Light-Cone Lattice from the Universal R-matrix

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Lessons from the first phase of the LHC

(in collaboration with Jörg Teschner)

Integrable Models

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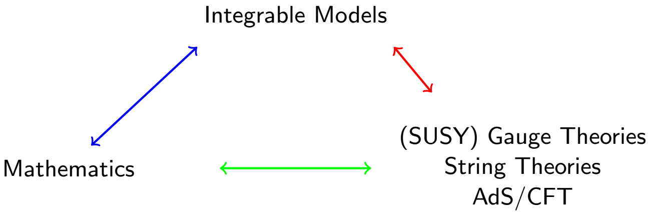
(SUSY) Gauge Theories
String Theories
AdS/CFT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Mathematics
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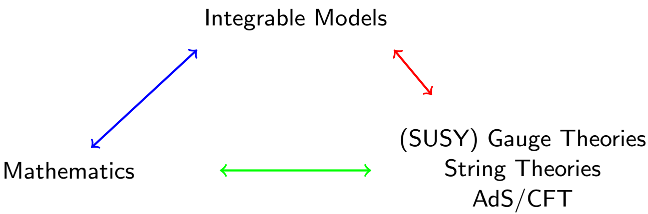
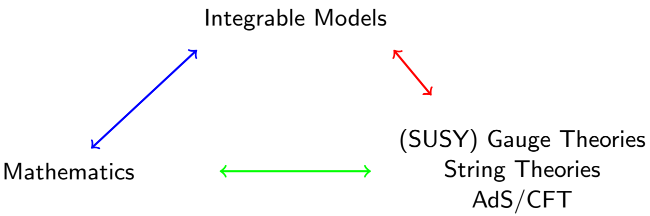


In Integrable Models: Proper Formulation = Solution



In Integrable Models: “Proper” Formulation = “Solution”

No fully general theory of (quantum) integrability exists to date.

Only a few 1+1 dimensional QFTs for which QI has been fully established.

Some systematics: non-compact, higher-rank, supersymmetric

Tailor-made lattice discretizations.

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Classical Integrability of Affine $\mathfrak{sl}_M$ Toda Theories I

$$\partial_+\partial_-\varphi_i = -\frac{m^2}{2b}\left(2e^{2b\varphi_i} - e^{2b\varphi_{i+1}} - e^{2b\varphi_{i-1}}\right), \qquad \varphi_i = \phi_i - \phi_{i+1}.$$

$$\left[\,\partial_- - A_-(\lambda)\,,\,\partial_+ - A_+(\lambda)\,\right] = 0\,.$$

$$A_+(\lambda) := +\sum_{i=1}^{M}\left(-b\,(\partial_+\phi_i)\,\mathsf{E}_{ii} + \lambda^{+1}\,m\,e^{b(\phi_i-\phi_{i+1})}\,\mathsf{E}_{ii+1}\right),$$

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Classical Integrability of Affine $\mathfrak{sl}_M$ Toda Theories II

Monodromy matrix: $$M(\lambda) = \mathcal{P} \exp\left(\int_0^R dx\, A_x(\lambda) \right)$$

$$T(\lambda) = \mathrm{Tr}(M(\lambda))$$

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Goal: Construct integrable discretizations

Discretization princpile: preserve as many features of the continuous theory as possible.

The Yang-Baxter equation

$$\mathcal{R}_{12}(\lambda,\mu)\,\mathcal{R}_{13}(\lambda,\nu)\,\mathcal{R}_{23}(\mu,\nu) \;=\; \mathcal{R}_{23}(\mu,\nu)\,\mathcal{R}_{13}(\lambda,\nu)\,\mathcal{R}_{12}(\lambda,\mu)$$

$$\mathcal{R}_{ij} \;:\; V_i \otimes V_j \rightarrow V_i \otimes V_j\,.$$

- YBE is too hard to solve in general.
- Framework of quantum groups (see later).
- Two inequivalent representations can correspond to the same vector space.

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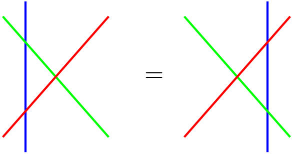

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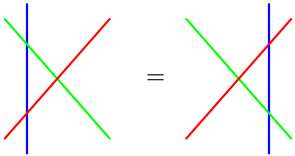

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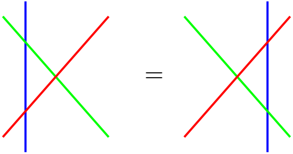

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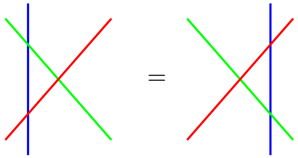

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Central objects in the QISM are (generalized) transfer matrices

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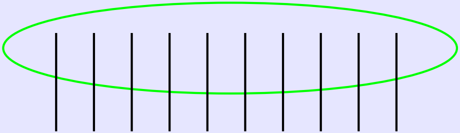

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aux = quantum at a site ,

- They contain shift operator and local charges (Hamiltonian).

Quantum Inverse Scattering Method

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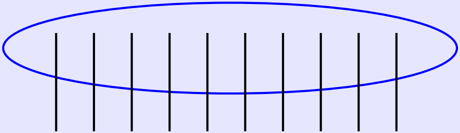

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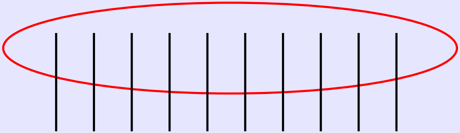

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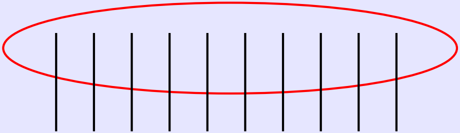

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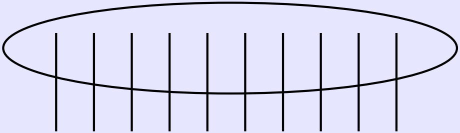

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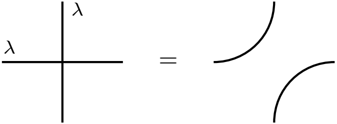

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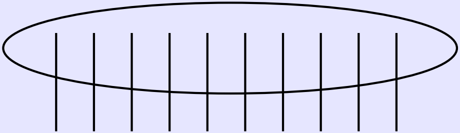

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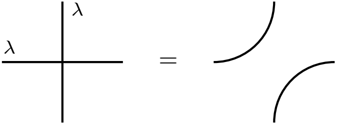

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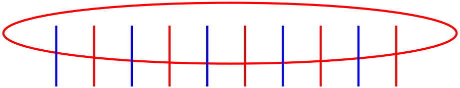
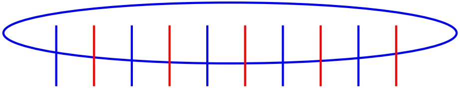
QISM and light-cone lattice: a staggered chain

$$\mathcal{Q}^{(+)}{}_{(\bar{\mu},\,\mu;\,\lambda)} \;=$$



$$\mathcal{Q}^{(-)}{}_{(\bar{\mu},\,\mu;\,\bar{\lambda})} \;=$$



$$\left(\mathsf{U}^{+}_{\kappa}\right)^{-1} := \mathcal{Q}^{(+)}{}_{(\bar{\mu},\,\mu;\,\mu)}\,, \qquad \mathsf{U}^{-}_{\kappa} := \mathcal{Q}^{(-)}{}_{(\bar{\mu},\,\mu;\,\bar{\mu})}\,, \qquad \kappa^{2} := \mu\,\bar{\mu}^{-1} = (m\Delta)^{2}\,.$$

$$= \;\mathsf{L}^{(-)}_{2L}\,\mathsf{L}^{(+)}_{2L-1}\ldots\mathsf{L}^{(-)}_{2}\,\mathsf{L}^{(+)}_{1} \rightarrow$$

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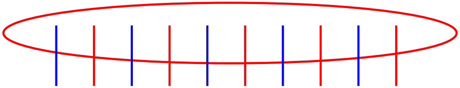

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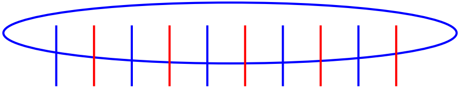

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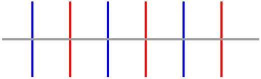

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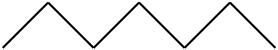
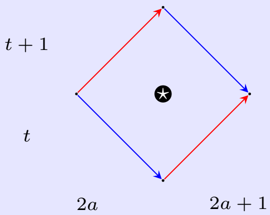


Quantum discrete zero-curvature representation

$$\mathsf{L}^{(+)}_{2a+1,t}\,\mathsf{L}^{(-)}_{2a,t} = \mathsf{L}^{(-)}_{2a+1,t+1}\,\mathsf{L}^{(+)}_{2a,t+1}$$



where

$$\mathcal{O}_{r+1,t+1} := (\mathsf{U}^{+})^{-1}\,\mathcal{O}_{r,t}\,\mathsf{U}^{+} \qquad \mathcal{O}_{r-1,t+1} := (\mathsf{U}^{-})^{-1}\,\mathcal{O}_{r,t}\,\mathsf{U}^{-}$$

Applying the strategy above to ATT

- Need to identify the relevant algebraic structure.
- Need to identify the relevant representations.
- Need to study fusion/factorization of representations.
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Generators: $k_i,\, e_i,\, f_i\,, \qquad i = 1, \ldots, M.$

Cartan Matrix ($M > 2$): $\begin{cases} A_{ij} = 2 & \text{if } i = j, \\ A_{ij} = -1 & \text{if } |i-j| = 1, \\ A_{ij} = 0 & \text{otherwise.} \end{cases}$

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$$k_i\, f_j \;=\; q^{-A_{ij}}\, f_j\, k_i$$

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Generators: $k_i,\, \underline{e_i},\, f_i\,, \qquad i = 1, \ldots, M.$

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Coproduct and the Universal R-matrix

$$\Delta(e_i) = e_i \otimes k_i + 1 \otimes e_i\,, \qquad \Delta(f_i) = f_i \otimes 1 + k_i^{-1} \otimes f_i\,, \qquad \Delta(k_i) = k_i \otimes k_i$$

Quasi-tringularity and the Yang-Baxter equation follow!

$\mathcal{R}$ and $\sigma\,\mathcal{R}^{-1}$, for example, are two *different* universal R-matrices.

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Some remarkable representations

$$\mathsf{L}^{+}(\lambda) \,=\, \sum_{i=1}^{M} \left(\mathsf{u}_i\, \mathsf{E}_{ii} \,+\, \lambda^{+1}\, \mathsf{v}_i\, \mathsf{E}_{ii+1}\right)$$

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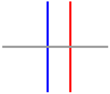
Fusion and physical degrees of freedom

$$\mathcal{L}_a(\lambda) := \bar{\mathsf{L}}^{-}_{2a}(\bar{\mu})\,\mathsf{L}^{+}_{2a-1}(\mu) =$$



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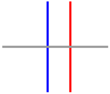

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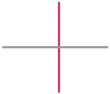

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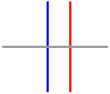

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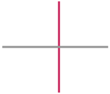

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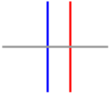

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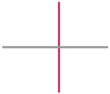

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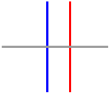
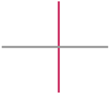
$$\mathcal{L}_a(\lambda) := \bar{\mathsf{L}}^{-}_{2a}(\bar{\mu})\,\mathsf{L}^{+}_{2a-1}(\mu) =$$  $$\in \mathcal{W} \otimes \mathcal{W} \otimes \mathsf{End}(\mathbb{C}^M)$$

$$\mathcal{L}_a(\lambda) \simeq \mathbb{L}_a(\bar{\mu}, \mu) =$$  $$\in \mathcal{W} \otimes \mathsf{End}(\mathbb{C}^M)$$

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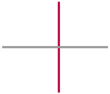
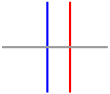
 $\simeq$ 

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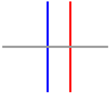

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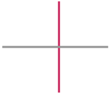

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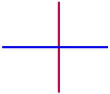

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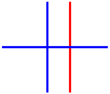
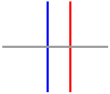
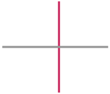


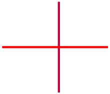
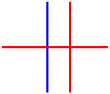
Fusion and physical degrees of freedom

$$\mathcal{L}_a(\lambda) := \bar{\mathsf{L}}^{-}_{2a}(\bar{\mu})\,\mathsf{L}^{+}_{2a-1}(\mu) =$$  $$\in \mathcal{W} \otimes \mathcal{W} \otimes \mathsf{End}(\mathbb{C}^M)$$

$$\mathcal{L}_a(\lambda) \simeq \quad \mathbb{L}_a(\bar{\mu}, \mu) \quad =$$  $$\in \quad \mathcal{W} \quad \otimes \mathsf{End}(\mathbb{C}^M)$$

$$\mathbb{L}_a(\bar{\mu}, \mu) := \left(1 + \bar{\mu} \textstyle\sum_{i=1}^{M} \mathsf{X}_{i+1,a}\,\mathsf{Y}_{i,a}\,\mathsf{E}_{i+1i}\right)^{-1} \left(\textstyle\sum_{i=1}^{M} \mathsf{X}_{i,a}\,\mathsf{E}_{ii}\right) \left(1 + \mu \textstyle\sum_{i=1}^{M} \mathsf{X}^{-1}_{i,a}\,\mathsf{Y}_{i,a}\,\mathsf{E}_{ii+1}\right)^{+1}$$

$$\mathcal{U}_q(\widehat{\mathfrak{gl}}_M) \to \mathcal{U}_q(\mathfrak{gl}_M) \to \mathcal{W}$$

 $\simeq$ 

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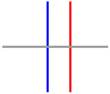

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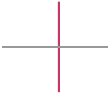

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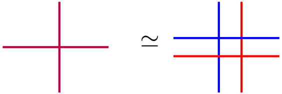
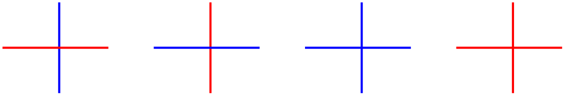


Universal R-matrix and basic building blocks

Construct: 

$$\mathcal{R} = \bar{\mathcal{R}}\, q^{-t}, \qquad \bar{\mathcal{R}} = \prod_{\alpha \in \vec{\Delta}_+} \Xi_\alpha \in \mathcal{U}_q^{>} \otimes \mathcal{U}_q^{<}$$

Remarkably, for the cases of interest only finitely many terms in the product contributes and one gets, e.g.

$$= \quad \left(\bar{\pi}^-_{\bar{\mu}} \otimes \pi^+_{\mu}\right)(\mathcal{R}) = \left(\prod_{i=1}^{M} \mathcal{J}_\kappa(\mathsf{Y}^2_{i,a})\right) q^{-t}$$

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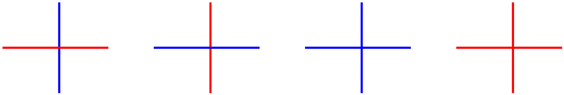
Construct: 

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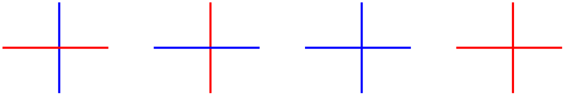
Construct: 

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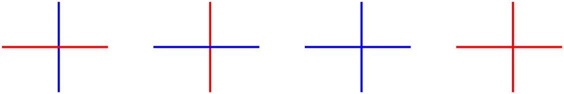
Construct: 

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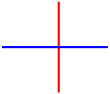

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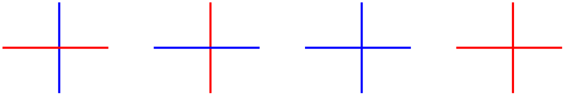
Construct: 

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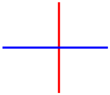

$$= \left(\bar{\pi}_{\bar{\mu}}^{-} \otimes \pi_{\mu}^{+}\right)(\mathcal{R}) = \left(\prod_{i=1}^{M} \mathcal{J}_\kappa(\mathsf{Y}_{i,a}^2)\right) q^{-t}$$

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Quantum discrete time evolution equations

$$\chi_{i,a}^{(t-1)}\,\chi_{i,a}^{(t+1)} = \chi_{i,a-1}^{(t)}\,\chi_{i,a+1}^{(t)}\,\frac{\left(1-\kappa^2\,\chi_{i-1,a}^{(t)}\right)\left(1-\kappa^2\,\chi_{i-1,a+1}^{(t)}\right)}{\left(1-\kappa^2\,\chi_{i,a}^{(t)}\right)\left(1-\kappa^2\,\chi_{i,a+1}^{(t)}\right)}$$

$$\chi_{i,2a-1}^{(t=0)} := \mathsf{Y}_{i,a}^2 \qquad \chi_{i,2a}^{(t=1)} := \mathsf{X}_{i+1,a}\,\mathsf{Y}_{i,a}\,\mathsf{Y}_{i,a+1}\,\mathsf{X}_{i,a+1}^{-1}$$

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Conclusions and Outlook

- Derivation of functional relations and systematics of Baxter's Q-operators.
- Remarkable fusion/factorization properties both in quantum and auxiliary space.
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- Hint at quantum group from the algebra of screening charges.
- Relation between integrability and "free field" description in general.

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Thank you!