DESY theory workshop: Lessons from the first phase of the LHC

Motivic multiple zeta values and superstring amplitudes

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based on: 1205.1516: OS, St. Stieberger

1106.2645, 1106.2646: C. Mafra, OS, St. Stieberger

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Introduction: Superstring N point disk amplitude

Color stripped tree amplitude for scattering N massless open string states

$$\mathcal{A}(1,2,\ldots,N;\alpha') \;=\; \sum_{\pi\in S_{N-3}} \mathcal{A}^{\rm YM}\big(1,2_\pi,\ldots,(N-2)_\pi,N-1,N\big)\; F^\pi(\alpha')$$

[Mafra, OS, Stieberger 1106.2645, 1106.2646]

- decomposes into $(N-3)!$ field theory subamplitudes $\mathcal{A}^{\rm YM}_{\pi\in S_{N-3}}$
- string effects (α' dependence) from generalized Euler integrals $F^\pi(\alpha')$

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- decomposes into $(N-3)!$ field theory subamplitudes $\mathcal{A}^{\text{YM}}_{\pi\in S_{N-3}}$
- string effects (α' dependence) from generalized Euler integrals $F^\pi(\alpha')$
- consistent with field theory limit: $F^\pi(\alpha' \to 0) = \delta^\pi_{(2,3,\ldots,N-2)}$
- valid for states of $\mathcal{N}=1$ SYM in $D=10$ (or $\mathcal{N}=4$ SYM in $D=4$)
- remain valid for the gluon's SUSY multiplet for $\mathcal{N}<4$ compactification

Euler integrals F^π only depend on dimensionless Mandelstam variables:

$$s_{ij} \;=\; \alpha' \, (k_i + k_j)^2$$

In a disk boundary parametrization $z \in \mathbb{R}$ with $(z_1, z_{N-1}, z_N) = (0, 1, \infty)$:

$$F^\pi(\alpha') \;=\; \underbrace{\prod_{k=2}^{N-2} \int_{z_i < z_{i+1}} \mathrm{d}z_k}_{N-3 \text{ integrations}} \; \underbrace{\prod_{i<j}^{N} |z_{ij}|^{s_{ij}} \prod_{k=2}^{N-2} \sum_{m=1}^{k-1} \frac{s_{\pi(m)\pi(k)}}{z_{\pi(m)\pi(k)}}}_{\text{result of CFT correlation function}} \,,$$

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$\forall$ color ordering $\Sigma \in S_N$ $\exists$ separate set of functions $\{F_\Sigma{}^\pi , \; \pi \in S_{N-3}\}$:

$$\mathcal{A}\big(\Sigma(1,2,\ldots,N);\alpha'\big) \quad \leftrightarrow \quad F_\Sigma{}^\pi(\alpha') \quad = \quad \underbrace{\prod_{k=2}^{N-2} \int_{z_{\Sigma(i)} < z_{\Sigma(i+1)}} \mathrm{d}z_k}_{\Sigma \text{ dependent integration range}} \quad \ldots$$

Any subamplitude $\mathcal{A}\big(\Sigma(1,2,\ldots,N);\alpha'\big)$ with $\Sigma \in S_N$ can be expanded in the basis $\left\{ \mathcal{A}_\sigma(\alpha') \equiv \mathcal{A}\big(1,\sigma(2,3,\ldots N-2),N-1,N;\alpha'\big),\ \sigma \in S_{N-3} \right\}$

[Bjerrum-Bohr, Damgaard, Vanhove 0907.1425; Stieberger 0907.2211]

$\Longrightarrow$ sufficient to compute $\mathcal{A}_\sigma(\alpha')$ with $\sigma \in S_{N-3}$:

$$\boxed{\mathcal{A}_\sigma(\alpha') \quad = \sum_{\pi \in S_{N-3}} F_\sigma{}^\pi(\alpha')\, \mathcal{A}_\pi^{\mathrm{YM}}}$$

α' dependent $(N-3)! \times (N-3)!$ matrix $F_\sigma{}^\pi(\alpha')$ acting on the $(N-3)!$ cpt. vector $\mathcal{A}_\pi^{\mathrm{YM}} \equiv \mathcal{A}^{\mathrm{YM}}\big(1,\pi(2,3,\ldots,N-2),N-1,N\big)$

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Aim of the talk: Investigate the structure of the matrix $F_\sigma{}^\pi(\alpha')$

... and the appearance of multiple zeta values $\zeta_{n_1,n_2,\ldots,n_r}$ therein.

Outline

I. Multiple zeta values and their $\mathbb{Q}$ relations

II. First look at disk amplitude: matrix multiplications

III. Motivic multiple zeta values

IV. Second look at disk amplitude: motivic structure

V. Main result

I. Multiple zeta values and their $\mathbb{Q}$ relations

α' expansion of string amplitudes give rise to multiple zeta values (MZV's)

$$\zeta_{n_1,\ldots,n_r} := \sum_{0<k_1<\ldots<k_r} \frac{1}{k_1^{n_1}\, k_2^{n_2} \ldots k_r^{n_r}} \;, \quad n_i \in \mathbb{N} \;,\; n_r \geq 2$$

of various weights (transcendentality degree) $w = \sum_{i=1}^{r} n_i$ and depth r.

MZV relations over $\mathbb{Q}$ leave the following conjectural weight w bases $\mathcal{Z}_w$:

w	0	1	2	3	4	5	6	7	8	9	10
$\mathcal{Z}_w$	1	$\emptyset$	ζ_2	ζ_3	ζ_4	ζ_5 $\zeta_3\zeta_2$	ζ_6 ζ_3^2	$\zeta_7,\ \zeta_2\zeta_5$ $\zeta_4\zeta_3$	$\zeta_8,\ \zeta_3\zeta_5$ $\zeta_{3,5},\ \zeta_2\zeta_3^2$	$\zeta_9,\ \zeta_3^3,\ \zeta_2\zeta_7$ $\zeta_4\zeta_5,\ \zeta_6\zeta_3$	$\zeta_{10},\ \zeta_5^2,\ \zeta_{3,7},\ \zeta_7\zeta_3$ $\zeta_4\zeta_3^2,\ \zeta_2\zeta_{3,5},\ \zeta_2\zeta_3\zeta_5$
d_w	1	0	1	1	1	2	2	3	4	5	7

where $d_w := \dim_{\mathbb{Q}} \mathcal{Z}_w = d_{w-2} + d_{w-3}$ with $d_0 = 1$ and $d_1 = 0$.

[Zagier]

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$\mathcal{Z}_w$	1	$\emptyset$	ζ_2	ζ_3	ζ_4	ζ_5 $\zeta_3\zeta_2$	ζ_6 ζ_3^2	ζ_7, $\zeta_2\zeta_5$ $\zeta_4\zeta_3$	ζ_8, $\zeta_3\zeta_5$ $\zeta_{3,5}$, $\zeta_2\zeta_3^2$	ζ_9, ζ_3^3, $\zeta_2\zeta_7$ $\zeta_4\zeta_5$, $\zeta_6\zeta_3$	ζ_{10}, ζ_5^2, $\zeta_{3,7}$, $\zeta_7\zeta_3$ $\zeta_4\zeta_3^2$, $\zeta_2\zeta_{3,5}$, $\zeta_2\zeta_3\zeta_5$
d_w	1	0	1	1	1	2	2	3	4	5	7

Unlike odd single zeta values ζ_{2n+1}, even single ζ_{2n} are related over $\mathbb{Q}$,

$$\zeta_2 \;=\; \frac{\pi^2}{6}\,, \qquad \zeta_{2n} \;=\; \frac{(-1)^{n+1}}{2\,(2n)!}\,B_{2n}\,(2\pi)^{2n} \;\in\; \mathbb{Q}\cdot\pi^{2n}$$

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MZVs of different depth related by shuffle & stuffle relations such as:

$$\zeta_m\,\zeta_n = \zeta_{m,n} + \zeta_{n,m} + \zeta_{m+n}$$

$$\zeta_{3,5} = -\tfrac{5}{2}\,\zeta_{2,6} - \tfrac{21}{25}\,\zeta_2^4 + 5\,\zeta_3\,\zeta_5$$

<u>MZVs of depth > 1</u> become inevitable starting from weight $w=8$:

II. First look at disk amplitude: matrix multiplications

Observe a matrix-multiplicative structure in the α' expansion of $F(\alpha')$

$$\mathcal{A}(\alpha')\Big|_{N=5} = \Big(1 + \zeta_2 P_2 + \zeta_3 M_3 + \zeta_4 P_4 + \zeta_5 M_5 + \underline{\zeta_2 P_2 \zeta_3 M_3}$$
$$+ \zeta_6 P_6 + \underline{\tfrac{1}{2}\zeta_3^2 M_3^2} + \zeta_7 M_7 + \underline{\zeta_2 P_2 \zeta_5 M_5} + \underline{\zeta_4 P_4 \zeta_3 M_3} + \mathcal{O}(\alpha'^8)\Big) A^{\rm YM}$$

where entries of P_w, M_w are degree w polynomials in α' or s_{ij}, e.g.

$$P_2 = \begin{pmatrix} s_{12}s_{34} - s_{34}s_{45} - s_{51}s_{12} & s_{13}s_{24} \\ s_{12}s_{34} & s_{13}s_{24} - s_{24}s_{45} - s_{51}s_{13} \end{pmatrix}$$

$$M_3 = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} ,$$

$$m_{11} = s_{34}s_{45}(s_{34}+s_{45}) + s_{51}s_{12}(s_{51}+s_{12}) - s_{12}s_{34}(s_{12}+s_{34}) - 2s_{12}s_{23}s_{34}$$

$$m_{12} = s_{13}s_{24}(s_{12}+s_{23}+s_{34}+s_{45}+s_{51})$$

$$m_{21} = m_{12}\Big|_{2\leftrightarrow 3} , \qquad m_{22} = m_{11}\Big|_{2\leftrightarrow 3}$$

Promising pattern up to weight $w \leq 7$ ...

$$\mathcal{A}(\alpha')\Big|_{N=5} = \Big(1 + \zeta_2 P_2 + \zeta_3 M_3 + \zeta_4 P_4 + \zeta_5 M_5 + \zeta_2 P_2 \zeta_3 M_3$$
$$+ \zeta_6 P_6 + \tfrac{1}{2}\zeta_3^2 M_3^2 + \zeta_7 M_7 + \zeta_2 P_2 \zeta_5 M_5 + \zeta_4 P_4 \zeta_3 M_3 + \mathcal{O}(\alpha'^8)\Big) A^{\mathrm{YM}}$$

... in contrast to higher weights with depth ≥ 2 MZVs. E.g. at $w = 11$:

$$\mathcal{A}\Big|_{\substack{w=11\\N=5}} = \Big(\zeta_{11} M_{11} + \tfrac{1}{5}\zeta_{3,3,5}[M_3,[M_5,M_3]] + \tfrac{1}{5}\zeta_{3,5}\zeta_3 [M_5,M_3]M_3 + \zeta_3 \zeta_2^4 P_8 M_3$$
$$+ \tfrac{1}{2}\zeta_3^2 \zeta_5 M_5 M_3^2 + \tfrac{1}{6}\zeta_3^3 \zeta_2 P_2 M_3^3 + \zeta_9 \zeta_2 \big(P_2 M_9 + 9[M_3,[M_5,M_3]]\big)$$
$$+ \zeta_7 \zeta_2^2 \big(P_4 M_7 + \tfrac{6}{25}[M_3,[M_5,M_3]]\big) + \zeta_5 \zeta_2^3 \big(P_6 M_5 - \tfrac{4}{35}[M_3,[M_5,M_3]]\big)\Big) A^{\mathrm{YM}}$$

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- ugly coefficients $\frac{1}{5}$, 9, $\frac{6}{25}$, $\frac{4}{35} \in \mathbb{Q}$ along with MZV's & M_i commutators
- preferred depth $r \geq 2$ element in the MZV bases $\mathcal{Z}_{w \geq 8}$ unclear, e.g.

$$\zeta_{3,5} = \zeta_3 \zeta_5 - \zeta_8 - \zeta_{5,3} = -\frac{5}{2} \zeta_{2,6} - \frac{21}{25} \zeta_2^4 + 5 \zeta_3 \zeta_5$$

III. Motivic MZVs

Lift the true MZVs $\in \mathbb{R}$ to motivic MZVs $\zeta^{\mathfrak{m}}_{n_1,\ldots n_r}$ which are defined purely algebraically, form a Hopf algebra and satisfy relations of $\zeta_{n_1,\ldots n_r} \in \mathbb{R}$:

- eliminate the ambiguity of MZV basis choice
- automatically build in $\mathbb{Q}$ relations among MZVs

[Brown 1102.1310]

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[Brown 1102.1310]

Can map them to another Hopf algebra with more transparent basis:

$$\mathcal{U} \; = \; \underbrace{\mathbb{Q}\langle f_3, f_5, f_7, \ldots \rangle}_{\text{Hopf algebra on cogenerators } f_{odd}} \; \otimes_{\mathbb{Q}} \; \underbrace{\mathbb{Q}[f_2]}_{\text{commutes with } f_{odd}}$$

$$= \; \text{span}\{ \, (\text{non-commutative polynomials in } f_{odd}) \; \otimes \; f_2^k, \; k \in \mathbb{N}_0 \, \}$$

$\Longrightarrow$ reproduces recursion $d_w = d_{w-2} + d_{w-3}$ for conjectural $\dim_{\mathbb{Q}} \mathcal{Z}_w$

Need isomorphism $\phi : \{\zeta^{\mathfrak{m}}_{n_1,\ldots n_r}\} \longrightarrow \mathcal{U}$ preserving Hopf algebra structures.

- on single zetas: pick normalization $\phi(\zeta^{\mathfrak{m}}_n) = f_n$ where $f_{2k} = \frac{\zeta_{2k}}{(\zeta_2)^k}(f_2)^k$
- on depth ≥ 2 MZVs: determined by the coproduct, see **[Brown 1102.1310]**
- on products: $\phi(\zeta^{\mathfrak{m}}_{n_1,\ldots n_r} \cdot \zeta^{\mathfrak{m}}_{p_1,\ldots p_s}) = \phi(\zeta^{\mathfrak{m}}_{n_1,\ldots n_r}) \sqcup\!\sqcup\, \phi(\zeta^{\mathfrak{m}}_{p_1,\ldots p_s})$ where

$$f_2^p\, f_{i_1} \ldots f_{i_r} \sqcup\!\sqcup\, f_2^q\, f_{i_{r+1}} \ldots f_{i_s} = f_2^{p+q} \sum_{\sigma \in \Sigma(r,s)} f_{i_{\sigma(1)}} \ldots f_{i_{\sigma(r+s)}} , \qquad i_j \in 2\mathbb{N}+1$$

shuffle product $\sqcup\!\sqcup$ preserves relative order in $\{f_{i_1} \ldots f_{i_r}\}$ and $\{f_{i_{r+1}} \ldots f_{i_s}\}$.

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Examples at weights $w = 8$ and $w = 11$:

$$\phi(\zeta^{\mathfrak{m}}_{3,5}) = -5\, f_5\, f_3 \, , \qquad \phi(\zeta^{\mathfrak{m}}_3\, \zeta^{\mathfrak{m}}_5) = f_3\, f_5 + f_5\, f_3$$

$$\phi(\zeta^{\mathfrak{m}}_{3,3,5}) = -5\, f_5\, f_3^2 + \frac{4}{7} f_5\, f_2^3 - \frac{6}{5} f_7\, f_2^2 - 45\, f_9\, f_2$$

$$\phi(\zeta^{\mathfrak{m}}_{3,5}\zeta^{\mathfrak{m}}_3) = -5\, f_5\, f_3 \sqcup\!\sqcup\, f_3 = -5\, f_3\, f_5\, f_3 - 10\, f_5\, f_3^2$$

IV. Second look at disk amplitude: motivic structure

To simplify weight $w = 11$, pass to motivic MZVs $\zeta_{n_1,\ldots n_r} \mapsto \zeta^{\mathfrak{m}}_{n_1,\ldots n_r}$ ...

$$\mathcal{A}^{\mathfrak{m}}\Big|_{\substack{w=11\\N=5}} = \Big(\zeta^{\mathfrak{m}}_{11}\, M_{11} + \tfrac{1}{5}\,\zeta^{\mathfrak{m}}_{3,3,5}\,[M_3,[M_5,M_3]] + \tfrac{1}{5}\,\zeta^{\mathfrak{m}}_{3,5}\,\zeta^{\mathfrak{m}}_3\,[M_5,M_3]M_3 + \zeta^{\mathfrak{m}}_3\,(\zeta^{\mathfrak{m}}_2)^4\,P_8M_3$$
$$+ \tfrac{1}{2}\,(\zeta^{\mathfrak{m}}_3)^2\,\zeta^{\mathfrak{m}}_5\,M_5M_3^2 + \tfrac{1}{6}\,(\zeta^{\mathfrak{m}}_3)^3\,\zeta^{\mathfrak{m}}_2\,P_2M_3^3 + \zeta^{\mathfrak{m}}_9\,\zeta^{\mathfrak{m}}_2\,\big(P_2M_9 + 9\,[M_3,[M_5,M_3]]\big)$$
$$+ \zeta^{\mathfrak{m}}_7\,(\zeta^{\mathfrak{m}}_2)^2\,\big(P_4M_7 + \tfrac{6}{25}\,[M_3,[M_5,M_3]]\big) + \zeta^{\mathfrak{m}}_5\,(\zeta^{\mathfrak{m}}_2)^3\,\big(P_6M_5 - \tfrac{4}{35}\,[M_3,[M_5,M_3]]\big)\Big)\,A^{\rm YM}$$

... and apply isomorphism $\phi : \{\zeta^{\mathfrak{m}}_{n_1,\ldots n_r}\} \rightarrow \mathcal{U}$

$$\phi(\mathcal{A}^{\mathfrak{m}})\Big|_{\substack{w=11\\N=5}} = \Big(f_{11}\, M_{11} + f_3^2 f_5\, M_5M_3^2 + f_3f_5f_3\, M_3M_5M_3 + f_5f_3^2\, M_3^2M_5$$
$$+ P_2\, f_2\,(f_9\,M_9 + f_3^3\,M_3^3) + P_4\,f_2^2\,f_7\,M_7 + P_6\,f_2^3\,f_5\,M_5 + P_8\,f_2^4\,f_3\,M_3\Big)\,A^{\rm YM}$$

- unit coefficients everywhere (instead of the $\frac{1}{5}$, 6, $\frac{6}{25}$, $\frac{4}{35}$ above)

- democratic treatment of (f_3, M_3) and (f_5, M_5)

To summarize everything we know up to weight $w = 11$:

$$\phi(\mathcal{A}^{\mathfrak{m}})\Big|_{N=5} = \Big(1 + f_2P_2 + f_2^2P_4 + f_2^3P_6 + f_2^4P_8 + f_2^5P_{10}\Big)$$
$$\times \Big\{1 + f_3M_3 + f_5M_5 + f_3^2M_3^2 + f_7M_7 + f_3f_5M_5M_3 + f_5f_3M_3M_5$$
$$+ f_9M_9 + f_3^3M_3^3 + f_5^2M_5^2 + f_3f_7M_7M_3 + f_7f_3M_3M_7 + f_{11}M_{11}$$
$$+ f_3^2f_5M_5M_3^2 + f_3f_5f_3M_3M_5M_3 + f_5f_3^2M_3^2M_5\Big\} A^{\rm YM} + \mathcal{O}(\alpha'^{12})$$

The f_2 powers with matrices P_{2k} act by left multiplication on

all non-commutative words in $\overrightarrow{f_{odd}}\,\overleftarrow{M_{odd}}$.

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$$\times \Big\{1 + f_3 M_3 + f_5 M_5 + f_3^2 M_3^2 + f_7 M_7 + f_3 f_5 M_5 M_3 + f_5 f_3 M_3 M_5$$
$$+ f_9 M_9 + f_3^3 M_3^3 + f_5^2 M_5^2 + f_3 f_7 M_7 M_3 + f_7 f_3 M_3 M_7 + f_{11} M_{11}$$
$$+ f_3^2 f_5 M_5 M_3^2 + f_3 f_5 f_3 M_3 M_5 M_3 + f_5 f_3^2 M_3^2 M_5 \Big\} A^{\mathrm{YM}} + \mathcal{O}(\alpha'^{12})$$

The f_2 powers with matrices P_{2k} act by left multiplication on

all non-commutative words in $\overrightarrow{f_{odd}}\,\overleftarrow{M_{odd}}$. Tempting to generalize:

$$\boxed{\phi(\mathcal{A}^{\mathfrak{m}}) = \Big(\sum_{k=0}^{\infty} f_2^k P_{2k}\Big)\Big\{\sum_{p=0}^{\infty} \sum_{\substack{i_1,\ldots,i_p \\ \in 2\mathbb{N}+1}} f_{i_1} f_{i_2} \ldots f_{i_p} M_{i_p} \ldots M_{i_2} M_{i_1}\Big\} A^{\mathrm{YM}}}$$

Explicit checks done up to $w \leq 16$ for $N = 5$ and $w \leq 8$ for $N = 6$.

V. Main result

α' expansion of N-point disk amplitude: [OS, Stieberger 1205.1516]

$$\phi(\mathcal{A}^{\mathfrak{m}}) = \left(\sum_{k=0}^{\infty} f_2^k P_{2k} \right) \left\{ \sum_{p=0}^{\infty} \sum_{\substack{i_1,\ldots,i_p \\ \in 2\mathbb{N}+1}} f_{i_1} f_{i_2} \ldots f_{i_p} M_{i_p} \ldots M_{i_2} M_{i_1} \right\} A^{\mathrm{YM}}$$

- "even sector": commutative element $f_2 = \phi(\zeta_2^{\mathfrak{m}})$ with matrices P_{2k}
- "odd sector": democratic sum over all non-commutative polynomials

$f_{i_1} f_{i_2} \ldots f_{i_p} M_{i_p} \ldots M_{i_2} M_{i_1} \equiv \overrightarrow{f_{odd}} \overleftarrow{M_{odd}}$ with unit coefficients

V. Main result

α' expansion of N-point disk amplitude: **[OS, Stieberger 1205.1516]**

$$\phi(\mathcal{A}^{\mathfrak{m}}) \ = \ \left(\sum_{k=0}^{\infty} f_2^k P_{2k} \right) \left\{ \sum_{p=0}^{\infty} \sum_{\substack{i_1,\ldots,i_p \\ \in 2\mathbb{N}+1}} f_{i_1} f_{i_2} \ldots f_{i_p} M_{i_p} \ldots M_{i_2} M_{i_1} \right\} A^{\mathrm{YM}}$$

- "even sector": commutative element $f_2 = \phi(\zeta_2^{\mathfrak{m}})$ with matrices P_{2k}
- "odd sector": democratic sum over all non-commutative polynomials

  $f_{i_1} f_{i_2} \ldots f_{i_p} M_{i_p} \ldots M_{i_2} M_{i_1} \equiv \overrightarrow{f_{odd}} \, \overleftarrow{M_{odd}}$ with unit coefficients
- at each weight w, only one matrix P_w, $M_w \sim s_{ij}^w$ to determine
- form of $\phi(\mathcal{A}^{\mathfrak{m}})$ independent on basis choice for $\mathcal{Z}_w$
- The isomorphism ϕ is invertible, so $\phi(\mathcal{A}^{\mathfrak{m}})$ carries complete information

Concluding remarks

- Complete superstring disk amplitudes follows via matrix-vector-product from $(N-3)!$ YM subamplitudes: $\mathcal{A}_\sigma(\alpha') = \sum_{\pi \in S_{N-3}} F_\sigma{}^\pi(\alpha') \mathcal{A}_\pi^{\rm YM}$
- α' expansion of $F(\alpha')$ involves MZVs of weight w at order α'^w.
- The structure of $F(\alpha')$ greatly simplifies when lifting $\zeta_{n_1,\ldots,n_r}$ to motivic version $\zeta^{\mathfrak{m}}_{n_1,\ldots,n_r}$ and making use of their Hopf algebra structure: Isomorphism $\phi : \{\zeta^{\mathfrak{m}}_{n_1,\ldots,n_r}\} \to \{\prod_j f_{i_j}\}$ allows for all-weight-formula.

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Thank you for your attention !

Since the ϕ map is invertible, $\phi(\mathcal{A}^{\mathfrak{m}})$ contains all information on $\mathcal{A}^{\mathfrak{m}}$, e.g.

$$\phi\begin{pmatrix} (\zeta_2^{\mathfrak{m}})^4 \\ \zeta_2^{\mathfrak{m}}\,(\zeta_3^{\mathfrak{m}})^2 \\ \zeta_3^{\mathfrak{m}}\,\zeta_5^{\mathfrak{m}} \\ \zeta_{3,5}^{\mathfrak{m}} \end{pmatrix} = {\color{red}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -5 \end{pmatrix}} \begin{pmatrix} f_2^4 \\ f_2\,f_3^2 \\ f_3\,f_5 \\ f_5\,f_3 \end{pmatrix}$$

$$\phi\begin{pmatrix} \zeta_{11}^{\mathfrak{m}} \\ \zeta_{3,3,5}^{\mathfrak{m}} \\ \zeta_{3,5}^{\mathfrak{m}}\,\zeta_3^{\mathfrak{m}} \\ (\zeta_3^{\mathfrak{m}})^2\,\zeta_5^{\mathfrak{m}} \\ \zeta_3^{\mathfrak{m}}\,(\zeta_2^{\mathfrak{m}})^4 \\ (\zeta_3^{\mathfrak{m}})^3\,\zeta_2^{\mathfrak{m}} \\ \zeta_9^{\mathfrak{m}}\,\zeta_2^{\mathfrak{m}} \\ \zeta_7^{\mathfrak{m}}\,(\zeta_2^{\mathfrak{m}})^2 \\ \zeta_5^{\mathfrak{m}}\,(\zeta_2^{\mathfrak{m}})^3 \end{pmatrix} = {\color{blue}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 & 0 & 0 & -45 & -\frac{6}{5} & \frac{4}{7} \\ 0 & -10 & -5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} f_{11} \\ f_5\,f_3^2 \\ f_3\,f_5\,f_3 \\ f_3^2\,f_5 \\ f_3\,f_2^4 \\ f_3^3\,f_2 \\ f_9\,f_2 \\ f_7\,f_2^2 \\ f_5\,f_2^3 \end{pmatrix}$$

The shown ${\color{red}d_8 \times d_8}$ and ${\color{blue}d_{11} \times d_{11}}$ matrices are non-singular!

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$$\phi\begin{pmatrix} (\zeta_2^{\mathfrak{m}})^4 \\ \zeta_2^{\mathfrak{m}}\,(\zeta_3^{\mathfrak{m}})^2 \\ \zeta_3^{\mathfrak{m}}\,\zeta_5^{\mathfrak{m}} \\ \zeta_{3,5}^{\mathfrak{m}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ X & 0 & 0 & -5 \end{pmatrix} \begin{pmatrix} f_2^4 \\ f_2\,f_3^2 \\ f_3\,f_5 \\ f_5\,f_3 \end{pmatrix}$$

$$\phi\begin{pmatrix} \zeta_{11}^{\mathfrak{m}} \\ \zeta_{3,3,5}^{\mathfrak{m}} \\ \zeta_{3,5}^{\mathfrak{m}}\,\zeta_3^{\mathfrak{m}} \\ (\zeta_3^{\mathfrak{m}})^2\,\zeta_5^{\mathfrak{m}} \\ \zeta_3^{\mathfrak{m}}\,(\zeta_2^{\mathfrak{m}})^4 \\ (\zeta_3^{\mathfrak{m}})^3\,\zeta_2^{\mathfrak{m}} \\ \zeta_9^{\mathfrak{m}}\,\zeta_2^{\mathfrak{m}} \\ \zeta_7^{\mathfrak{m}}\,(\zeta_2^{\mathfrak{m}})^2 \\ \zeta_5^{\mathfrak{m}}\,(\zeta_2^{\mathfrak{m}})^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ Y & -5 & 0 & 0 & 0 & 0 & -45 & -\frac{6}{5} & \frac{4}{7} \\ 0 & -10 & -5 & 0 & X & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_{11} \\ f_5\,f_3^2 \\ f_3\,f_5\,f_3 \\ f_3^2\,f_5 \\ f_3\,f_2^4 \\ f_3^3\,f_2 \\ f_9\,f_2 \\ f_7\,f_2^2 \\ f_5\,f_2^3 \end{pmatrix}$$

The shown $d_8 \times d_8$ and $d_{11} \times d_{11}$ matrices are non-singular $\forall\ X, Y$

However: Coproduct cannot detect f_w in $\phi(\zeta^{\mathfrak{m}}_{n_1,\ldots,n_r})$ with $r \geq 2$

$\Longrightarrow$ ambiguity in P_w or M_w $\sim$ commutators of lower weight M_{odd}.