available in (private) codes.
  - Example: Polarized massless 3-loop Wilson coefficients for DIS.

These calculations are modern adventures.

One enters a terra incognita with rough ideas but insufficient means and one has to develop new technologies all the way along to get through. In this way one lifts the whole field to new levels, which allows to perform many more calculations.

One has to pass many intermediate stops (no-goes) to arrive at the final complete solution: the strategic goal.

Copernican turns:

[J. Reinhardt scan & Deutsche Bundespost]

I. Kant, Vorwort zu Kritik der Reinen Vernunft:

Es iſt hiermit ebenſo, als mit den erſten Gedanken des Kopernikus bewandt, der, nachdem es mit der Erklärung der Himmelsbewegungen nicht gut fort wollte, wenn er annahm, das ganze Sternenheer drehe ſich um den Zuſchauer, verſuchte, ob es nicht beſſer gelingen möchte, wenn er den Zuſchauer ſich drehen, und dagegen die Sterne in Ruhe ließ. In der Metaphyſik kann man nun, was die Anſchauung der Gegenſtände betrifft, es auf ähnliche Weiſe verſuchen.

If one view [method] does not solve (a part of) a problem, try the adjoint view.

The solution to any problem is simple, after the correct Meta-Language has been found.

This is very much related to Gell-Mann's question: **Why not ?**

Copernican turns:

- Change from N-space methods to x-space methods (and probably back).
- Use local information (simpler equations), if the solution of global equations are to difficult or even impossible at the time.
- Don't forget about the Principle of Simplicity.

[BWV 846, from Wikipedia]

The Wilson Coefficients at large Q^2

$$L_{q,(2,L)}^{\rm NS}(N_F+1) = a_s^2\left[A_{qq,Q}^{(2),\rm NS}(N_F+1)\delta_2 + \hat{C}_{q,(2,L)}^{(2),\rm NS}(N_F)\right] + a_s^3\left[A_{qq,Q}^{(3),\rm NS}(N_F+1)\delta_2 + A_{qq,Q}^{(2),\rm NS}(N_F+1)C_{q,(2,L)}^{(1),\rm NS}(N_F+1) + \hat{C}_{q,(2,L)}^{(3),\rm NS}(N_F)\right]$$

$$L_{q,(2,L)}^{\rm PS}(N_F+1) = a_s^3\left[A_{qq,Q}^{(3),\rm PS}(N_F+1)\delta_2 + N_F A_{gq,Q}^{(2),\rm NS}(N_F)\tilde{C}_{g,(2,L)}^{(1),\rm NS}(N_F+1) + N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\rm PS}(N_F)\right]$$

$$\begin{aligned} L_{g,(2,L)}^{\rm S}(N_F+1) = {}& a_s^2 A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1) + a_s^3\left[A_{qg,Q}^{(3)}(N_F+1)\delta_2 + A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\right.\\ &\left.+ A_{gg,Q}^{(2)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + A_{Qg}^{(1)}(N_F+1)N_F\tilde{C}_{q,(2,L)}^{(2),\rm PS}(N_F+1) + N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\right] \end{aligned}$$

$$\begin{aligned} H_{q,(2,L)}^{\rm PS}(N_F+1) = {}& a_s^2\left[A_{Qq}^{(2),\rm PS}(N_F+1)\delta_2 + \tilde{C}_{q,(2,L)}^{(2),\rm PS}(N_F+1)\right]\\ &+ a_s^3\left[A_{Qq}^{(3),\rm PS}(N_F+1)\delta_2 + A_{gq,Q}^{(2)}(N_F+1)\tilde{C}_{g,(1,L)}^{(2)}(N_F+1) + A_{Qq}^{(2),\rm PS}(N_F+1)\tilde{C}_{q,(2,L)}^{(1),\rm NS}(N_F+1) + \tilde{C}_{q,(2,L)}^{(3),\rm PS}(N_F+1)\right] \end{aligned}$$

$$\begin{aligned} H_{g,(2,L)}^{\rm S}(N_F+1) = {}& a_s\left[A_{Qg}^{(1)}(N_F+1)\delta_2 + \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\right]\\ &+ a_s^2\left[A_{Qg}^{(2)}(N_F+1)\delta_2 + A_{Qg}^{(1)}(N_F+1)\tilde{C}_{q,(2,L)}^{(1)}(N_F+1) + A_{gg,Q}^{(1)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + \tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\right]\\ &+ a_s^3\left[A_{Qg}^{(3)}(N_F+1)\delta_2 + A_{Qg}^{(2)}(N_F+1)\tilde{C}_{q,(2,L)}^{(1)}(N_F+1) + A_{gg,Q}^{(2)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\right.\\ &\left.+ A_{Qg}^{(1)}(N_F+1)\tilde{C}_{q,(2,L)}^{(2),\rm S}(N_F+1) + A_{gg,Q}^{(1)}(N_F+1)\tilde{C}_{g,(2,L)}^{(2)}(N_F+1) + \tilde{C}_{g,(2,L)}^{(3)}(N_F+1)\right] \end{aligned}$$

- The case for two different masses obeys an analogous representation.
- Note the contributions of the massless Wilson coefficients.

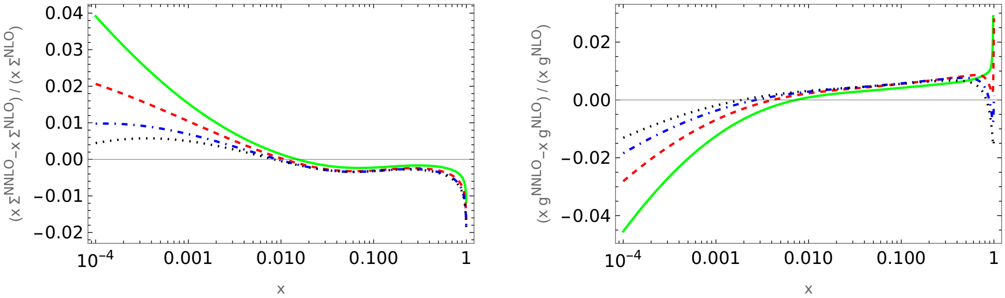
The variable flavor number scheme

- Matching conditions for parton distribution functions:

$$
\begin{aligned}
f_k(N_F+2) + f_{\overline{k}}(N_F+2) &= A_{qq,Q}^{\rm NS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot \left[f_k(N_F) + f_{\overline{k}}(N_F)\right] + \frac{1}{N_F} A_{qq,Q}^{\rm PS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot \Sigma(N_F) \\
&+ \frac{1}{N_F} A_{qg,Q}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot G(N_F)\,, \\
f_Q(N_F+2) + f_{\overline{Q}}(N_F+2) &= A_{Qq}^{\rm PS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2},\right) \cdot \Sigma(N_F) + A_{Qg}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot G(N_F)\,, \\
\Sigma(N_F+2) &= \left[A_{qq,Q}^{\rm NS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) + A_{qq,Q}^{\rm PS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) + A_{Qq}^{\rm PS}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right)\right] \cdot \Sigma(N_F) \\
&+ \left[A_{qg,Q}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) + A_{Qg}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right)\right] \cdot G(N_F)\,, \\
G(N_F+2) &= A_{gq,Q}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot \Sigma(N_F) + A_{gg,Q}\left(N_F+2, \frac{m_1^2}{\mu^2}, \frac{m_2^2}{\mu^2}\right) \cdot G(N_F)\,.
\end{aligned}
$$

The charm and bottom quark masses are not that much different.

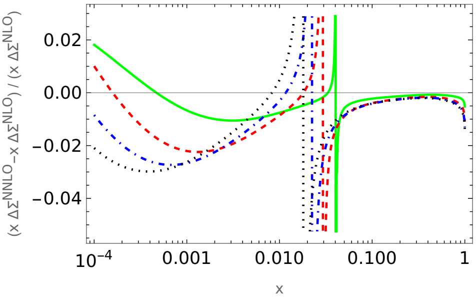
Relative effect in unpolarized NNLO evolution

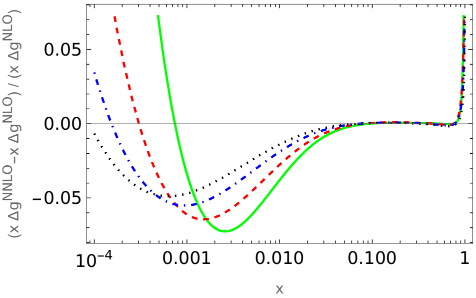


$Q^2 = 10, 10^2, 10^3, 10^4$ GeV^2 dotted, dash-dotted, dashed, full lines. [M. Saragnese, 2022]

The unpolarized world deep-inelastic data have a precision of $O(1\%)$.

Relative effect in polarized NNLO evolution





$Q^2 = 10, 10^2, 10^3, 10^4$ GeV^2 dotted, dash-dotted, dashed, full lines. [M. Saragnese, 2022]

The future polarized data at the EIC will reach a precision of $O(1\%)$.

Calculation methods

- Diagram generation: `QGRAF` [Nogueira, 1993]
- Lorentz and Dirac algebra: `Form` [Vermaseren, 2000]
- Color algebra: `Color` [van Ritbergen, Schellekens, Vermaseren, 1999]
- IBP reduction: `Reduze 2` [von Manteuffel, Studerus 2009,2012]
- *N* space calculations:
  - Method of arbitrary large moments [JB, Schneider, 2017]
  - Summation theory and solving first-order factorizing recurrences: `Sigma` [Schneider, 2007,2013]
  - Reduce the results in the respective function spaces: `HarmonicSums` [Ablinger, 2009, 2012, etc.]
- *x* space calculations
  - solve 1st order factorizing differential equations
  - transform from $N \to t$-space, solve the respective systems of differential equations (not necessarily factorizing to first order) [Behring, JB, Schönwald, 2023]
  - Reduce the results in the respective function spaces; iterated integrals over alphabets containing also higher transcendental letters [Ablinger et al. 2017]
  - The higher transcendental letters have to be known in analytic form for $z \in \mathbb{C}$.
- Both *N* and *x* space techniques are needed to solve the present problem. The recurrences for $A_{Qg}^{(3)}$ need far more than 15000 moments to be found & there are no technologies yet to solve non-first order factorizing recurrences analytically.
- Currently we have technologies to generate $O(25000)$ moments $\Longrightarrow$ massive 3-loop form factor and we guess differential equations based on $O(50000)$.

Mathematical Background

- massless and massive contributions to two-loops: harmonic sums
- all pole terms to three-loops: harmonic sums
- all massless Wilson coefficients to three-loops: harmonic sums

Single-mass OMEs

- all N_F of the massive OMEs three-loops: harmonic sums
- $(\Delta)A_{qq,Q}^{(3),\mathrm{NS}}, (\Delta)A_{gq,Q}^{(3)}, (\Delta)A_{qg,Q}^{(3)}, (\Delta)A_{qq,Q}^{(3),\mathrm{PS}}$ to three-loops: harmonic sums
- $(\Delta)A_{Qq}^{(3),\mathrm{PS}}$ to three-loops: generalized harmonic sums and also $H_{\vec{a}}(1-2x)$
- $(\Delta)A_{gg,Q}^{(3)}$ to three-loops: finite binomial sums and square-root valued iterated integrals
- $(\Delta)A_{Qg}^{(3)}$ to three-loops:
  - first-order factorizing contributions: finite binomial sums; all iterated integrals in x-space can be rationalized
  - non-first-order factorizing contributions: ${}_2F_1$ letters in iterated integrals in x-space

Inverse Mellin transform via analytic continuation: $a_{Qg}^{(3)}$

Resumming Mellin N into a continuous variable t, observing crossing relations. Ablinger et al. 2012

$$\sum_{k=0}^{\infty} t^k (\Delta.p)^k \frac{1}{2}[1 \pm (-1)^k] = \frac{1}{2}\left[\frac{1}{1-t\Delta.p} \pm \frac{1}{1+t\Delta.p}\right]$$

$$\mathfrak{A} = \{f_1(t), ..., f_m(t)\}, \quad \mathrm{G}(b, \vec{a}; t) = \int_0^t dx_1 f_b(x_1) \mathrm{G}(\vec{a}; x_1), \quad \left[\frac{d}{dt}\frac{1}{f_{a_{k-1}}(t)}\frac{d}{dt}\cdots\frac{1}{f_{a_1}(t)}\frac{d}{dt}\right]\mathrm{G}(\vec{a}; t) = f_{a_k}(t).$$

The $f_i(t)$ include higher transcendental letters. Regularization for $t \to 0$ needed.

$$\begin{aligned} F(N) &= \int_0^1 dx x^{N-1}[f(x) + (-1)^{N-1} g(x)] \\ \tilde{F}(t) &= \sum_{N=1}^{\infty} t^N F(N) \\ f(x) + (-1)^{N-1} g(x) &= \frac{1}{2\pi i}\left[\mathrm{Disc}_x \tilde{F}\left(\frac{1}{x}\right) + (-1)^{N-1} \mathrm{Disc}_x \tilde{F}\left(-\frac{1}{x}\right)\right]. \end{aligned} \qquad (1)$$

t-space is still Mellin space. One needs closed expressions to perform the analytic continuation (1). Analytic continuation is needed to calculate the small x behavior. The final expansion maps the problem into a very large number of G-constants, including those with higher transcendental letters.

Iterative non-iterative Integrals

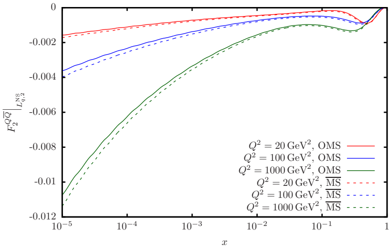
- Master integrals, solving differential equations not factorizing to 1st order
- ${}_2F_1$ solutions Ablinger et al. [2017]
- Mapping to complete elliptic integrals: duplication of the higher transcendental letters.
- Complete elliptic integrals, modular forms Sabry, Broadhurst, Weinzierl, Remiddi, Tancredi, Duhr, Broedel et al. and many others.
- Abel integrals
- K3 surfaces Brown, Schnetz [2012]
- Calabi-Yau motives Klemm, Duhr, Weinzierl et al. [2022]

Refer to as few as possible higher transcendental functions, the properties of which are known in full detail.

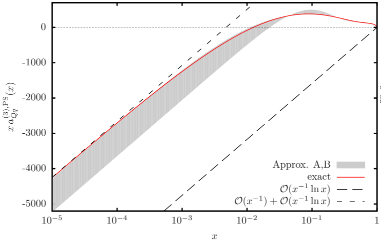
- $A_{Qg}^{(3)}$: effectively only one 3×3 system of this kind.
- The system is connected to that occurring in the case of ρ parameter. Ablinger et al. [2017], JB et al. [2018], Abreu et al. [2019]
- Most simple solution: two ${}_2F_1$ functions.

Quantitative Results

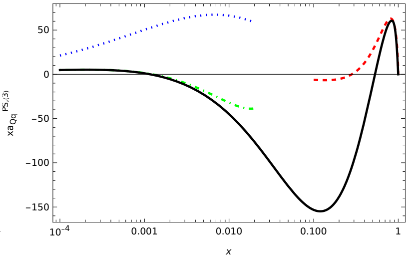
Non-singlet and pure singlet



NS sHQ F_2



PS unpolarized

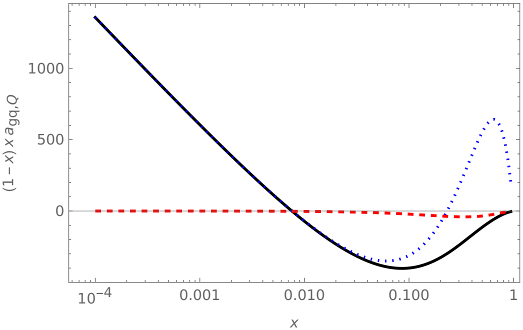


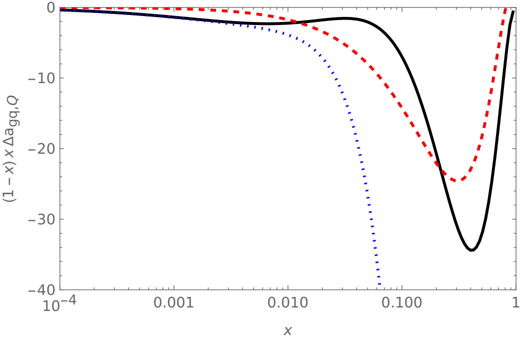
PS polarized

[J. Ablinger et al., 2014 a,b; 2020]

Unpolarized PS: failure of the BFKL approach due to missing subleading terms.

$a_{gq}^{(3)}$ and $\Delta a_{gq}^{(3)}$

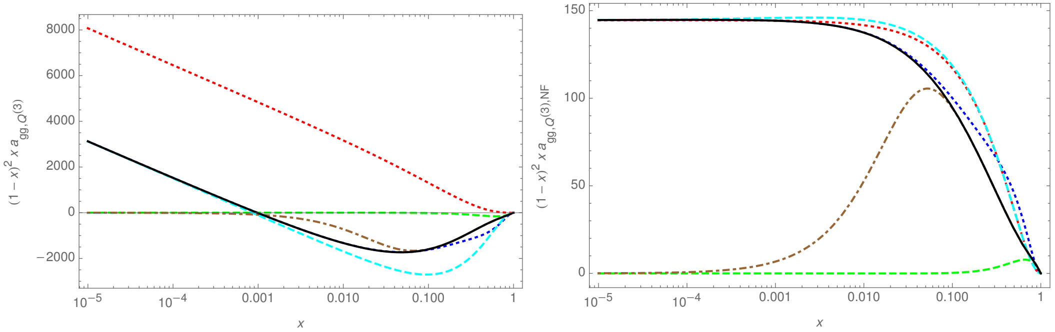




$N_F = 3$. Full line (black): complete result; dotted line (blue): small x expansion; dashed line (red): large x expansion.

[J. Ablinger et al., 2014; 2020]

$a_{gg}^{(3)}$

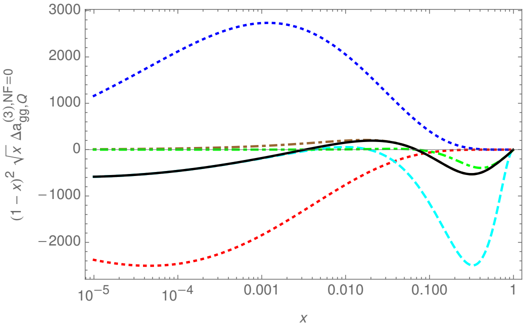


non N_F terms $\qquad$ N_F terms

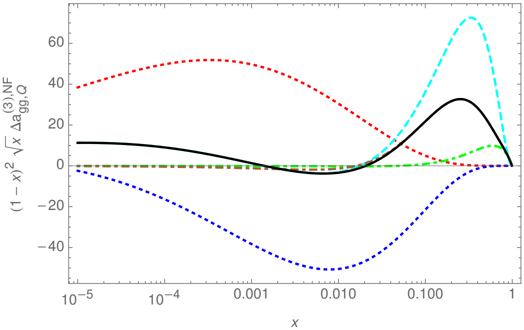
Left panel: The non–N_F terms of $a_{gg,Q}^{(3)}(N)$ (rescaled) as a function of x. Full line (black): complete result; upper dotted line (red): term $\propto \ln(x)/x$; lower dashed line (cyan): small x terms $\propto 1/x$; lower dotted line (blue): small x terms including all $\ln(x)$ terms up to the constant term; upper dashed line (green): large x contribution up to the constant term; dash-dotted line (brown): complete large x contribution. Right panel: the same for the N_F contribution.

[J. Ablinger et al., 2022]

$\Delta a_{gg}^{(3)}$



non N_F terms



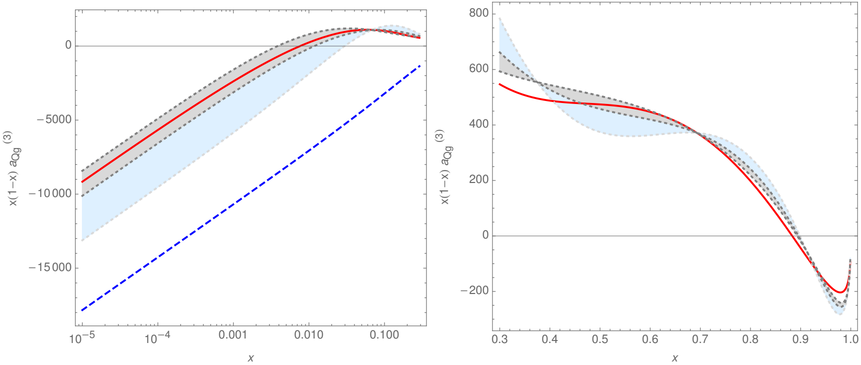
N_F terms

The non–N_F terms of $\Delta a_{gg,Q}^{(3)}(N)$ (rescaled) as a function of x. Full line (black): complete result; lower dotted line (red): term $\ln^5(x)$; upper dotted line (blue): small x terms $\propto \ln^5(x)$ and $\ln^4(x)$; upper dashed line (cyan): small x terms including all $\ln(x)$ terms up to the constant term; lower dash-dotted line (green): large x contribution up to the constant term; dash-dotted line (brown): full large x contribution. Right panel: the same for the N_F contribution.

[J. Ablinger et al., 2022]

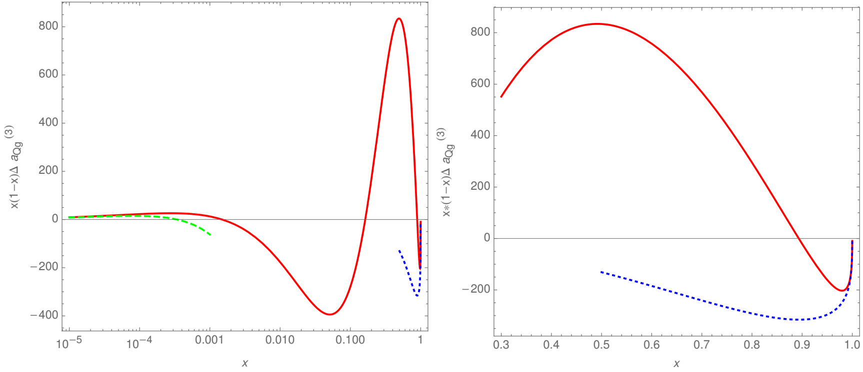
$a_{Qg}^{(3)}$

1009 of the total 1233 Feynman diagrams have first-order factorizing contributions only and are given by G-functions up to root-values letters. The letters for all constants can be rationalized.



$a_{Qg}^{(3)}(x)$ as a function of x, rescaled by the factor $x(1-x)$. Left panel: smaller x region. Full line (red): $a_{Qg}^{(3)}(x)$; dashed line (blue): leading small-x term $\propto \ln(x)/x$ [Catani, Ciafaloni, Hautmann, 1990]; light blue region: estimates of [Kawamura et al., 2012]; gray region: estimates of [ABMP 2017]. Right panel: larger x region. Full line (red): $a_{Qg}^{(3)}(x)$; light blue region: estimates of [Kawamura et al., 2012] gray region: estimates of [ABMP 2017]. [J. Ablinger et al., 2022; 2023]

$\Delta a_{Qg}^{(3)}$

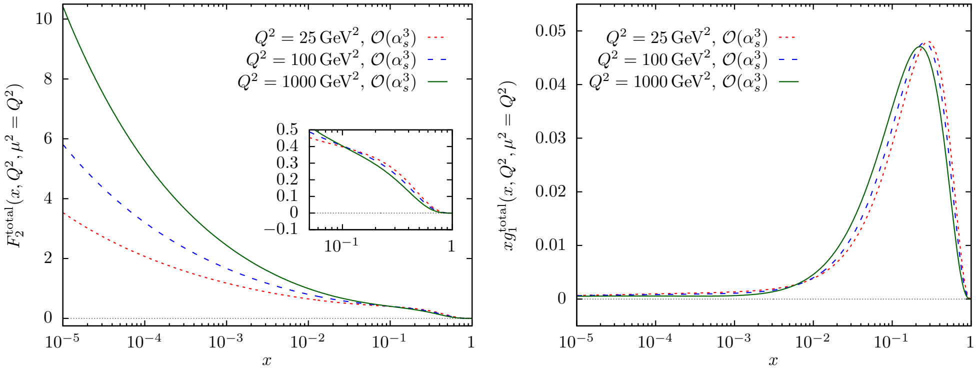


$\Delta a_{Qg}^{(3)}(x)$ as a function of x, rescaled by the factor $x(1-x)$. Left panel: full line (red): $\Delta a_{Qg}^{(3)}(x)$; dashed line (green): the small-x terms $\ln^k(x)$, $k \in \{1, \ldots, 5\}$; dotted line (blue): the large-x terms $\ln^l(1-x)$, $l \in \{1, \ldots, 4\}$. Right panel: larger x region. Full line (red): $\Delta a_{Qg}^{(3)}(x)$; dotted line (blue): the large-x terms $\ln^l(1-x)$, $l \in \{1, \ldots, 4\}$.

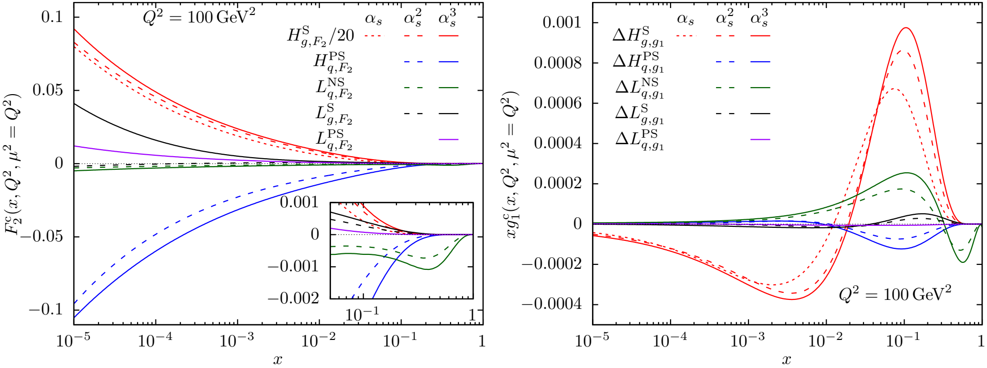
[J. Ablinger et al., 2022; 2023]

Precise and fast numerical representations of all single mass VFNS OMEs are given in [arXiv:2510.02175].

F_2 and xg_1: massless and single-mass HQ contributions

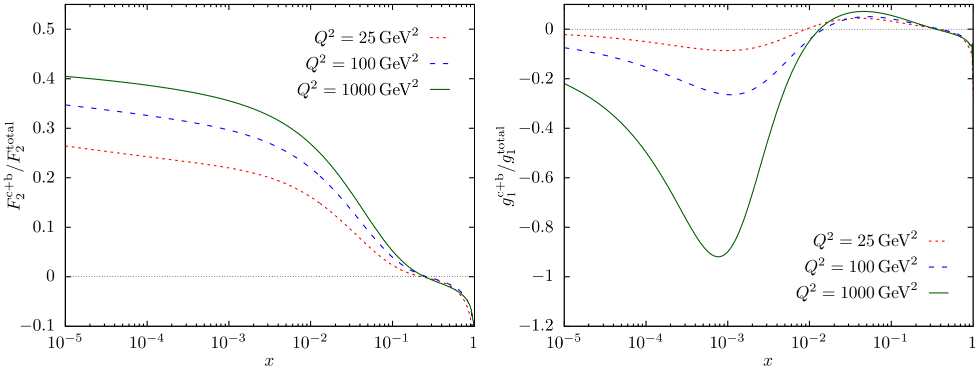


Contributions of the different Wilson coefficients to $F_2^{c,b}$ and $xg_1^{c,b}$ for the different orders in α_s

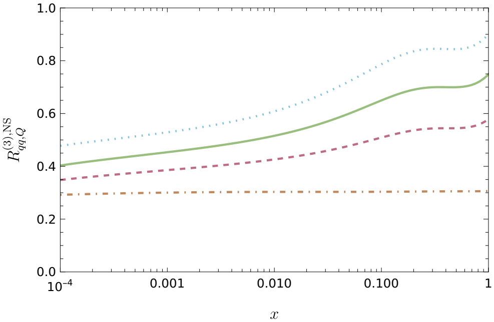


charm contributions at $Q^2 = 100$ GeV^2: cumulative in α_s

Single mass heavy flavor ratios for F_2 and xg_1



Two-mass Corrections: $(\Delta) A_{qq,Q}^{NS,2m}$

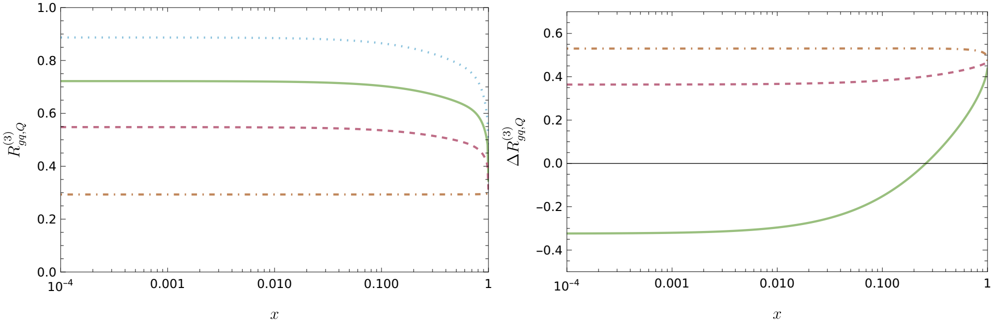


[Dotted line: $Q^2 = 30$ GeV^2, full line: $Q^2 = 50$ GeV^2, dashed line: $Q^2 = 100$ GeV^2, dash-dotted line: $Q^2 = 1000$ GeV^2]

[Ablinger, JB, De Freitas, Hasselhuhn, Schneider, Wißbrock '17]

$$R_{ij} = \frac{A_{ij}^{2m,irred}}{A_{ij,tot}^{T_F^2}}$$

$(\Delta)A_{gq}^{2m}$

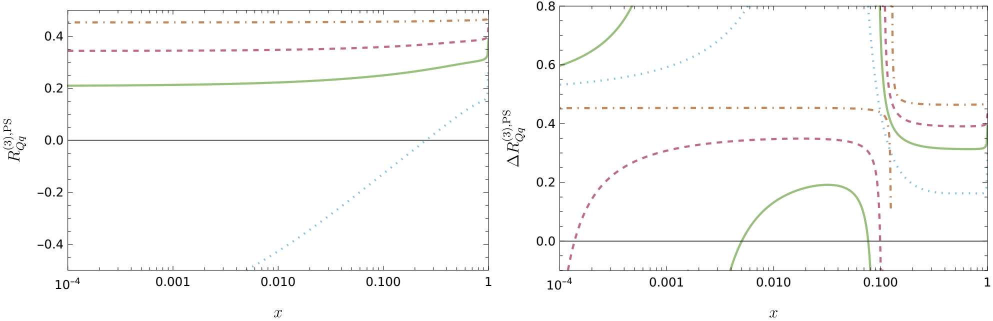


[Dotted line: $Q^2 = 30$ GeV^2, full line: $Q^2 = 50$ GeV^2, dashed line: $Q^2 = 100$ GeV^2, dash-dotted line: $Q^2 = 1000$ GeV^2]

[Ablinger, JB, De Freitas, Hasselhuhn, Schneider, Wißbrock '17]

[Behring, JB, De Freitas, von Manteuffel, Schönwald, Schneider '21]

$(\Delta)A_{Qq}^{PS,2m}$

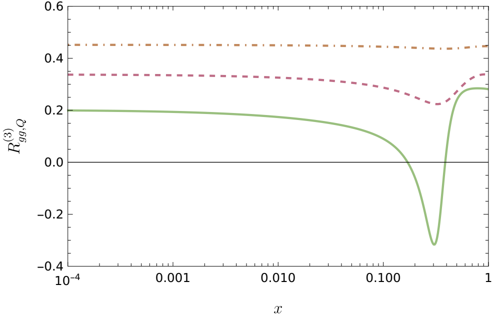


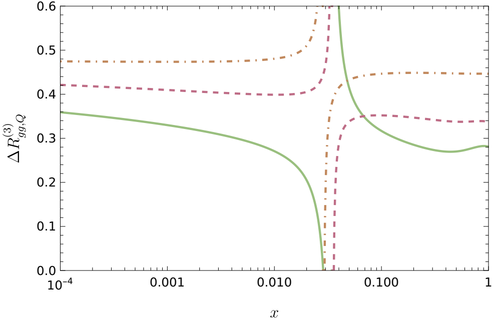
[Dotted line: $Q^2 = 30$ GeV^2, full line: $Q^2 = 50$ GeV^2, dashed line: $Q^2 = 100$ GeV^2, dash-dotted line: $Q^2 = 1000$ GeV^2]

[Ablinger, JB, De Freitas, Schneider, Schönwald '17]

[Ablinger, JB, De Freitas, Saragnese, Schneider, Schönwald '19]]

$(\Delta)A_{ggQ}^{2m}$



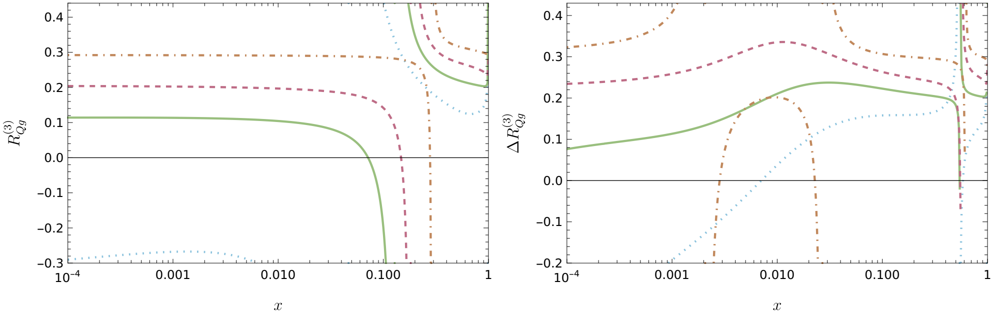


[Dotted line: $Q^2 = 30$ GeV^2, full line: $Q^2 = 50$ GeV^2, dashed line: $Q^2 = 100$ GeV^2, dash-dotted line: $Q^2 = 1000$ GeV^2]

[Ablinger, JB, De Freitas, Schneider, Schönwald ’18]

[Ablinger, JB, Goedicke, De Freitas, Saragnese, Schneider, Schönwald ’20]

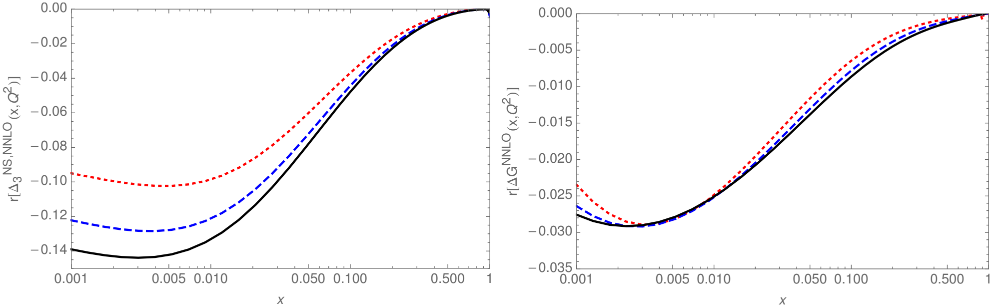
$(\Delta)A_{Qg}^{2m}$



[Dotted line: $Q^2 = 30$ GeV^2, full line: $Q^2 = 50$ GeV^2, dashed line: $Q^2 = 100$ GeV^2, dash-dotted line: $Q^2 = 1000$ GeV^2]
[J.Ablinger et al. 2025]

In summary: the 3-loop 2-mass OMEs form ~ 50 % of the $T_F^2 C_i$ contributions in all cases.

Polarized PDF evolution in the Larin Scheme



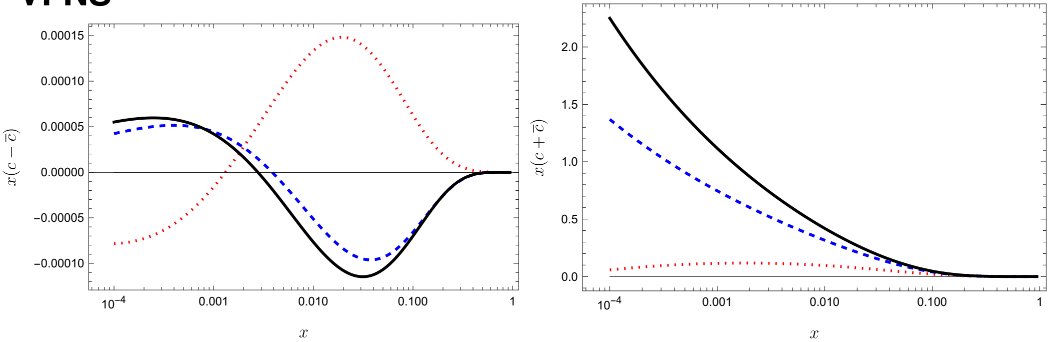
[Dotted line: $Q^2 = 100$ GeV^2, dashed line: $Q^2 = 1000$ GeV^2, full line: $Q^2 = 10000$ GeV^2]

$$r(x, Q^2) = \frac{f^L(x, Q^2)}{f^M(x, Q^2)} - 1$$

The pdfs are necessary to match HO Larin-scheme calculations.

[JB and M. Saragnese, 2024]

The difference and the sum of heavy flavor pdfs in the VFNS

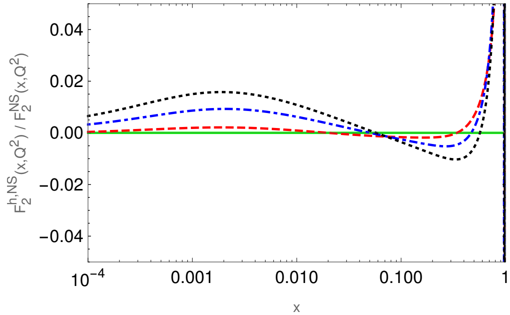


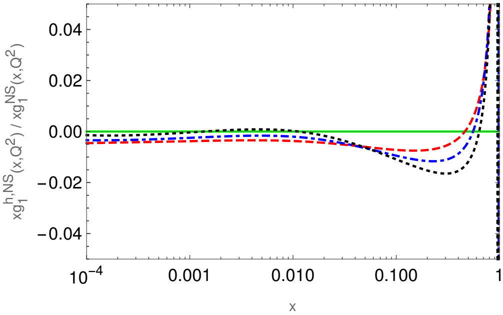
The unpolarized distributions $x[c(x,Q^2)-\bar{c}(x,Q^2)]$ (left panel) and $x[c(x,Q^2)+\bar{c}(x,Q^2)]$ (right panel). Dotted lines: $Q^2 = 4$ GeV^2. Dashed lines: $Q^2 = 30$ GeV^2. Full lines: $Q^2 = 100$ GeV^2. [A. Behring, JB, A. De Freitas, A. von Manteuffel, C. Schneider, K. Schönwald, DESY 25–185].

$$
\begin{aligned}
\Delta\gamma_{qq}^{\mathrm{NS,s},(2)} \quad = \quad & 4\frac{d_{abc}d^{abc}}{N_c}N_F\frac{1}{2}[1+(-1)^N]\Biggl\{\frac{S_1 Q_4}{N^4(1+N)^4}+\Biggl[-\frac{2(1+N+N^2)(2+N+N^2)}{N^3(1+N)^3} \\
& -\frac{4(N-1)(2+N)}{N^2(1+N)^2}S_1\Biggr]S_{-2}-\frac{(2+N+N^2)}{N^2(1+N)^2}[S_3-2S_{-3}+4S_{-2,1}]\Biggr\}.
\end{aligned}
$$

The relative contribution of HQ to non-singlet structure functions at N^{3}LO

Scheme-invariant evolution: the way to high-precision $\alpha_s(M_Z^2)$





The relative contribution of the heavy flavor contributions due to c and b quarks to the structure function $F_2^{\rm NS}$ and $xg_1^{\rm NS}$; dashed lines: 100 GeV^2; dashed-dotted lines: 1000 GeV^2; dotted lines: 10000 GeV^2. massless: N^{3}LO; single+two-mass massive contributions: $O(a_s^3)$. [JB, M. Saragnese, 2021].

Conclusions

- All unpolarized and polarized single-mass and two-mass OMEs and the associated massive Wilson coefficients for $Q^2 \gg m_Q^2$ have been calculated. The unpolarized and polarized massless three-loop Wilson coefficients were calculated and contribute to the present results.
- Analytic all Q^2 results are available in the unpolarized and polarized NS- and PS-cases at 2 loop order.
- Various new mathematical and technological methods were developed during the present project. They are available for use in further single- and two-mass calculations in other QFT projects.
- Both the single- and two-mass terms of the VFNS at 3-loop order are finished and the single-mass contributions were made available in form of numerical codes.
- The results in the polarized case prepare the analysis of the precision data, which will be taken at the EIC starting at the end of this decade.
- For all sub-processes it turned out that the small x BFKL approaches fail to present the physical result due to quite a series of missing subleading terms, which substantially correct the LO behavior. The correct description requires the full calculation.
- Very soon new precision analyses of the world DIS-data to measure $\alpha_s(M_Z)$ and m_c at higher precision can be carried out.