# Integrability and scattering amplitudes in $\mathcal{N} = 4$ SYM

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**Amazing progress in planar $\mathcal{N} = 4$ SYM in recent years**

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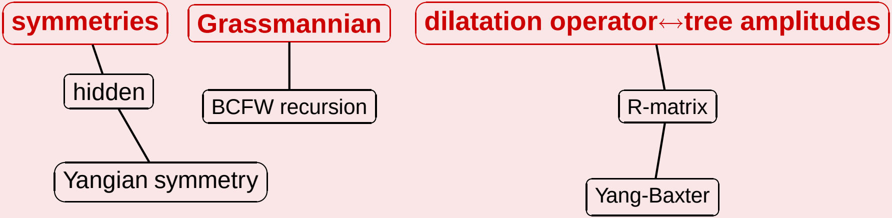



**Scattering amplitudes and integrability**

**spectral parameter?**

On-shell supermultiplet of $\mathcal{N} = 4$ SYM described by a superfield $\Phi$

$$\Phi = G^{+} + \eta^{A}\Gamma_{A} + \frac{1}{2!}\eta^{A}\eta^{B}S_{AB} + \frac{1}{3!}\eta^{A}\eta^{B}\eta^{C}\epsilon_{ABCD}\overline{\Gamma}^{D} + \frac{1}{4!}\eta^{A}\eta^{B}\eta^{C}\eta^{D}\epsilon_{ABCD}G^{-}$$

- $p^2 = 0 \Longleftrightarrow p^{\alpha\dot{\alpha}} = \lambda^{\alpha}\tilde{\lambda}^{\dot{\alpha}}, \quad q^{\alpha A} = \lambda^{\alpha}\eta^{A}$
- on-shell superspace: $(\lambda^{\alpha}, \tilde{\lambda}^{\dot{\alpha}}, \eta^{A})$

Expansion in powers of Grassmann parameters $\eta$:

$$\mathcal{A}(\Phi_1, \dots, \Phi_n) = \mathcal{A}_n^{\mathrm{MHV}} + \mathcal{A}_n^{\mathrm{NMHV}} + \dots + \mathcal{A}_n^{\overline{\mathrm{MHV}}}$$

MHV tree-level[Parke, Taylor]

$$\mathcal{A}_n^{\mathrm{MHV}} = \frac{\delta^4(p)\delta^8(q)}{\langle 12\rangle\langle 23\rangle \dots \langle n1\rangle}, \quad \langle \mathrm{ij}\rangle = \lambda_i^{\alpha}\lambda_{j\alpha}$$

# Symmetries

$\mathcal{N} = 4$ SYM symmetries:

- Superconformal symmetry: expected [Witten]

$$j_a \mathcal{A}_n^{\text{tree}} = 0, \qquad j_a \in \mathfrak{psu}(2,2|4)$$

- Dual superconformal symmetry: hidden [Drummond, Henn, Korchemsky, Sokatchev]
  - dual space: $x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}} = \lambda_i^{\alpha}\tilde{\lambda}_i^{\dot{\alpha}}, \qquad \theta_i^{\alpha A} - \theta_{i+1}^{\alpha A} = \lambda_i^{\alpha}\eta_i^A$
  - not related to ordinary superconformal symmetry

$$J'_a \mathcal{A}_n^{\text{tree}} = 0, \qquad J'_a \in \mathfrak{psu}(2,2|4)^{\text{dual}}$$

Superconformal + dual superconformal algebras: [Drummond, Henn, Plefka]

**Yangian structure**

↓

**Hint of integrability**

Full symmetry of the **tree-level** amplitudes

$$y \mathcal{A}_n^{\text{tree}} = 0, \qquad y \in Y(\mathfrak{psu}(2,2|4))$$

Broken beyond tree level.

# Integrability

- Another realization of $Y(\mathfrak{g})$ is given by the RTT definition
- Another approach to the Yangian algebra: monodromy matrices

$$\mathcal{M}(z) = R_1(z)...R_L(z)$$

are generating functions for all levels of the Yangian generators

$$\mathcal{M}(z)^{\mathcal{A}}_{\mathcal{B}} = \sum_{i=0}^{\infty} \frac{1}{z^i} (j^{(i)})^{\mathcal{A}}_{\mathcal{B}}$$

- R-matrices are the essential ingredients for all integrable models
- In order to find $R(z)$ we have to solve the Yang-Baxter equation

**The Question:**
is this something more than a hint?

Recent relation between the dilatation operator for the spin chain and tree-level scattering amplitudes [Zwiebel]

$$\langle \Lambda_1 \Lambda_2 | \mathcal{D}_{L\to 2} | \Lambda_3 \ldots \Lambda_{L+2} \rangle = \mathcal{A}_{L+2}(\Lambda_1, \ldots, \Lambda_{L+2})$$

Nearest-neighbor Hamiltonian $\mathcal{H}$ of spin chain $\equiv$ one-loop planar dilatation generator of $\mathcal{N} = 4$ SYM

| Spin chain | | Amplitudes |
|---|---|---|
| $\mathcal{H} = \mathcal{D}_{2\to 2}$ | $\overset{\textit{Zwiebel}}{\longleftrightarrow}$ | $\mathcal{A}_4$ |
| $\uparrow d \log$ | | $\uparrow$ |
| $R(z)$ | $\longleftrightarrow$ | ? |

- How is this related to the Grassmannian formulation of amplitudes?

# The Grassmannian language

**In twistor space** $\mathcal{Z}_i^{\mathcal{A}} = (\tilde{\mu}_i^{\alpha}, \tilde{\lambda}_i^{\dot{\alpha}}, \eta_i^{A})$

$$\mathcal{A}_{n,k}^{\text{tree}} = \oint \frac{\prod_{a=1}^{k} \prod_{i=1}^{n} dc_{ai}}{\mathcal{M}_1 \mathcal{M}_2 \ldots \mathcal{M}_n} \prod_{a=1}^{k} \delta^{4|4}\left(\sum_{i=1}^{n} c_{ai} \mathcal{Z}_i^{\mathcal{A}}\right)$$

[Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Cheung, Goncharov, Hodges, Kaplan, Postnikov, Trnka] [Mason, Skinner]

- $c_{ai}$: complex parameters forming a $(k \times n)$ matrix $\mathcal{C}$
- $\mathcal{M}_p = (p \ldots p+k-1)$: determinant of $(k \times k)$ submatrix of the $c_{ai}$'s
- Yangian invariant up to total derivatives [Drummond, L.F.]

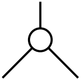
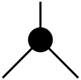
Example: **MHV four points**

$$\mathcal{A}_{4,2} = \oint \frac{dc_{13} dc_{14} dc_{23} dc_{24}}{c_{13} c_{24} (c_{13} c_{24} - c_{14} c_{23})} \delta^{4|4}(\mathcal{Z}_1 + c_{13} \mathcal{Z}_3 + c_{14} \mathcal{Z}_4) \delta^{4|4}(\mathcal{Z}_2 + c_{23} \mathcal{Z}_3 + c_{24} \mathcal{Z}_4)$$

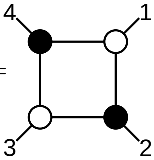
$$\mathcal{C} = \begin{pmatrix} 1 & 0 & c_{13} & c_{14} \\ 0 & 1 & c_{23} & c_{24} \end{pmatrix}$$

# Building blocks for amplitudes

- All-loop recursion relation [Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka]
- Using BCFW recursion relations, all amplitudes can be written employing only two on-shell diagrams:

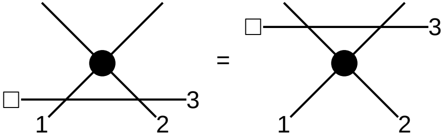
$\mathcal{A}_{3,1} =$  $\mathcal{A}_{3,2} =$ 

- Example: **MHV four points tree level**

$\mathcal{A}_{4,2} =$ 

# Yang-Baxter equation and Grassmannian

Yang-Baxter equation:




$$R_{12}(z_1 - z_2)R_{13}(z_1)R_{23}(z_2) = R_{23}(z_2)R_{13}(z_1)R_{12}(z_1 - z_2)$$

with $R^{A}_{i3,B}(z) = z\delta^{A}_{B} + J^{A}_{i\,B}$. Solution in the Grassmannian form

$$\mathcal{R}(z) = \oint \frac{dc_{13}dc_{14}dc_{23}dc_{24}}{c_{13}c_{24}(c_{13}c_{24} - c_{14}c_{23})} F(C;z)\delta^{4|4}(\mathcal{Z}_1 + c_{13}\mathcal{Z}_3 + c_{14}\mathcal{Z}_4)\delta^{4|4}(\mathcal{Z}_2 + c_{23}\mathcal{Z}_3 + c_{24}\mathcal{Z}_4)$$

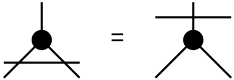
and the fact that all particles have vanishing central charge leads to the answer

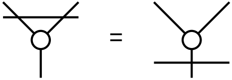
$$F(C;z) = \left(\frac{c_{13}c_{24}}{c_{13}c_{24} - c_{14}c_{23}}\right)^{z}$$

Spectral parameter deformation of $\mathcal{A}_{4,2}$

# Building blocks for R(z)

MHV R-matrix – $R_\bullet(z_1, z_2)$ and $\overline{\text{MHV}}$ R-matrix – $R_\circ(z_1, z_2)$





They satisfy equations similar to Yang-Baxter:

$$
\begin{aligned}
z_1 (J_1)^A_C R_\bullet(z_1, z_2) &= R_\bullet(z_1, z_2)(J_1)^A_B (z_1 \delta^B_C + (J_2)^B_C) \\
(J_1)^A_B (z_1 \delta^B_C + (J_2)^B_C) R_\circ(z_1, z_2) &= z_1 R_\circ(z_1, z_2)(J_1)^A_C
\end{aligned}
$$

A second set of equations with $(1 \leftrightarrow 2)$ leads to a second spectral parameter $z_2$

$$
\begin{aligned}
\mathcal{R}_\bullet(z_1, z_2) &= \oint \frac{dc_1 dc_2}{c_1 c_2} \frac{1}{c_1^{z_1} c_2^{z_2}} \delta^{4|4}(\mathcal{Z}_1^{\mathcal{A}} + c_1 \mathcal{Z}_3^{\mathcal{A}}) \delta^{4|4}(\mathcal{Z}_2^{\mathcal{A}} + c_2 \mathcal{Z}_3^{\mathcal{A}}) \\
\mathcal{R}_\circ(z_1, z_2) &= \oint \frac{dc_1 dc_2}{c_1 c_2} \frac{1}{c_1^{z_1} c_2^{z_2}} \delta^{4|4}(\mathcal{Z}_3^{\mathcal{A}} + c_1 \mathcal{Z}_1^{\mathcal{A}} + c_2 \mathcal{Z}_2^{\mathcal{A}})
\end{aligned}
$$

# Interpretation of the spectral parameters

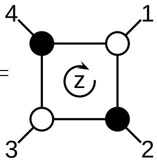
## Building blocks:

- 3-point amplitudes are very singular objects
- non-trivial 3-point amplitudes: complexify momenta
- non-trivial 3-point R-matrices: relax also the central charge constraints

**The spectral parameters have the interpretation of unphysical-particle helicities**

- For 4 points we get one non-vanishing spectral parameter after demanding that outer particles have vanishing central charge

$\mathcal{R}_{4,2} =$




$$= \oint \frac{dc_{13}dc_{14}dc_{23}dc_{24}}{c_{13}c_{24}(c_{13}c_{24}-c_{14}c_{23})}\left(\frac{c_{13}c_{24}}{c_{13}c_{24}-c_{14}c_{23}}\right)^{z} \delta^{4|4}(\mathcal{C}\cdot\mathcal{Z})$$

- For higher-point R-matrices the number of parameters grows and is equal to the number of loops in the on-shell diagram

# Loop amplitudes: spectral regularization

- One can write any loop amplitude using on-shell diagrams

$$\mathcal{A}_{n,k}^{\ell} \longrightarrow \mathcal{A}_{n-2,k-1}^{\ell+1}$$

- The simplest example is $\mathcal{A}_{6,3}^{\text{tree}} \longrightarrow \mathcal{A}_{4,2}^{\text{1-loop}}$. One gets:

$$\mathcal{A}_{4,2}^{\text{1-loop}} = \mathcal{A}_{4,2}^{\text{tree}} \int \frac{d^4 q}{q^2 (q+p_1)^2 (q+p_1+p_2)^2 (q+p_1+p_2+p_3)^2}$$

- The box integral is known to be divergent $\rightarrow$ one uses dimensional or mass regularization to calculate it

$$B \sim \frac{2}{\epsilon^2}\left(\left(\frac{s}{\mu^2}\right)^{-\epsilon} + \left(\frac{t}{\mu^2}\right)^{-\epsilon}\right) - \log^2\left(\frac{s}{t}\right) - \frac{4\pi^2}{3}$$

- Use the spectral parameter(s) to regulate loop amplitudes in a novel way: replace dimensional regularization by spectral regularization staying in exactly four dimensions
- Currently under active investigation

# Summary and conclusions

- We constructed a new class of objects which are deformations of the $\mathcal{N} = 4$ amplitudes
- We introduced spectral parameters into the scattering amplitude problem
- The spectral parameter is related to the central charge

- Possibility to provide a novel regularization scheme
- Should allow to establish the exact link between the amplitude and the spectral problem