Towards a Holographic Realization of Homes' Law

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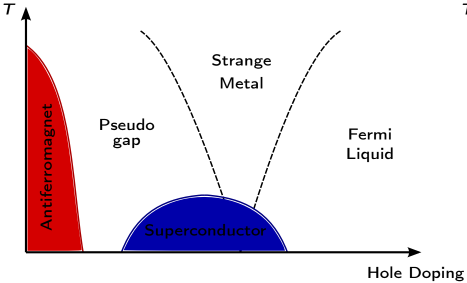
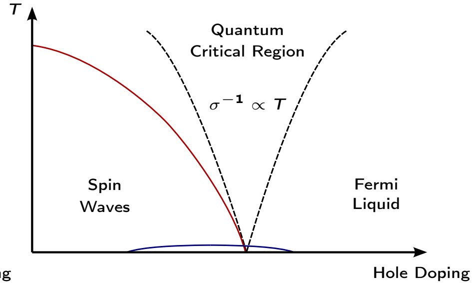
based on work of
Johanna Erdmenger, Patrick Kerner and SM
arXiv:1206.5305

Outline

High-Temperature Superconductors

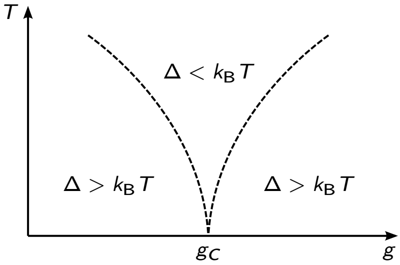




Unresolved Problems in High T_c Superconductors

- Unknown pairing mechanism between electrons
- Pseudo-gap between insulator/superfluid phase
- QCP below superconducting dome/strange metal phase a QCR?

Continuous Quantum Phase Transitions



- long-range quantum fluctuations/entanglement
- Non-analyticity in $E_{GS}(g_c)$
- Near critical coupling g_c:

$$\Delta \sim J\,|g - g_c|^{z\nu} \qquad \xi^{-1} \sim \Lambda\,|g - g_c|^{\nu} \qquad \Delta \sim \xi^{-z}$$

- Quantum critical point $\longrightarrow$ quantum critical region for $T > 0$ with two different regimes:

$$\Delta > k_B T \qquad \tau \gg \frac{\hbar}{k_B T} \qquad \text{(effectively classical)}$$

$$\Delta < k_B T \qquad \tau \sim \frac{\hbar}{k_B T} \qquad \text{(quantum critical)}$$

Quantum Criticality & Planckian Dissipation

- Planckian dissipation:
  - Shortest possible timescale for finite temperature

$$\tau_\hbar = \frac{\hbar}{k_B T}$$

  - if $\tau < \tau_\hbar$ superfluid motion is purely quantum mechanical

    $\longrightarrow$ no dissipation (entropy production) possible

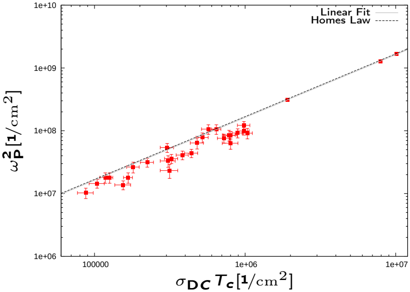
    $\longrightarrow$ Perfect fluid without quasi particle excitations

- Comparison with Quark-Gluon-Plasma:
  - Viscosity density $\eta \sim \epsilon\tau$

$$\tau_\eta = \frac{\eta}{Ts} = \frac{1}{4\pi}\frac{\hbar}{k_B T}$$

Homes' Law for Superconductors

- All superconductors seem to show an universal scaling law
  [Homes et al., Nature 2004]



Homes' Law

$$\rho_s \propto \sigma_{DC}(T_c)\, T_c$$

- The superfluid density ρ_s is defined as

$$\rho_s \equiv \omega_{Ps}^2 \equiv \lambda_L^{-2} = \frac{4\pi n_s e^2}{m^*}$$

  - Superconductor becomes "transparent" for $\omega > \omega_{Ps}$
  - Photons can penetrate the superconductor to $l < \lambda_L$

Homes' Law for Superconductors

- The optical sum rule can be used to obtain ω_P^2 [Homes et al., cond-mat/0410719]

$$\frac{\omega_P^2}{8} = \int_0^\infty \mathrm{d}\omega\, \sigma^{(1)}(\omega),$$

and determine the difference in spectral weights

$$N_n = \int_{0^+}^\infty \mathrm{d}\omega\, \sigma(\omega)\Big|_{T>T_c} \simeq \frac{\omega_P}{8}$$

$$N_s = \int_{0^+}^\infty \mathrm{d}\omega\, \sigma(\omega)\Big|_{T<T_c}$$

Superfluid Density

$$\rho_s = 8\,(N_n - N_s) = \omega_{Pn}^2 - 8N_s$$

Homes' Law for Superconductors

- Using the Drude-Sommerfeld form of the optical conductivity

Conductivity

$$\sigma_{\mathrm{DC}} = \frac{\omega_{\mathrm{Pn}}^2 \tau}{4\pi} = \frac{n_{\mathrm{n}} e^2 \tau}{m^*}$$

we can follow to different paths:

1. Assuming that all charge carriers condense in the superconducting phase, i.e. $N_{\mathrm{s}} = 0$

Plasma Frequencies

$$\omega_{\mathrm{Ps}}^2 = \omega_{\mathrm{Pn}}^2$$

2. Only valid for High T_c superconductors

Tanner's law

$$n_{\mathrm{s}} \approx \frac{1}{4} n_{\mathrm{n}}$$

Holographic Version of Homes' Law

- Problem:
  - Due to translational invariance/momentum conservation

$$\mathrm{Re}\,\sigma(\omega) \sim \delta(\omega)\Big|_{T>T_c}$$

  - Inserting 1 or 2 into Homes' law yields

Universal Timescale

$$\tau_c T_c = \frac{4\pi}{C}$$

  - Assume a simple proportionality $D(T) \propto \tau$:

Holographic Homes' Law

$$D(T_c)\,T_c = \text{const.}$$

Holographic S-Wave Superconductor [Hartnoll,Herzog,Horowitz, 2008]

Einstein-Maxwell Action

$$S = \frac{1}{2\kappa^2} \int [d+1]x \sqrt{-g} \left[R - 2\lambda + \frac{2\kappa^2}{e^2} \left(F^2 - |\nabla\Phi - \mathrm{i}A\psi|^2 - V(|\psi|) \right) \right]$$

Equations of Motion

Einstein: $$R_{ab} - \frac{1}{2} R\, g_{ab} + \frac{d(d-1)}{2L^2} g_{ab} = \alpha^2 L^2 T_{ab}[A, \Phi]$$

Maxwell: $$\nabla^a F_{ab} = j_b[A, \Phi]$$

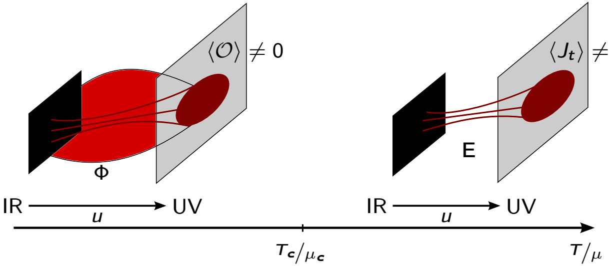
Scalar Field: $$(\nabla - \mathrm{i}A)^2 \Phi - \frac{1}{2} V'(|\Phi|) \frac{\Phi}{|\Phi|} = 0$$

Solutions & Phase Diagram

Solutions to Equations of Motion

	$\alpha = 0$ (probe limit)	$\alpha \neq 0$ (backreaction)
$\Phi = 0$	AdS Schwarzschild BH	AdS Reissner Nordström BH
$\Phi \neq 0$	AdS Schwarzschild BH	Scalar Hair AdS RN BH

- Phase transition at critical T_c/μ_c



Momentum & R-Charge Diffusion

- Momentum diffusion D_{M} related to $\langle T_{xy} T_{xy} \rangle$
  $\longrightarrow$ Solution of h_{xy} fluctuation equation

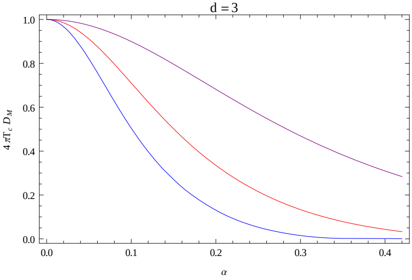
$$D_{\mathrm{M}} = \frac{1}{4\pi T} \left(1 + \frac{\mu}{T}\frac{n}{s}\right)^{-1}$$
$$= \frac{1}{4\pi T} \left(1 + \frac{2(d-1)\bar{Q}^2}{d-(d-2)\bar{Q}^2}\right)^{-1}$$

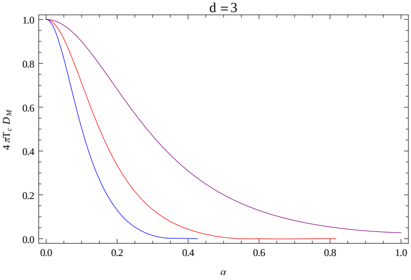
- R–charge diffusion D_{R} related to $\langle J_t J_t \rangle$
  $\longrightarrow$ Solution of a_t fluctuation equation (only known in $d = 4$)

$$D_{\mathrm{R}} = \frac{1}{4\pi T} \frac{(2-\bar{Q}^2)(2+\bar{Q}^2)}{2(1+\bar{Q}^2)}$$

[Banerjee, Bhattacharya, Bhattacharyya, Dutta, Loganayagam et al., 2011]

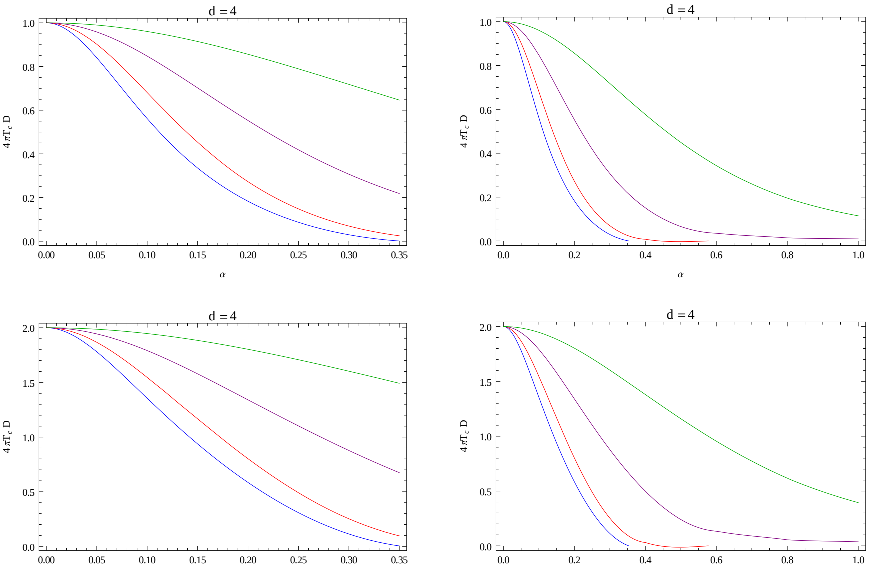
Numerical Results for $d = 3$





- For each value of α we can determine a critical $T_c/\mu_c \longrightarrow \alpha\left(T_c/\mu_c\right)$
- Furthermore, we can tune the mass of the scalar field m^2L^2
- For $m^2L^2 < 0$ we approach AdS_2 for $T \rightarrow 0$
  $\longrightarrow$ For $\alpha > \alpha_c$ scalar field fluctuations shows instability at $T = 0$

Numerical Results for $d = 4$



Conclusion & Outlook

- Conclusion:
  - In the case of $\alpha = 0$ we recover

$$D_{\mathrm{M}} = \frac{1}{4\pi T} \qquad D_{\mathrm{R}} = \frac{1}{4\pi T}\frac{d}{d-2} \qquad \longrightarrow \qquad \tau T = \text{const.}$$

  - Assumption N_{s} might not hold for holographic superconductors

$$\omega_{\mathrm{Ps}}^2 = \omega_{\mathrm{Pn}}^2 - 8N_{\mathrm{s}} \qquad \Leftrightarrow \qquad \tau_c T_c = \frac{4\pi}{C}\left(1 - \frac{N_{\mathrm{s}}}{N_{\mathrm{n}}}\right).$$

  - Tanner's law could be modified by backreaction

$$n_{\mathrm{s}} = B(\alpha) n_{\mathrm{n}} \qquad \Leftrightarrow \qquad \tau_c T_c = 4\pi \frac{B(\alpha)}{C}$$

- Outlook:
  - Calculate explicitly ρ_{s} near $T = 0$
  - Investigate the influence of normal state degrees of freedom $\longrightarrow$ Coexistence of ideal and superconductor
  - Check Tanner's law holographically

Thank You

for

Listening!

Einstein-Maxwell Theory [Hartnoll,Herzog,Horowitz, 2008]

Einstein-Maxwell Action

$$S = \frac{1}{2\kappa^2} \int [d{+}1]x\sqrt{-g}\left[R - 2\Lambda + \frac{2\kappa^2}{e^2}\left(\frac{1}{4}F^2 - |\nabla\Phi - \mathrm{i}A\Phi|^2 - V(|\Phi|)\right)\right]$$

- The backreaction of the gauge and matter fields is described by

$$\alpha^2 L^2 = \frac{\kappa^2}{e^2} \sim \frac{c_{\langle JJ\rangle}}{c_{\langle TT\rangle}} \sim \frac{\#(\text{charged degrees of freedom})}{\#(\text{total degrees of freedom})}$$

- Gauge/Gravity Duality tells us

Cooper Pair $\mathcal{O} = \Psi\Psi$	$\Leftrightarrow$	charged scalar field Φ
global $U(1)$ current J_μ		locale $U(1)$ gauge field A_a

Einstein-Maxwell Theory [Hartnoll,Herzog,Horowitz, 2008]

Equations of Motion for the Background Fields

Einstein: $$R_{ab} - \frac{1}{2} R\, g_{ab} + \frac{d(d-1)}{2L^2} g_{ab} = \alpha^2 L^2 T_{ab}[A, \Phi]$$

Maxwell: $$\nabla^a F_{ab} = j_b[A, \Phi]$$

Scalar Field: $$(\nabla - \mathrm{i}A)^2 \Phi - \frac{1}{2} V'(|\Phi|) \frac{\Phi}{|\Phi|} = 0$$

Solutions to Equations of Motion

	$\alpha = 0$ (probe limit)	$\alpha \neq 0$ (backreaction)
$\Phi = 0$	AdS Schwarzschild BH	AdS Reissner Nordström BH
$\Phi \neq 0$	AdS Schwarzschild BH	Scalar Hair AdS RN BH

AdS–Reissner–Nordström Black Brane

- Solution in the normal conducting phase $\Phi = 0 \Leftrightarrow \langle \mathcal{O} \rangle = 0$:

$$\mathrm{d}s^2 = \frac{L^2}{u^2}\left(-f(u)\,\mathrm{d}t^2 + \mathrm{d}\mathbf{x}^2 + \frac{\mathrm{d}u^2}{f(u)}\right)$$

$$f(u) = 1 - \bar{M}\frac{u^d}{u_{\mathrm{H}}^d} + \bar{Q}^2\frac{u^{2(d-1)}}{u_{\mathrm{H}}^{2(d-1)}}$$

$$A = \mu\left(1 - \frac{u^{d-2}}{u_{\mathrm{H}}^{d-2}}\right)\mathrm{d}t$$

- Relations between dimensionless quantities

$$\bar{M} = 1 + \bar{Q}^2 \qquad \bar{Q} = \sqrt{\frac{d-2}{d-1}}\,\bar{\mu}\alpha \qquad \bar{\mu} = \mu u_{\mathrm{H}}$$

Thermodynamics of AdS–Reissner–Nordström Black Branes

- Thermodynamic quantities are derived from the grand canonical potential via the regularized Euclidean on–shell action I

$$\Omega(T, V, \mu) = \frac{1}{\beta} I = k_\mathrm{B} T I,$$

using the thermodynamic relations

$$s = \frac{4\pi}{2\kappa^2} \frac{L^{d-1}}{u_\mathrm{H}^{d-1}} \sim T^{d-1} \mathcal{F}_s\left(\alpha, \frac{T}{\mu}\right)$$

$$\varepsilon = \frac{d-1}{2\kappa^2 L^{d-1}} \frac{1+\bar{Q}^2}{u_\mathrm{H}^d} \sim T^d \mathcal{F}_\varepsilon\left(\alpha, \frac{T}{\mu}\right)$$

$$n = \sqrt{(d-1)(d-2)} \frac{L^{d-1}}{\kappa^2} \frac{\alpha \bar{Q}}{u_\mathrm{H}^{d-1}} \sim T^{d-1} \mathcal{F}_n\left(\alpha, \frac{T}{\mu}\right)$$

Thermodynamics of AdS–Reissner–Nordström Black Brane

Temperature of the AdS–RN Black Brane

$$T = \frac{1}{2\pi}\left[\frac{1}{\sqrt{g_{uu}}}\frac{\mathrm{d}}{\mathrm{d}u}\sqrt{-g_{tt}}\right]_{u=u_{\mathrm{H}}} = \frac{d-(d-2)\bar{Q}^2}{4\pi u_{\mathrm{H}}}$$

- UV scale invariant theory with the dimensionless quantity

$$\frac{T}{\mu} = \frac{d-(d-2)\bar{Q}^2}{\frac{4\pi}{\alpha}\sqrt{\frac{d-1}{d-2}}\,\bar{Q}} = \frac{d-\frac{(d-2)^2}{d-1}\bar{\mu}^2\alpha^2}{4\pi\bar{\mu}}$$

- Relations between field theory quantities and gravitational quantities

$$u_{\mathrm{H}} = \frac{d-(d-2)\bar{Q}^2}{4\pi T} \qquad \bar{Q} = \mathcal{F}\left(\alpha, \frac{T}{\mu}\right)$$

Linear Response Theory

- Weak and short perturbations:

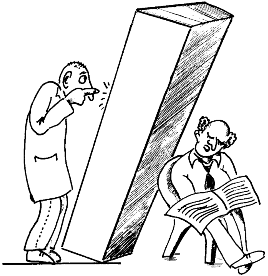
$$\delta\mathcal{H} = \int [\underset{.}{d}]x\varphi_i(t,x)\mathcal{O}^i(t,x)$$

- The response of the system is

$$\delta\left\langle\mathcal{O}^i(x)\right\rangle = \int [\underset{.}{d}+1]yG_{\mathrm{R}}^{ij}(x,y)\varphi_j(y)$$

- Causal structure:

$$G_{\mathrm{R}}^{ij}(x,y) = \mathrm{i}\theta(t_x - t_y)\left\langle\left[\mathcal{O}^i(x),\mathcal{O}^j(y)\right]\right\rangle$$



[Alexei Tsvelik - QFT in CMP]

- Fluctuation-Dissipation theorem:
  - Calculate correlators of fluctuations about background solution
  - Charged black hole with finite temperature provides damping

    $\longrightarrow$ determines dissipative transport coefficients

Holographic Fluctuation/Dissipation Theorem

- Non–equilibrium physics described by fluctuations

$$\varphi = \Phi + \delta\varphi \qquad\qquad A_a = A_a^{\mathrm{B}} + a_a$$

- Quadratic action in fluctuations
  $\longrightarrow$ equations of motion for fluctuations $\delta\varphi$

- Solved with in–falling wave boundary conditions at the black brane horizon and asymptotic UV behavior

$$\delta\varphi = \delta\varphi^{(0)}(\omega, \mathbf{k})u^{\Delta_-} + \delta\varphi^{(1)}(\omega, \mathbf{k})u^{\Delta_+} + \dots$$

- Dissipation into the black brane $\longrightarrow$ retarded Green's function

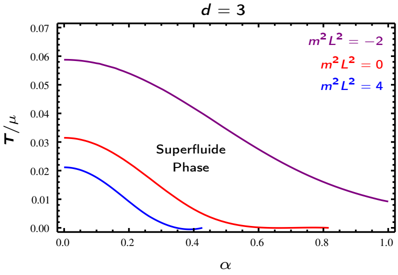
$$G^{\mathrm{R}}(\omega, \mathbf{k}) \propto \frac{\delta\psi^{(1)}}{\delta\psi^{(0)}}$$

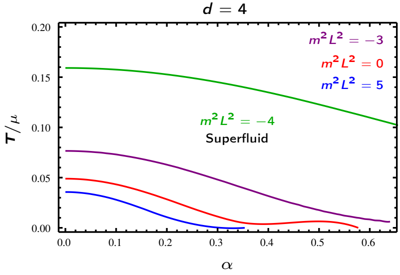
Approaching the Instability from the Normal Phase

- Solving scalar field fluctuations equations numerically

$$\delta\varphi''(u) + \left(\frac{f'(u)}{f(u)} - \frac{d-1}{u}\right)\delta\varphi'(u) + \left[\frac{(\omega+A_t)^2}{f(u)^2} - \frac{k^2}{f(u)} - \frac{L^2 m^2}{u^2 f(u)}\right]\delta\varphi(u) = 0$$

- 2nd order phase transition $\longrightarrow$ signaled by poles of $G^R(\omega, \mathbf{0})$ going through the origin of complex ω–plane





$T = 0$ Criticality

- For $T = 0$ IR-Geometry reduces to $AdS_2 \times \mathbb{R}^{d-1}$
  - Condition for condensation [Gubser, 2008]

$$L^2_{AdS_2} m^2_{\text{eff}} = \frac{1}{d(d-1)} \left(m^2 L^2 - \frac{1}{\alpha^2} \right) \leq L^2_{AdS_2} m^2_{BF} = -\frac{1}{4}$$

- Conditions for the existence of α_c