Recent Results on Three Loop Corrections to Heavy Quark Contributions to DIS Structure Functions

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- Introduction
- 3-Loop OMEs with m_c and m_b
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- Conclusions

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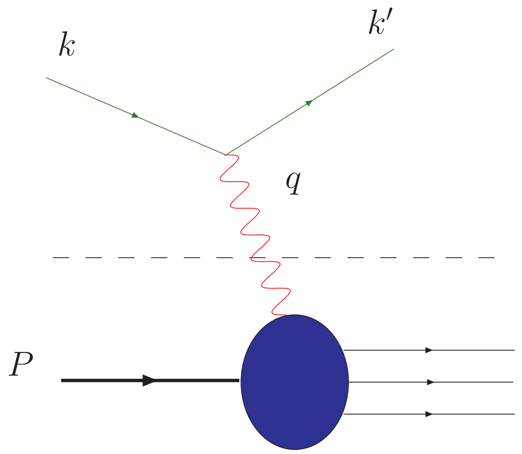
Introduction

Unpolarized Deep–Inelastic Scattering (DIS):



$\longrightarrow \quad L_{\mu\nu}$

$\longrightarrow \quad W_{\mu\nu}$

$$Q^2 := -q^2, \quad x := \frac{Q^2}{2P.q}$$ Bjorken–x

$$\frac{d\sigma}{dQ^2\,dx} \sim W_{\mu\nu} L^{\mu\nu}$$

$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s\rangle$$

$$= \frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x,Q^2)\ .$$

Structure Functions: $F_{2,L}$

contain light and heavy quark contributions.

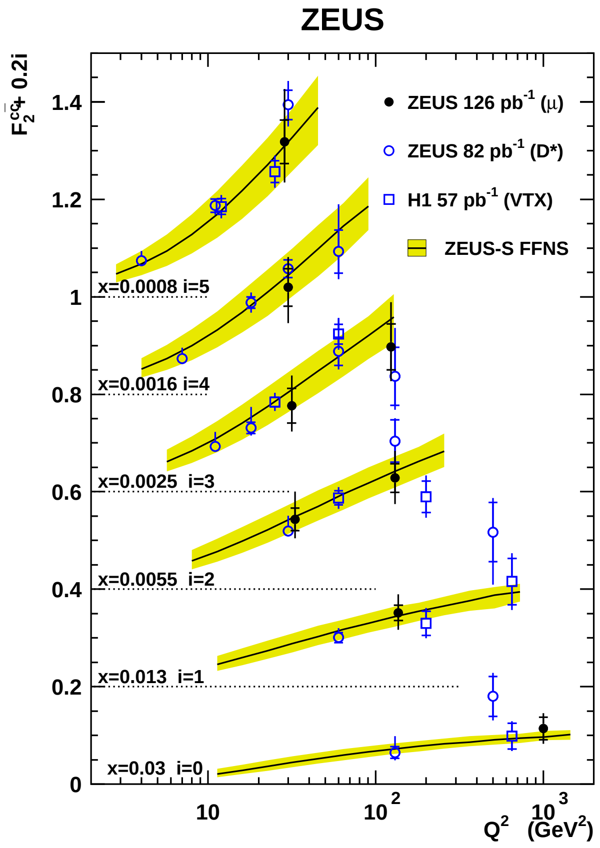
Need for full NNLO result

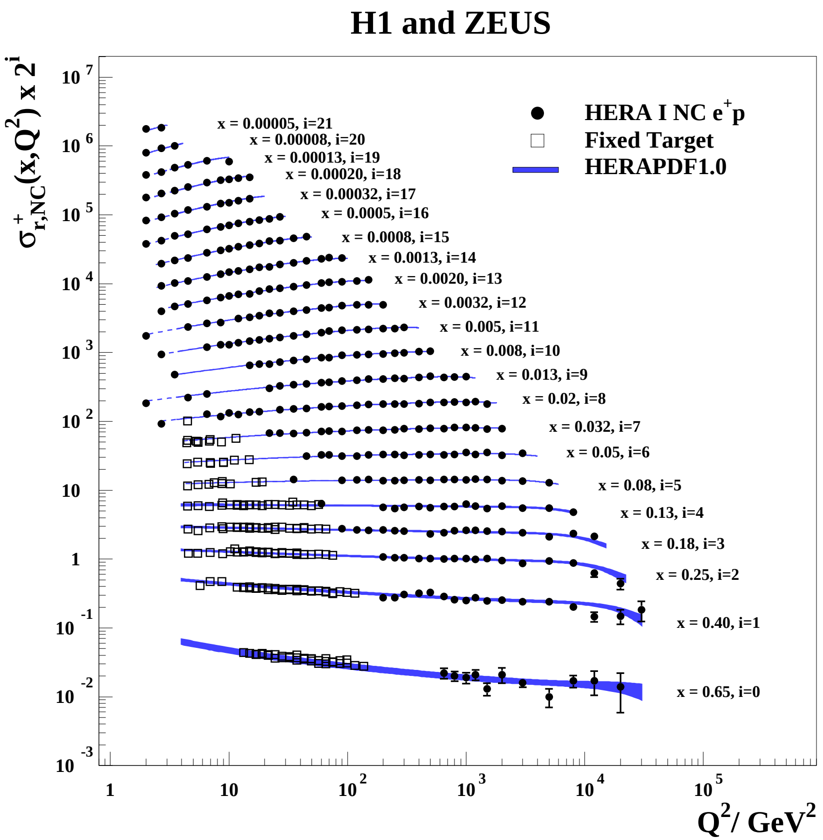
- The knowledge of heavy flavor DIS contributions to $O(\alpha_s^3)$ allows to measure Λ_{QCD} and $\alpha_s(M_Z)$ with increased precision, $|\Delta\alpha_s| < 1\%$.
- The unification of forces depends crucially on the value of α_s at low scales.
- The detailed understanding of parton densities in necessary to interpret the hadron induced processes at HERA, TEVATRON & LHC.

$\longrightarrow$ Important case:

hadronic Higgs Boson production $\propto \alpha_s^2 G^2(x, Q^2)$.

$F_2^{c\bar{c}}(x, Q^2)$





→ different scaling violations compared to massless $F_2(x, Q^2)$

→ massive contributions at lower values of x are of order $20\% - 35\%$.

→ Massive contributions needed!

Factorization of the Structure Functions

At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x,Q^2) = \sum_j \underbrace{\mathbb{C}_{j,(2,L)}\left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right)}_{perturbative} \otimes \underbrace{f_j(x,\mu^2)}_{nonpert.}$$

into (pert.) Wilson coefficients and (nonpert.) parton distribution functions (PDFs).
$\otimes$ denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz \, \delta(x - yz) f(y) g(z) \; .$$

The subsequent calculations are performed in Mellin space, where $\otimes$ reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) := \int_0^1 dx \, x^{N-1} f(x) \; .$$

Wilson coefficients:

$$\mathbb{C}_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=C_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)+H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)\ .$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=\sum_i C_{i,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)A_{ij}\left(\frac{m^2}{\mu^2},N\right)$$

[Buza, Matiounine, Smith, van Neerven 1996 Nucl.Phys.B]

factorizes into the light flavor Wilson coefficients C and the massive operator matrix elements (OMEs) of local operators O_i between partonic states j

$$A_{ij}\left(\frac{m^2}{\mu^2},N\right)=\langle j\mid O_i\mid j\rangle\ .$$

$\rightarrow$ additional Feynman rules with local operator insertions for partonic matrix elements.

The unpolarized light flavor Wilson coefficients are known up to NNLO

[Moch, Vermaseren, Vogt, 2005 Nucl.Phys.B].

For $F_2(x,Q^2)$: at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

The Heavy Flavor Wilson Coefficients

$$
\begin{aligned}
L_{2,q}^{\mathsf{NS}}(n_f) &= a_s^2\left[A_{qq,Q}^{\mathsf{NS},(2)}(n_f) + \hat{C}_{2,q}^{\mathsf{NS},(2)}(n_f)\right] \\
&+ a_s^3\left[A_{qq,Q}^{\mathsf{NS},(3)}(n_f) + A_{qq,Q}^{\mathsf{NS},(2)}(n_f)C_{2,q}^{\mathsf{NS},(1)}(n_f) + \hat{C}_{2,q}^{\mathsf{NS},(3)}(n_f)\right] \\
\tilde{L}_{2,q}^{\mathsf{PS}}(n_f) &= a_s^3\left[\ \tilde{A}_{qq,Q}^{\mathsf{PS},(3)}(n_f) + A_{gq,Q}^{(2)}(n_f)\ \tilde{C}_{2,g}^{(1)}(n_f+1) + \hat{\tilde{C}}_{2,q}^{\mathsf{PS},(3)}(n_f)\right] \\
\tilde{L}_{2,g}^{\mathsf{S}}(n_f) &= a_s^2 A_{gg,Q}^{(1)}(n_f)\tilde{C}_{2,g}^{(1)}(n_f+1) \\
&+ a_s^3\Big[\ \tilde{A}_{qg,Q}^{(3)}(n_f) + A_{gg,Q}^{(1)}(n_f)\ \tilde{C}_{2,g}^{(2)}(n_f+1) + A_{gg,Q}^{(2)}(n_f)\ \tilde{C}_{2,g}^{(1)}(n_f+1) \\
&\quad + A_{Qg}^{(1)}(n_f)\ \tilde{C}_{2,q}^{\mathsf{PS},(2)}(n_f+1) + \hat{\tilde{C}}_{2,g}^{(3)}(n_f)\Big] \\
H_{2,q}^{\mathsf{PS}}(n_f) &= a_s^2\left[\ A_{Qq}^{\mathsf{PS},(2)}(n_f) +\ \tilde{C}_{2,q}^{\mathsf{PS},(2)}(n_f+1)\right] \\
&+ a_s^3\Big[\ A_{Qq}^{\mathsf{PS},(3)}(n_f) +\ \tilde{C}_{2,q}^{\mathsf{PS},(3)}(n_f+1) + A_{gq,Q}^{(2)}(n_f)\ \tilde{C}_{2,g}^{(1)}(n_f+1) \\
&\quad + A_{Qq}^{\mathsf{PS},(2)}(n_f)\ C_{2,q}^{\mathsf{NS},(1)}(n_f+1)\Big] \\
H_{2,g}^{\mathsf{S}}(n_f) &= a_s\left[\ A_{Qg}^{(1)}(n_f) +\ \tilde{C}_{2,g}^{(1)}(n_f+1)\right] \\
&+ a_s^2\Big[\ A_{Qg}^{(2)}(n_f) +\ A_{Qg}^{(1)}(n_f)\ C_{2,q}^{\mathsf{NS},(1)}(n_f+1) +\ A_{gg,Q}^{(1)}(n_f)\ \tilde{C}_{2,g}^{(1)}(n_f+1) \\
&\quad + \tilde{C}_{2,g}^{(2)}(n_f+1)\Big] \\
&+ a_s^3\Big[\ A_{Qg}^{(3)}(n_f) +\ A_{Qg}^{(2)}(n_f)\ C_{2,q}^{\mathsf{NS},(1)}(n_f+1) +\ A_{gg,Q}^{(2)}(n_f)\ \tilde{C}_{2,g}^{(1)}(n_f+1) \\
&\quad + A_{Qg}^{(1)}(n_f)\left[C_{2,q}^{\mathsf{NS},(2)}(n_f+1) +\ \tilde{C}_{2,q}^{\mathsf{PS},(2)}(n_f+1)\right] +\ A_{gg,Q}^{(1)}(n_f)\ \tilde{C}_{2,g}^{(2)}(n_f+1) \\
&\quad + \tilde{C}_{2,g}^{(3)}(n_f+1)\Big]\ .
\end{aligned}
$$

Status of OME calculations

Leading Order: [Witten, 1976 Nucl.Phys.B; Babcock, Sivers, 1978 Phys.Rev.D; Shifman, Vainshtein, Zakharov, 1978 Nucl.Phys.B; Leveille, Weiler, 1979 Nucl.Phys.B; Glück, Reya, 1979 Phys.Lett.B; Glück, Hoffmann, Reya, 1982 Z.Phys.C.]

Next-to-Leading Order : [Laenen, van Neerven, Riemersma, Smith, 1993 Nucl. Phys. B]
[Large Q^2/m^2: Buza, Matiounine, Smith, Migneron, van Neerven, 1996 Nucl.Phys.B] IBP
[Bierenbaum, Blümlein, Klein, 2007 Nucl.Phys.B] via ${}_pF_q$'s, more compact results
[Bierenbaum, Blümlein, Klein 2008 Nucl.Phys.B, 2009 Phys.Lett.B]:$O(\alpha_s^2\varepsilon)$ contributions (all N)

NNLO:[Bierenbaum, Blümlein, Klein 2009 Nucl.Phys.B] Moments for F_2: $N = 2...10(14)$
[Ablinger, Blümlein, Klein, Schneider, Wißbrock 2011 Nucl.Phys.B] contrib. $\propto n_f$ to F_2 (all N):

At 3-loop order known:

- $A_{qq,Q}^{\rm PS}$, $A_{qg,Q}$: complete; A_{Qg}, $A_{Qq}^{\rm PS}$, $A_{qq,Q}^{\rm NS} A_{qq,Q}^{\rm NS,TR}$: all terms of $O(n_f T_F^2 C_{A/F})$
- $A_{gq,Q}$, $A_{gg,Q}$: see this talk $\longrightarrow$ all terms of $O(n_f T_F^2 C_{A/F})$
- Also $O(n_f T_F^2 C_{A/F})$ contributions to all polarized OMEs have been computed.
- $A_{Qq}^{\rm PS}$, $A_{qq,Q}^{\rm NS} A_{qq,Q}^{\rm NS,TR}$: all terms of $O(T_F^2 C_{A/F})$
- Ladder and Benz topologies with a single massive line: first results this talk.

Graphs with m_c and m_b

- At 3–loop order graphs containing both c- and b- quarks contribute.
- They do neither belong to the pure $c-$ or $b-$ contribution to the structure function.
- Note that:

$$\frac{m_c}{m_b} \simeq \frac{1.3GeV}{4.2GeV} \qquad \longrightarrow \boxed{x^3 := \left(\frac{m_c}{m_b}\right)^6 \simeq 0.0001}$$

$\rightarrow$ Expand in m_c/m_b

- for fixed values of N the diagrams can be mapped onto tadpole diagrams by projection operators [Bierenbaum, Blümlein, Klein 2009.]
- e.g. $N = 2$

$$\Pi_{\mu\nu} \quad = \quad \frac{1}{d-1}\left(\frac{-g_{\mu\nu}}{p^2} + d\frac{p(\mu)p(\nu)}{p^4}\right)$$

- more complex structures occur for higher Moments
- expansion in masses was performed using EXP [Harlander, Seidensticker, Steinhauser 1998, Seidensticker 1999]

$$
\begin{aligned}
a_{Qg}^{(3)}(N=6) \quad = \quad & T_F^2 C_A \Bigg\{ \frac{69882273800453}{367569090000} - \frac{395296}{19845}\zeta_3 + \frac{1316809}{39690}\zeta_2 + \frac{832369820129}{14586075000}x + \frac{1511074426112}{624023544375}x^2 - \frac{84840004938801319}{690973782403905000}x^3 \\
& + \ln\left(\frac{m_b^2}{\mu^2}\right)\left[\frac{11771644229}{194481000} + \frac{78496}{2205}\zeta_2 - \frac{1406143531}{69457500}x - \frac{105157957}{180093375}x^2 + \frac{2287164970759}{7669816654500}x^3\right] \\
& + \ln^2\left(\frac{m_b^2}{\mu^2}\right)\left[\frac{2668087}{79380} + \frac{112669}{661500}x - \frac{49373}{51975}x^2 - \frac{31340489}{34054020}x^3\right] + \ln^3\left(\frac{m_b^2}{\mu^2}\right)\frac{324148}{19845} + \ln^2\left(\frac{m_b^2}{\mu^2}\right)\ln\left(\frac{m_c^2}{\mu^2}\right)\frac{156992}{6615} \\
& + \ln\left(\frac{m_b^2}{\mu^2}\right)\ln\left(\frac{m_c^2}{\mu^2}\right)\left[\frac{128234}{3969} - \frac{112669}{330750}x + \frac{98746}{51975}x^2 + \frac{31340489}{17027010}x^3\right] + \ln\left(\frac{m_b^2}{\mu^2}\right)\ln^2\left(\frac{m_c^2}{\mu^2}\right)\frac{68332}{6615} \\
& + \ln\left(\frac{m_c^2}{\mu^2}\right)\left[\frac{83755534727}{583443000} + \frac{78496}{2205}\zeta_2 + \frac{1406143531}{69457500}x + \frac{105157957}{180093375}x^2 - \frac{2287164970759}{7669816654500}x^3\right] \\
& + \ln^2\left(\frac{m_c^2}{\mu^2}\right)\left[\frac{2668087}{79380} + \frac{112669}{661500}x - \frac{49373}{51975}x^2 - \frac{31340489}{34054020}x^3\right] + \ln^3\left(\frac{m_c^2}{\mu^2}\right)\frac{412808}{19845}\Bigg\} \\
& + T_F^2 C_F \Bigg\{ - \frac{3161811182177}{71471767500} + \frac{447392}{19845}\zeta_3 + \frac{9568018}{4862025}\zeta_2 - \frac{64855635472}{2552563125}x + \frac{1048702178522}{97070329125}x^2 + \frac{1980566069882672}{2467763508585375}x^3 \\
& + \ln\left(\frac{m_b^2}{\mu^2}\right)\left[\frac{1786067629}{204205050} - \frac{111848}{15435}\zeta_2 - \frac{128543024}{24310125}x - \frac{22957168}{3361743}x^2 - \frac{2511536080}{2191376187}x^3\right] \\
& + \ln^2\left(\frac{m_b^2}{\mu^2}\right)\left[\frac{3232799}{4862025} + \frac{752432}{231525}x + \frac{177944}{40425}x^2 + \frac{127858928}{42567525}x^3\right] - \ln^3\left(\frac{m_b^2}{\mu^2}\right)\frac{111848}{19845} - \ln^2\left(\frac{m_b^2}{\mu^2}\right)\ln\left(\frac{m_c^2}{\mu^2}\right)\frac{223696}{46305} \\
& + \ln\left(\frac{m_b^2}{\mu^2}\right)\ln\left(\frac{m_c^2}{\mu^2}\right)\left[\frac{22238456}{4862025} - \frac{1504864}{231525}x - \frac{355888}{40425}x^2 - \frac{255717856}{42567525}x^3\right] + \ln\left(\frac{m_b^2}{\mu^2}\right)\ln^2\left(\frac{m_c^2}{\mu^2}\right)\frac{223696}{46305} \\
& + \ln\left(\frac{m_c^2}{\mu^2}\right)\left[-\frac{24797875607}{1021025250} - \frac{111848}{15435}\zeta_2 + \frac{128543024}{24310125}x + \frac{22957168}{3361743}x^2 + \frac{2511536080}{2191376187}x^3\right] \\
& + \ln^2\left(\frac{m_c^2}{\mu^2}\right)\left[\frac{3232799}{4862025} + \frac{752432}{231525}x + \frac{177944}{40425}x^2 + \frac{127858928}{42567525}x^3\right] - \ln^3\left(\frac{m_c^2}{\mu^2}\right)\frac{1230328}{138915}\Bigg\} + O\left(x^4\ln^3(x)\right)
\end{aligned}
$$

These moments have been calculated referring qexp by Steinhauser et al. with operator insertions. Despite being universal, these contribution do not belong to the charm or bottom PDF. This is not compatible with the VFNF.

Renormalization of the OMEs:

[Bierenbaum, Blümlein, Klein 2009 Nucl.Phys.B]

1. include contributions from reducible diagrams
2. perform on-shell mass renormalization
3. renormalize the coupling in a MOM-scheme, using the background field method
4. remove remaining UV singularities through the Z-factors of the local operators
5. remove collinear singularities via coll. factorization (being different from the former one)
6. transform coupling constant to $\overline{\mathsf{MS}}$
7. choice: m on-shell or $m_{\overline{\mathsf{MS}}}$

$\rightarrow$ Generalization for OMEs with two different masses is needed.

Renormalized result for the OME $A_{qqQ}^{\mathrm{NS},(3)}(m_b, m_c)$

$$
\begin{aligned}
A_{qq,Q}^{\mathrm{NS},(3)}\left(m_c^2, m_b^2, Q^2, N\right) \quad = \quad & T_F^2 C_F \Biggl\{ \left(\frac{2+3\,N+3\,N^2}{N\,(1+N)} - 4S_1 \right) \times \\
& \Biggl(\left[\frac{128}{2835} \ln^2(x) - \frac{9014}{99225} \ln(x) + \frac{283396}{3472875} \right] x^3 \\
& + \left[\frac{8}{35} \ln^2(x) - \frac{4232}{3675} \ln(x) + \frac{897044}{385875} \right] x^2 + \left[-\frac{64}{15} \ln(x) + \frac{1504}{225} \right] x \\
& \frac{496}{27} \ln\left(\frac{m_b^2}{Q^2}\right) + \frac{496}{81} \ln\left(\frac{m_c^2}{Q^2}\right) + \frac{32}{27} \ln^3\left(\frac{m_c^2}{Q^2}\right) + \frac{16}{9} \ln^3\left(\frac{m_b^2}{Q^2}\right) \\
& + \frac{16}{9} \ln\left(\frac{m_b^2}{Q^2}\right) \ln^2\left(\frac{m_c^2}{Q^2}\right) \Biggr) \\
& + \left(\ln\left(\frac{m_c^2}{Q^2}\right) + \ln\left(\frac{m_b^2}{Q^2}\right) \right)^2 \left(-\frac{320}{27} S_1 + \frac{64}{9} S_2 + \frac{8}{27} \frac{P_1(N)}{N^2\,(1+N)^2} \right) \\
& + \frac{64}{729} S_1 - \frac{1280}{27} \zeta_3 S_1 + \frac{256}{81} S_2 + \frac{1280}{81} S_3 - \frac{256}{27} S_4 \\
& + \frac{320}{27} \frac{\left(2+3\,N+3\,N^2\right)\zeta_3}{N\,(1+N)} - \frac{8}{729} \frac{P_2(N)}{N^4\,(1+N)^4} \Biggr\}
\end{aligned}
$$

- Results for contributions with two fermions of equal and non-equal mass have also been computed for the OMEs $A_{Qq}^{\mathrm{PS},(3)}$ and $A_{qq,Q}^{\mathrm{NS,Tr},(3)}$.
- The computation of these contributions to the OME A_{Qg} is in progress.

New topologies:

New topologies are studied using various methods:

- Representation in terms of higher hypergeometric functions (e.g.: Appell functions)
- Mellin-Barnes techniques
- Integration in terms of hyperlogarithms

Various functions appear in intermediary and final results:

- Hyperlogarithms
- Generalized harmonic Sums:

$$S_{a_1,\ldots,a_m}(x_1,\cdots,x_m)(N) = \sum_{n_1=1}^{N}\sum_{n_2=1}^{n_1}\cdots\sum_{n_m=1}^{n_{m-1}} \frac{(\mathrm{sign}(a_1))^{n_1} x_1^{n_1}}{n_1^{|a_1|}} \frac{(\mathrm{sign}(a_2))^{n_2} x_2^{n_2}}{n_2^{|a_2|}} \cdots \frac{(\mathrm{sign}(a_m))^{n_m} x_m^{n_m}}{n_m^{|a_m|}}$$

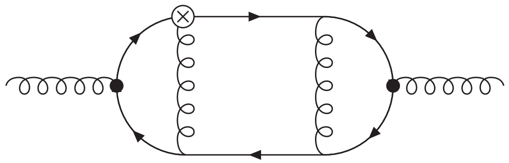
[Moch, Uwer, Weinzierl, 2002]

- Cyclotomic sums and cyclotomic HPLs [Ablinger, Blümlein, Schneider, 2011]

In very recent calculations of diagrams with two massive fermions also a new class of sums has been observed:

$$\sum_{i=1}^{N} \frac{4^i}{i} \frac{1}{\binom{2i}{i}} S_2(i-1) \quad = \quad \int_0^1 \frac{dz}{\sqrt{1-z}} \left\{ \ln^2\left(\frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right) - \zeta_2 \right\} \frac{z^N-1}{z-1}$$

Calculation of Convergent Massive 3-Loop Graphs



$$
\begin{aligned}
I_4(N) &= \int \cdots \int d\alpha_1\ d\alpha_2\ d\alpha_3\ d\alpha_4\ d\alpha_5\ d\alpha_6\ d\alpha_7\ d\alpha_8\ \frac{\sum_{j=0}^{N} T_{4\alpha}^{N-j} T_{4b}^{j}}{U^2 V^2} \\
T_{4\alpha} &= \alpha_5\alpha_7\alpha_4 + \alpha_2\alpha_3\alpha_5 + \alpha_2\alpha_5\alpha_4 + \alpha_3\alpha_5\alpha_7 + \alpha_2\alpha_5\alpha_8 + \alpha_8\alpha_5\alpha_4 + \alpha_5\alpha_7\alpha_8 + \alpha_2\alpha_3\alpha_8 \\
&\quad +\alpha_7\alpha_2\alpha_8 + \alpha_6\alpha_2\alpha_8 + \alpha_3\alpha_7\alpha_2 + \alpha_2\alpha_3\alpha_6 + \alpha_4\alpha_2\alpha_8 + \alpha_2\alpha_6\alpha_4 + \alpha_4\alpha_7\alpha_2 \\
T_{4b} &= +\alpha_2\alpha_5\alpha_4 + \alpha_4\alpha_2\alpha_8 + \alpha_4\alpha_7\alpha_2 + \alpha_2\alpha_5\alpha_8 + \alpha_2\alpha_3\alpha_5 + \alpha_7\alpha_2\alpha_8 + \alpha_3\alpha_7\alpha_2 + \alpha_8\alpha_5\alpha_4 \\
&\quad +\alpha_5\alpha_7\alpha_4 + \alpha_4\alpha_1\alpha_8 + \alpha_1\alpha_7\alpha_4 + \alpha_3\alpha_5\alpha_7 + \alpha_5\alpha_7\alpha_8 + \alpha_8\alpha_1\alpha_7 + \alpha_1\alpha_3\alpha_7 \\
U &= \alpha_2\alpha_5\alpha_4 + \alpha_2\alpha_3\alpha_5 + \alpha_1\alpha_3\alpha_5 + \alpha_5\alpha_7\alpha_4 + \alpha_1\alpha_6\alpha_4 + \alpha_1\alpha_3\alpha_6 + \alpha_2\alpha_3\alpha_6 + \alpha_2\alpha_6\alpha_4 \\
&\quad +\alpha_5\alpha_6\alpha_4 + \alpha_1\alpha_5\alpha_4 + \alpha_3\alpha_5\alpha_7 + \alpha_1\alpha_3\alpha_7 + \alpha_1\alpha_7\alpha_4 + \alpha_3\alpha_7\alpha_2 + \alpha_4\alpha_7\alpha_2 + \alpha_3\alpha_5\alpha_6 \\
&\quad +\alpha_2\alpha_3\alpha_8 + \alpha_2\alpha_5\alpha_8 + \alpha_5\alpha_7\alpha_8 + \alpha_8\alpha_5\alpha_4 + \alpha_8\alpha_5\alpha_6 + \alpha_5\alpha_3\alpha_8 + \alpha_1\alpha_8\alpha_5 + \alpha_1\alpha_8\alpha_6 \\
&\quad +\alpha_6\alpha_2\alpha_8 + \alpha_1\alpha_8\alpha_3 + \alpha_4\alpha_1\alpha_8 + \alpha_4\alpha_2\alpha_8 + \alpha_7\alpha_2\alpha_8 + \alpha_8\alpha_1\alpha_7 \\
V &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_7
\end{aligned}
$$

- The integral above is a projective integral, one α-parameter may be set 1
- The operators sit on on-shell diagrams which obey specific symmetries. These are generally not obeyed by the operator insertion.
- For the above example : after applying symmetry transformations $\alpha_1 \to x_1 - \alpha_2$, $\alpha_3 \to x_2 - \alpha_4$, $\alpha_5 \to x_5 - \alpha_6$ $\alpha_2, \alpha_4, \alpha_6$ are only contained in the operator polynomials and may be integrated out at this stage.

Calculation of Convergent Massive 3-Loop Graphs

- Feynman parameter integrals are performed in terms of Hyperlogarithms,
  [Brown 2008 Comm. Math. Phys.]
  $L(\overrightarrow{w}, z) : \mathbb{C}\backslash\Sigma \to \mathbb{C}$, where
  - $\Sigma = \{\sigma_0, \sigma_1, ..., \sigma_N\}$ are distinct points in $\mathbb{C}$ which may contain variables
  - $\overrightarrow{w}$ is a word over the alphabet $\mathfrak{A} = \{a_0, a_1, ..., a_N\}$ where each letter a_i corresponds to a point σ_i
- $L\left(\overrightarrow{w}, z\right)$ is uniquely defined by the following properties

1. $L\left(\{\}, z\right) = 1$, and $L\left(0^n, z, \right) = \frac{1}{n!}\log^n(z)$ for $n \geq 1$

2. $\frac{\partial}{\partial z} L\left(\{a_i \overrightarrow{w}\}, z\right) = \frac{1}{z - \sigma_i} L\left(\overrightarrow{w}, z\right)$ for $z \in \mathbb{C}\backslash\Sigma$

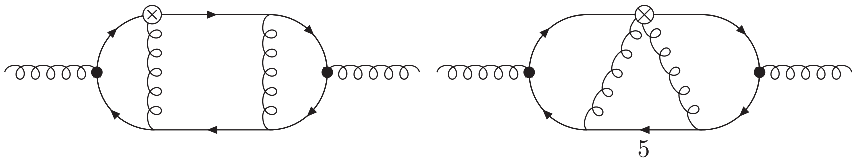
3. If $\overrightarrow{w}$ is not of the form $w = (0, 0, \cdots, 0)$, then $\lim_{z\to 0} L\left(\overrightarrow{w}, z\right) = 0$.

- e.g. $L\left(a_i, z\right) = \log(z - \sigma_i) - \log(\sigma_i)$
- The weight of $L\left(\overrightarrow{w}, z\right)$ is given by the number of letters in $\overrightarrow{w}$

- The hyperlogarithms satisfy shuffle relations $L\left(\overrightarrow{w_1}, z\right) L\left(\overrightarrow{w_2}, z\right) = L\left(\overrightarrow{w_1} ⧢ \overrightarrow{w_2}, z\right)$, e.g.:
  $L\left(\{a_1, a_2\}, z\right) L\left(\{a_3\}, z\right) = L\left(\{a_3, a_1, a_2\}, z\right) + L\left(\{a_1, a_3, a_2\}, z\right) + L\left(\{a_1, a_2, a_3\}, z\right)$
- The indices a_i contain further integration variables.
- Using these properties after partial fractioning and integration by parts, one can express any primitive for expressions consisting of rational and hyperlogarithmic functions in terms of different hyperlogarthmic functions
- These primitives have to be evaluated at the respective integration limits
  - The limit at $z \to 0$ is trivially obtained by computing the regularized Taylor series for the hyperlogarithmic functions
  - The limit at $z \to \infty$ is more sophisticated. General idea:
    1. Choose the integration order (Fubini).
    2. Compute the derivative with respect to the next integration variable x, (this lowers the weight by one).
    3. Perform the series expansion of the derivative.
    4. Perform the indefinite integration with respect to x.
    5. Determine the respective integration constant.

Fixed Mellin Moments

- Using this method we have computed a number of fixed Mellin-Moments from $N = 0..19$ e.g.:



N	Diag 4	Diag 5_a	Diag 5_b
0	$2 - 2\zeta_3$	$2\zeta_3$	$2\zeta_3$
1	$-2 + 2\zeta_3$	$-\frac{5}{2} - \zeta_3$	$-2 - 2\zeta_3$
2	$\frac{29}{12} - \frac{83}{36}\zeta_3$	$\frac{133}{72} + \frac{41}{8}\zeta_3$	$\frac{71}{24} + \frac{5}{2}\zeta_3$
3	$-\frac{17}{6} + \frac{47}{18}\zeta_3$	$-\frac{1735}{432} - \frac{35}{36}\zeta_3$	$-\frac{905}{216} - \frac{5}{2}\zeta_3$
...	...	...	...
19	$-\frac{58251582368792530944134896585 69181}{250356223589570838110891520 0000}$ $-\frac{104899807174743864253}{5419237599135360 0}\zeta_3$	$-\frac{1280902668906280290626432157 83549}{1335233192477711136591421440 00}$ $+\frac{2383887939492174 97}{301068755507520}\zeta_3$	$-\frac{254116903575797385411050257769}{2528850743328998364756480 0000}$ $-\frac{1968329}{635040}\zeta_3$

General Values of N

- Due to the operator-insertions leading to power-type functions, the integrals do not fit directly into the framework of the algorithm for general values of N.
- In order to use the algorithm also on integrals with general values of N, a generating function is constructed e.g. by the mapping

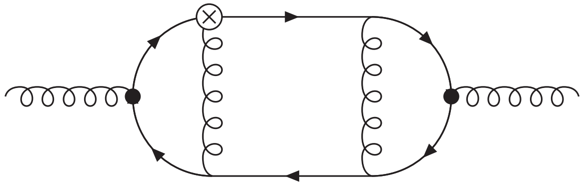
$$p\left(\alpha_1, \cdots, \alpha_n\right)^N \rightarrow \frac{1}{1 - x\, p\left(\alpha_1, \cdots, \alpha_n\right)} \ .$$

- Performing the Feynman-parameter integrations then leads to an expression which contains hyperlogarithms $L_w(x)$ in the variable x.
- Finally the Nth coefficient of this expression in x has to be extracted analytically. This has been done with the package `HarmonicSums` by J.Ablinger.
- Generalized harmonic sums occur:

$$S_{n_1,\ldots,n_k}(a_1, \ldots, a_k)(N), \quad n_i \in \mathbb{N}, a_i \in \mathbb{Q}$$

.

Six Massive Lines & Vertex Insertion

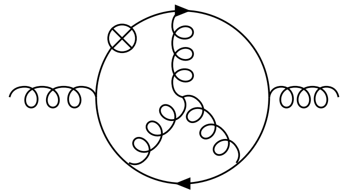


$$
\begin{aligned}
\hat{I}_4 \;=\;& \frac{Q_1(N)}{2(1+N)^5(2+N)^5(3+N)^5} + \frac{Q_2(N)}{(1+N)^2(2+N)^2(3+N)^2}\zeta_3 + \frac{(-1)^N\,(65+101N+56N^2+13N^3+N^4)}{2(1+N)^2(2+N)^2(3+N)^2}S_{-3} \\
& + \frac{(-24-5N+2N^2)}{12(2+N)^2(3+N)^2}S_1^3 - \frac{1}{2(1+N)(2+N)(3+N)}S_2^2 + \frac{1}{(2+N)(3+N)}S_1^2 S_2 \\
& + \frac{Q_4(N)}{4(1+N)^3(2+N)^2(3+N)^2}S_1^2 - \frac{3}{2}S_5 - \frac{Q_5(N)}{6(1+N)^2(2+N)^2(3+N)^2}S_3 - 2S_{-2,-3} - 2\zeta_3 S_{-2} - S_{-2,1}S_{-2} \\
& + \frac{(-1)^N\,(65+101N+56N^2+13N^3+N^4)}{(1+N)^2(2+N)^2(3+N)^2}S_{-2,1} + \frac{(59+42N+6N^2)}{2(1+N)(2+N)(3+N)}S_4 + \frac{(5+N)}{(1+N)(3+N)}\zeta_3 S_1\,(2) \\
& - \frac{Q_6(N)}{4(1+N)^3(2+N)^2(3+N)^2}S_2 - \zeta_3 S_2 - \frac{3}{2}S_3 S_2 - 2S_{2,1}S_2 + \frac{(99+225N+190N^2+65N^3+7N^4)}{2(1+N)^2(2+N)^2(3+N)}S_{2,1} \\
& + \frac{Q_3(N)}{(1+N)^4(2+N)^4(3+N)^4}S_1 - \frac{(11+5N)}{(1+N)(2+N)(3+N)}\zeta_3 S_1 - \frac{Q_7(N)}{4(1+N)^2(2+N)^2(3+N)^2}S_2 S_1 - S_{2,3} \\
& + \frac{(53+29N)}{2(1+N)(2+N)(3+N)}S_3 S_1 - \frac{3(3+2N)}{(1+N)(2+N)(3+N)}S_1 S_{2,1} + \frac{(-79-40N+N^2)}{2(1+N)(2+N)(3+N)}S_{3,1} - 3S_{4,1} \\
& + S_{-2,1,-2} + \frac{\mathbf{2^{N+1}}\,(-28-25N-4N^2+N^3)}{(1+N)^2(2+N)(3+N)^2}S_{1,2}\left(\frac{1}{2},1\right) - \frac{(-7+2N^2)}{(1+N)(2+N)(3+N)}S_{2,1,1} \\
& + 5S_{2,2,1} + 6S_{3,1,1} + \frac{\mathbf{2^{N}}\,(-28-25N-4N^2+N^3)}{(1+N)^2(2+N)(3+N)^2}S_{1,1,1}\left(\frac{1}{2},1,1\right) \\
& - \frac{(5+N)}{(1+N)(3+N)}S_{1,1,2}\left(2,\frac{1}{2},1\right) - \frac{(5+N)}{2(1+N)(3+N)}S_{1,1,1,1}\left(2,\frac{1}{2},1,1\right)
\end{aligned}
$$

The 2^N factors cancel in the large N limit:

$$
\begin{aligned}
\hat{I}_4 \approx\ & \zeta_2^2 \left[\frac{1115231}{20N^{10}} - \frac{74121}{4N^9} + \frac{122951}{20N^8} - \frac{40677}{20N^7} + \frac{13391}{20N^6} - \frac{873}{4N^5} + \frac{1391}{20N^4} - \frac{417}{20N^3} + \frac{101}{20\,N^2} \right] \\
& + \zeta_3 \left[\left(-\frac{95855}{2N^{10}} + \frac{31525}{2N^9} - \frac{10295}{2N^8} + \frac{3325}{2N^7} - \frac{1055}{2N^6} + \frac{325}{2N^5} - \frac{95}{2N^4} + \frac{25}{2N^3} - \frac{5}{2\,N^2} \right) \ln(N) \right. \\
& \left. - \frac{23280115}{2016N^{10}} + \frac{2093041}{1008N^9} - \frac{177251}{1008N^8} - \frac{25843}{336N^7} + \frac{2569}{48N^6} - \frac{155}{8N^5} + \frac{91}{24N^4} + \frac{2}{3N^3} - \frac{11}{12\,N^2} \right] \\
& + \zeta_2 \left[\left(\frac{19171}{N^{10}} - \frac{6305}{N^9} + \frac{2059}{N^8} - \frac{665}{N^7} + \frac{211}{N^6} - \frac{65}{N^5} + \frac{19}{N^4} - \frac{5}{N^3} + \frac{1}{N^2} \right) \ln^2(N) \right. \\
& + \left(\frac{103016863}{2520N^{10}} - \frac{3091261}{315N^9} + \frac{2571839}{1260N^8} - \frac{6215}{21N^7} - \frac{293}{20N^6} + \frac{2071}{60N^5} - \frac{103}{6N^4} + \frac{67}{12N^3} - \frac{1}{N^2} \right) \ln(N) \\
& \left. + \frac{292993001621}{302400N^{10}} - \frac{4402272031}{30240N^9} + \frac{22261739}{840N^8} - \frac{78507473}{14112N^7} + \frac{180961}{144N^6} - \frac{111807}{400N^5} + \frac{629}{12N^4} - \frac{319}{72N^3} - \frac{7}{4\,N^2} \right] \\
& + \left(\frac{249223}{6N^{10}} - \frac{145015}{12N^9} + \frac{10295}{3N^8} - \frac{11305}{12N^7} + \frac{1477}{6N^6} - \frac{715}{12N^5} + \frac{38}{3N^4} - \frac{25}{12N^3} + \frac{1}{6\,N^2} \right) \mathbf{ln^3(N)} \\
& + \left(\frac{193493767}{10080N^{10}} + \frac{210658237}{10080N^9} - \frac{21541697}{2520N^8} + \frac{243269}{96N^7} - \frac{30539}{48N^6} + \frac{2123}{16N^5} - \frac{59}{3N^4} + \frac{5}{8N^3} + \frac{1}{2\,N^2} \right) \ln^2(N) \\
& + \left(-\frac{2207364771673}{4233600N^{10}} + \frac{1390655509}{352800N^9} + \frac{285594061}{22050N^8} - \frac{67234111}{14400N^7} + \frac{8617073}{7200N^6} - \frac{35209}{144N^5} + \frac{116}{3N^4} - \frac{119}{24N^3} + \frac{1}{N^2} \right) \ln(N) \\
& + \frac{1344226725047831}{889056000N^{10}} - \frac{165849841805771}{889056000N^9} + \frac{808151260279}{27783000N^8} - \frac{708430537}{120960N^7} + \frac{304474703}{216000N^6} \\
& - \frac{606811}{1728N^5} + \frac{1867}{24N^4} - \frac{1813}{144N^3} + \frac{1}{N^2} + O(N^{-11})
\end{aligned}
$$

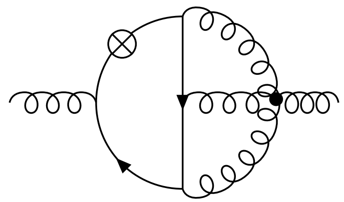
General Values of N: Higher Topologies



$$I(x) = \frac{1}{(1+N)(2+N)x}\Big\{\zeta_3\Big[2L(\{-1\},x) - 2(-1+2x)L(\{1\},x) - 4L(\{1,1\},x)\Big] - 3L(\{-1,0,0,1\},x)$$
$$+2L(\{-1,0,1,1\},x) - 2xL(\{0,0,1,1\},x) + 3xL(\{0,1,0,1\},x) - xL(\{0,1,1,1\},x)$$
$$+(-3+2x)L(\{1,0,0,1\},x) + 2xL(\{1,0,1,1\},x) - (-1+5x)L(\{1,1,0,1\},x) + xL(\{1,1,1,1\},x)$$
$$-2L(\{1,0,0,1,1\},x) + 3L(\{1,0,1,0,1\},x) - L(\{1,0,1,1,1\},x) + 2L(\{1,1,0,0,1\},x)$$
$$+2L(\{1,1,0,1,1\},x) - 5L(\{1,1,1,0,1\},x) + L(\{1,1,1,1,1\},x)\Big\}$$

$$I(N) = \frac{1}{(N+1)(N+2)(N+3)}\Big\{\frac{648+1512N+1458N^2+744N^3+212N^4+32N^5+2N^6}{(1+N)^3(2+N)^3(3+N)^3}$$
$$-\frac{2\left(-1+(-1)^N+N+(-1)^N N\right)}{(1+N)}\zeta_3 - (-1)^N S_{-3} - \frac{N}{6(1+N)}S_1^3 + \frac{1}{24}S_1^4$$
$$-\frac{\left(7+22N+10N^2\right)}{2(1+N)^2(2+N)}S_2 - \frac{19}{8}S_2^2 - \frac{1+4N+2N^2}{2(1+N)^2(2+N)}S_1^2 + \frac{9}{4}S_2 - \frac{(-9+4N)}{3(1+N)}S_3$$
$$-\frac{1}{4}S_4 - 2(-1)^N S_{-2,1} + \frac{(-1+6N)}{(1+N)}S_{2,1} + \frac{54+207N+246N^2+130N^3+32N^4+3N^5}{(1+N)^3(2+N)^2(3+N)^2}S_1$$
$$+4\zeta_3 S_1 - \frac{(-2+7N)}{2(1+N)}S_2 S_1 + \frac{13}{3}S_3 S_1 - 7S_{2,1}S_1 - 7S_{3,1} + 10S_{2,1,1}\Big\}$$

General Values of N: Higher Topologies



$$
\begin{aligned}
I(N) \quad = \quad & \frac{1}{(N+1)(N+2)}\Bigg\{\frac{2\left(1-13(-1)^N+(-1)^N 2^{3+N}+N-7(-1)^N N+3(-1)^N 2^{1+N}N\right)}{(1+N)(2+N)}\zeta_3 \\
& +\frac{1}{(2+N)}S_3+\frac{(-1)^N}{2(2+N)}S_1^3-\frac{(-1)^N(3+2N)}{2(1+N)^2(2+N)}S_2+\frac{5(-1)^N}{2}S_2^2 \\
& +\frac{(-1)^N(3+2N)}{2(1+N)^2(2+N)}S_1^2-\frac{(-1)^N}{2}S_2S_1^2+\frac{3(-1)^N(4+3N)}{(1+N)(2+N)}S_3+3(-1)^N S_4+\frac{2}{(2+N)}S_{-2,1} \\
& +2(-1)^N\zeta_3 S_1,(2)+\frac{2(-1)^N(3+N)}{(1+N)(2+N)}S_{2,1}-12(-1)^N S_1\zeta_3 \\
& +\frac{(-1)^N(5+7N)}{2(1+N)(2+N)}S_1S_2+3(-1)^N S_1S_3+4(-1)^N S_{2,1}S_1-4(-1)^N S_{3,1} \\
& -\frac{4\left((-1)^N 2^{2+N}-3(-2)^N N+3(-1)^N 2^{1+N}N\right)}{(1+N)(2+N)}S_{1,2}\left(\frac{1}{2},1\right)-5(-1)^N S_{2,1,1} \\
& +\frac{2\left(-(-1)^N 2^{2+N}-13(-2)^N N+5(-1)^N 2^{1+N}N\right)}{(1+N)(2+N)}S_{1,1,1}\left(\frac{1}{2},1,1\right) \\
& -2(-1)^N S_{1,1,2}\left(2,\frac{1}{2},1\right)-(-1)^N S_{1,1,1,1}\left(2,\frac{1}{2},1,1\right)\Bigg\}
\end{aligned}
$$

Conclusions

- A series of moments for the transition matrix elements A_{ij} at 3-loop order were given in [Bierenbaum, Blümlein, Klein 2009 Nucl. Phys. B].

- The corresponding quarkonic 3-loop contributions of $O(n_f T_F^2 C_{A,F})$ to A_{qq} and A_{qg} were calculated in [Ablinger, Blümlein, Klein, Schneider, Wißbrock 2011 Nucl. Phys. B]. Recently also $A_{gg,Q}$ and $A_{gq,Q}$ have been obtained for these color coefficients at general N.

- A series of OMEs were fully calculated A_{qq} and A_{qg} in $O(T_F^2 C_{A,F})$

- The moments N = 2,4,6 have been calculated for graphs depending on both m_c and m_b; general N results in the NS and PS case have been obtained already. Starting with 3-loops, graphs exist which conflict with the ideology of the VFNS.

- Using hyperlogarithms non-divergent 3-loop graphs can be calculated, if moments are considered. For general values of N first analytic results have been obtained, including Benz- and ladder-topologies, performing the calculation automatically.

- 3-loop moments of polarized massive OMEs up to the constant terms have been calculated. The $O(n_F T_F^2)$ contributions to these OMEs have been computed for general values of N.