Towards a metric-like higher-spin gauge theory in 2+1 dimensions

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Why 2+1 dimensions?

Gravity in 2+1 dimensions as toy model:

- No propagating degrees of freedom

$$R_{\mu\nu\lambda\sigma} = C_{\mu\nu\lambda\sigma} + \frac{2}{d-2}(g_{\mu[\lambda}R_{\sigma]\nu} - g_{\nu[\lambda}R_{\sigma]\mu})$$

$$\equiv 0 \qquad - \frac{2}{(d-1)(d-2)} R\, g_{\mu[\lambda}g_{\sigma]\nu}$$

- Black holes exist for $\Lambda < 0$ [Bañados, Teitelboim, Zanelli]

Black hole entropy can be studied.

AdS/CFT concepts can be tested.

Asymptotic symmetries: $\text{Virasoro} \oplus \text{Virasoro}$

central charge $c = \frac{3\ell}{2G}$ (ℓ → AdS radius)

[Brown,Henneaux]

Higher spins

Higher spin extensions of gravity simpler in 2+1 than in higher dimensions:
no propagating degrees of freedom

- asymptotic symmetries: [Henneaux,Rey] [Campoleoni,S.F., Pfenninger,Theisen]

  $$\text{Virasoro} \longrightarrow \mathcal{W}_N, \mathcal{W}_\infty$$

- generalisations of black holes

  [Gutperle,Kraus] [Ammon,Gutperle,Kraus,Perlmutter]

- AdS/CFT proposal: duality between HS gravity and minimal models [Gaberdiel, Gopakumar]

Frame-like vs. Metric-like

Most of this work is based on a frame-like formulation:

- Generalisation of $SO(2,2)$ Chern-Simons description of 2+1 gravity with $\Lambda < 0$

A metric-like formulation would be nice:

- better understand geometry of BHs
- use standard field theoretic tools
- study relation between frame-like and metric-like formulation

Content

The relation between frame- and metric-like formulation of a spin 3 field coupled to 2+1-gravity

- Review of pure gravity in 2+1 dim.
- Higher spin gauge theories in 2+1 dim.
- From frame-like to metric-like
- Applications and developments

2+1 gravity

Einstein-Hilbert action $S_{\rm EH}[g] = \frac{1}{16\pi G}\int d^3x\,\sqrt{g}\,R$

Introduce vielbeine e^a_μ with $e^a_\mu\, e^b_\nu\, \kappa_{ab} = g_{\mu\nu}$,

and spin connections $\omega^{ab}_\mu \longrightarrow \omega^a_\mu = f^a{}_{bc}\,\omega^{bc}_\mu$.

$$S[e,\omega] = \frac{1}{8\pi G}\int \mathrm{tr}(e\wedge R) \quad \text{with} \quad R = d\omega + \omega\wedge\omega$$

reproduces the Einstein-Hilbert action

for $de + \omega\wedge e + e\wedge\omega = 0$ (e.o.m. for ω).

Diffeomorphisms

Gauge invariance:

$$\delta e = [e, \Lambda]$$
$$\delta \omega = d\Lambda + [\omega, \Lambda]$$

Local Lorentz rotation

$$\delta e = d\xi + [\omega, \xi]$$
$$\delta \omega = 0$$

Local translation

For $\xi^a = e^a_\mu v^\mu$ we have

$$\delta e^a_\mu = e^a_\lambda \partial_\mu v^\lambda + v^\lambda \partial_\lambda e^a_\mu \quad \longleftarrow \text{Diffeo}$$
$$+ v^\lambda (\partial_{[\mu} e^a_{\lambda]} + f^a{}_{bc}\, \omega^b_{[\mu}\, e^c_{\lambda]}) \quad \longleftarrow 0 \text{ by e.o.m.}$$
$$+ f^a{}_{bc}\, v^\lambda\, \omega^b_\lambda\, e^c_\mu \quad \longleftarrow \text{Lorentz}$$

Note: Never used that Lie algebra was so(2,1).

Higher Spins

Two approaches:

- Second order/Metric-like

  $g_{\mu\nu} \longrightarrow$ tensor field $\varphi_{\mu_1 \ldots \mu_s}$

- First order/Frame-like

  $e^a_\mu \longrightarrow$ generalised Vielbein $e^{a_1 \ldots a_{s-1}}_\mu$

Metric-like

Free massless spin s particle [Fronsdal]

$\longrightarrow$ fully symmetric tensor $\varphi_{\mu_1 \ldots \mu_s}$

- $\varphi_{\mu_1 \ldots \mu_{s-4} \lambda}{}^{\lambda}{}_{\nu}{}^{\nu} = 0$ (double traceless)
- $$\Box \varphi_{\mu_1 \ldots \mu_s} - \partial_{(\mu_1|} \partial^{\lambda} \varphi_{|\mu_2 \ldots \mu_s)\lambda}{}^{\lambda} + \partial_{(\mu_1} \partial_{\mu_2} \varphi_{\mu_3 \ldots \mu_s)\lambda}{}^{\lambda} = 0$$ (e.o.m.)
- $\delta \varphi_{\mu_1 \ldots \mu_s} = \partial_{(\mu_1} \xi_{\mu_2 \ldots \mu_s)}$ (gauge invariance)
  where $\xi_{\mu_1 \ldots \mu_{s-3} \lambda}{}^{\lambda} = 0$

Interactions: add term by term while retaining gauge invariance.

[Bengtsson,Bengtsson,Brink]
[Metsaev]
[Manvelyan,Mkrtchyan,Rühl]
[Sagnotti,Taronna]
[Joung,Lopez,Taronna]

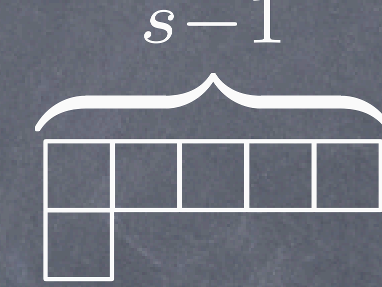
Frame-like

[Vasiliev]

Generalised vielbein $e_\mu^{a_1 \dots a_{s-1}}$ such that

$$\varphi_{\mu_1 \dots \mu_s} = \bar{e}^{a_1}_{(\mu_1} \dots \bar{e}^{a_{s-1}}_{\mu_{s-1}} \, e^{b_1 \dots b_{s-1}}_{\mu_s)} \, \kappa_{a_1 b_1} \dots \kappa_{a_{s-1} b_{s-1}}$$

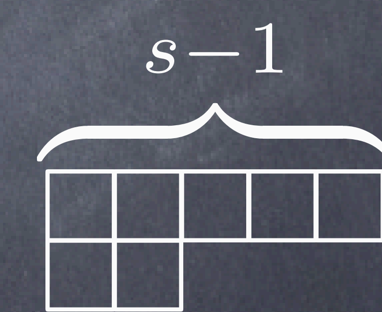
↖ background vielbein



Gauge transformations

$$\delta e_\mu^{a_1 \dots a_{s-1}} = \bar{D}_\mu \xi^{a_1 \dots a_{s-1}} + \bar{e}_{\mu, b} \, \Lambda^{b, a_1 \dots a_{s-1}}$$

with $\xi^{a_1 \dots a_{s-1}} = \xi^{\mu_1 \dots \mu_{s-1}} \, \bar{e}^{a_1}_{\mu_1} \dots \bar{e}^{a_{s-1}}_{\mu_{s-1}}$.



Spin connections

$$\delta \omega_\mu^{b, a_1 \dots a_{s-1}} = \bar{D}_\mu \Lambda^{b, a_1 \dots a_{s-1}} + \bar{e}_{\mu, c} \, \Lambda^{bc, a_1 \dots a_{s-1}}$$

~~$\delta \omega_\mu^{bc, a_1 \dots a_{s-1}} = \bar{D}_\mu \Lambda^{bc, a_1 \dots a_{s-1}} + \bar{e}_{\mu, d} \, \Lambda^{bcd, a_1 \dots a_{s-1}}$~~

$\vdots$

vanish in 3 dimensions !

Action

Dualise: $\omega_\mu^{a_1 \dots a_{s-1}} = \omega_\mu^{b,c(a_2 \dots a_{s-1}} f^{a_1)}{}_{bc}$

$$S = \frac{1}{8\pi G} \int \mathrm{tr}(e \wedge (d\omega + \omega \wedge \omega)) \qquad ?$$

$$\begin{aligned} e &= (e_\mu^a J_a + e_\mu^{a_1 a_2} J_{a_1 a_2} + \dots)dx^\mu \\ \omega &= (\omega_\mu^a J_a + \omega_\mu^{a_1 a_2} J_{a_1 a_2} + \dots)dx^\mu \end{aligned}$$

Lie algebra structure e.g. for spin 3:

$$\begin{aligned} [J_a, J_b] &= f_{ab}{}^c J_c \\ [J_a, J_{bc}] &= f^d{}_{a(b} J_{c)d} \\ [J_{ab}, J_{cd}] &= -(\kappa_{a(c} f_{d)v}{}^f + \kappa_{b(c} f_{d)a}{}^f) J_f \end{aligned}$$

sl(3)

From frame- to metric-like

Can we translate these consistent interacting theories to the metric-like formulation?

First step: Identify the fields [Campoleoni,S.F., Pfenninger,Theisen]

$$g_{\mu\nu} = \bar{e}^a_\mu \, \bar{e}^b_\nu \, \kappa_{ab} + \dots \qquad \varphi_{\mu\nu\rho} = e^{ab}_{(\mu} \, \bar{e}^c_\nu \, \bar{e}^d_{\rho)} \, \kappa_{ac} \, \kappa_{bd} + \dots$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$g_{\mu\nu} = e^{\mathcal{A}}_\mu \, e^{\mathcal{B}}_\nu \, \kappa_{\mathcal{AB}} \qquad \varphi_{\mu\nu\rho} = e^{\mathcal{A}}_\mu \, e^{\mathcal{B}}_\nu \, e^{\mathcal{C}}_\rho \, d_{\mathcal{ABC}}$$

$$d_{\mathcal{ABC}} \sim \mathrm{tr}(t_{(\mathcal{A}} \, t_{\mathcal{B}} \, t_{\mathcal{C})})$$

Next: solve for the spin connection and express everything in metric-like fields

Solving for the spin connection

For gravity, the invertibility of the vielbein was crucial to solve for ω_μ^a .

Here, one can solve for it perturbatively:

$$e_\mu^A \to \begin{pmatrix} e_\mu^a \\ e_\mu^A \end{pmatrix}$$

a, b, c sl(2) directions

A, B, C non-sl(2) directions

Then expand in e_μ^A .

All higher spin fields contain at least one e_μ^A

$\longrightarrow$ Perturbative expansion in φ

Solving for the spin connection

$$\partial_{[\mu} e^{\mathcal{C}}_{\nu]} + f_{\mathcal{D}\mathcal{E}}{}^{\mathcal{C}}\, \omega^{\mathcal{D}}_{[\mu}\, e^{\mathcal{E}}_{\nu]} \;=\; 0$$

$$\longrightarrow \; \partial_{[\mu} e^{\mathcal{C}}_{\nu]} + f_{\mathcal{D}E}{}^{\mathcal{C}}\, \omega^{\mathcal{D}}_{[\mu}\, e^{E}_{\nu]} + f_{\mathcal{D}e}{}^{\mathcal{C}}\, \omega^{\mathcal{D}}_{[\mu} e^{e}_{\nu]} \;=\; 0$$

Result:

$$(C_2)^{\mathcal{D}}_{\mathcal{A}}\, \omega^{\mathcal{A}}_{\mu} \;=\; e^{b}_{\mu}\big(\kappa_{ab}\, \delta^{\mathcal{D}}_{\mathcal{C}} + f_{ab}{}^{e}\, f_{e\mathcal{C}}{}^{\mathcal{D}} + f_{a\mathcal{C}}{}^{\mathcal{E}}\, f_{\mathcal{E}b}{}^{\mathcal{D}}\big) V^{\mathcal{C},a}$$

$$(C_2)^{\mathcal{D}}_{\mathcal{A}} \;=\; f^{\mathcal{D}}{}_{m\mathcal{C}}\, f^{\mathcal{C}m}{}_{\mathcal{A}}$$ quadratic Casimir

$$V^{\mathcal{C},a} \;=\; -f_{mn}{}^{p}\, g_0^{\mu\rho}\, g_0^{\nu\sigma}\, e^{m}_{\rho}\, e^{n}_{\sigma}(\partial_{[\mu} e^{\mathcal{C}}_{\nu]} + f_{\mathcal{D}E}{}^{\mathcal{C}}\, \omega^{\mathcal{D}}_{[\mu}\, e^{E}_{\nu]})$$

$$g_{0,\mu\nu} \;=\; e^{a}_{\mu}\, e^{b}_{\nu}\, \kappa_{ab}$$ sl(2) part of metric

Gauge parameters

Diffeomorphisms:

$$\xi^{\mathcal{A}} = e^{\mathcal{A}}_{\mu}\,\xi^{\mu}$$

consistent

$$\xi^{\mu} = g^{\mu\nu}\,\bar{e}^{a}_{\nu}\,\xi^{b}\,\kappa_{ab} + \ldots$$

$$\xi^{\mu} = g^{\mu\nu}\,e^{\mathcal{A}}_{\nu}\,\xi^{\mathcal{B}}\,\kappa_{\mathcal{A}\mathcal{B}}$$

Projector on diffeomorphisms:

$$P^{\mathcal{A}}_{\mathcal{C}} = e^{\mathcal{A}}_{\mu}\,g^{\mu\nu}\,e^{\mathcal{B}}_{\nu}\,\kappa_{\mathcal{B}\mathcal{C}} \qquad P^{\mathcal{A}}_{\mathcal{B}}\,P^{\mathcal{B}}_{\mathcal{C}} = P^{\mathcal{A}}_{\mathcal{C}}$$

spin 3 transformations:

$$\xi^{ab} = \bar{e}^{a}_{\mu}\,\bar{e}^{b}_{\nu}\,\xi^{\mu\nu} + \ldots$$

$$\longrightarrow \quad \xi^{\mathcal{A}} = d^{\mathcal{A}}{}_{\mathcal{B}\mathcal{C}}\,e^{\mathcal{B}}_{\mu}\,e^{\mathcal{C}}_{\nu}\,\xi^{\mu\nu} + \ldots$$

$$\longrightarrow \quad \xi^{\mathcal{A}} = (\mathbf{1} - P)^{\mathcal{A}}_{\mathcal{A}'}\,d^{\mathcal{A}'}{}_{\mathcal{B}\mathcal{C}}\,e^{\mathcal{B}}_{\mu}\,e^{\mathcal{C}}_{\nu}\,\xi^{\mu\nu} + \ldots$$

Spin 3 gauge transformation

With this ansatz:

$$\delta\varphi_{\mu\nu\rho} \;=\; \nabla_{(\mu}\,\xi_{\nu\rho)} + \mathcal{O}(\varphi^2)$$

$$\begin{aligned}\delta g_{\mu\nu} \;=\; 6\xi^{\rho\sigma}\Big(&\nabla_\rho\varphi_{\mu\nu\sigma} - \nabla_{(\mu}\varphi_{\nu)\rho\sigma} + \frac{1}{2}g_{\rho\mu}\,g_{\sigma\nu}\,\nabla\cdot\varphi_\lambda{}^\lambda \\ &+ g_{\rho(\mu}\big(\nabla\cdot\varphi_{\nu)\sigma} - \nabla_{|\sigma|}\varphi_{\nu)\lambda}{}^\lambda - \nabla_{\nu)}\varphi_{\sigma\lambda}{}^\lambda\big) \\ &- g_{\mu\nu}\big(\nabla\cdot\varphi_{\rho\sigma} - 2\nabla_\rho\varphi_{\sigma\lambda}{}^\lambda\big)\Big) + \mathcal{O}(\varphi^3)\end{aligned}$$

Looks ugly, but only half of the possible terms appear!

Action

There is a unique action up to quartic terms that is invariant under these transformations:

$$\mathcal{L} = \sqrt{g}\Big(R + \frac{1}{2}\varphi^{\mu\nu\rho}\big(\mathcal{F}_{\mu\nu\rho} - \frac{1}{2}g_{(\mu\nu}\,\mathcal{F}_{\rho)\lambda}{}^{\lambda}\big) + \frac{9}{4}R_{\rho\sigma}\big(\varphi^{\rho}{}_{\mu\nu}\,\varphi^{\sigma\mu\nu} - \frac{1}{2}\varphi^{\rho}{}_{\lambda}{}^{\lambda}\varphi^{\sigma}{}_{\kappa}{}^{\kappa}\big) - \frac{3}{4}R\,\varphi_{\mu\nu\rho}\,\varphi^{\mu\nu\rho}\Big) + \mathcal{O}(\varphi^{4})$$

with $$\mathcal{F}_{\mu\nu\rho} = \Box\varphi_{\mu\nu\rho} + \nabla_{(\mu}\nabla_{\nu}\varphi_{\rho)\lambda}{}^{\lambda} - \frac{1}{2}\big(\nabla^{\lambda}\nabla_{(\mu}\varphi_{\nu\rho)\lambda} + \nabla_{(\mu}\nabla^{\lambda}\varphi_{\nu\rho)\lambda}\big)$$

Application: Black hole entropy

- $\Lambda > 0$: BTZ black holes in 2+1 gravity
- HS gravity: black holes with non-trivial higher spin charge
- Characterised by a specific holonomy of the frame-like fields
- Relations between temperature, mass, chemical pot. and charge
  $\longrightarrow$ Entropy
- In a gauge with a smooth horizon we can determine Wald's entropy from metric-like description
  $\longrightarrow$ Disagreement

[Gutperle,Kraus]
[Ammon,Gutperle, Kraus,Perlmutter]

Higher Spin Geometry?

What is the principle behind the action? We would expect

$$S = \frac{1}{16\pi G}\int d^3x\,\sqrt{g}\,\hat{R}$$

with some generalised $\hat{R}$.

Recent progress by [Fujisawa,Nakayama] :

Introduce more vielbeine $e^{\mathcal{A}}_{(\mu\nu)} \sim d^{\mathcal{A}}{}_{\mathcal{B}\mathcal{C}}\, e^{\mathcal{B}}_{\mu}\, e^{\mathcal{C}}_{\nu}$

to get an invertible vielbein matrix $\begin{pmatrix} e^{\mathcal{A}}_{\mu} & e^{\mathcal{A}}_{(\mu\nu)} \end{pmatrix}$

Tangent bundle is extended to higher dimensional vector bundle.

Summary

- 2+1 dim. good testing ground for higher spin gauge theories
- Hope for a complete interacting metric-like theory?
- We determined the quadratic part in φ for metric-like action corresponding to frame-like sl(3) theory
- Next order more complicated - need geometric insights?
- Application to black hole entropy