Form factors in N=4 SYM
from weak to strong coupling

Gang Yang

University of Hamburg

Based on works with Brandhuber, Gurdogan, Mooney, Spence, Travaglini

and work in progress at strong coupling with Zhiquan Gao

“Lessons from the first phase of the LHC” Sep 27, DESY Hamburg

Outline

- What are form factors?
- Form factors at weak coupling
  - surprising relation to Higgs amplitudes in QCD
- Form factors at strong coupling
- Outlook

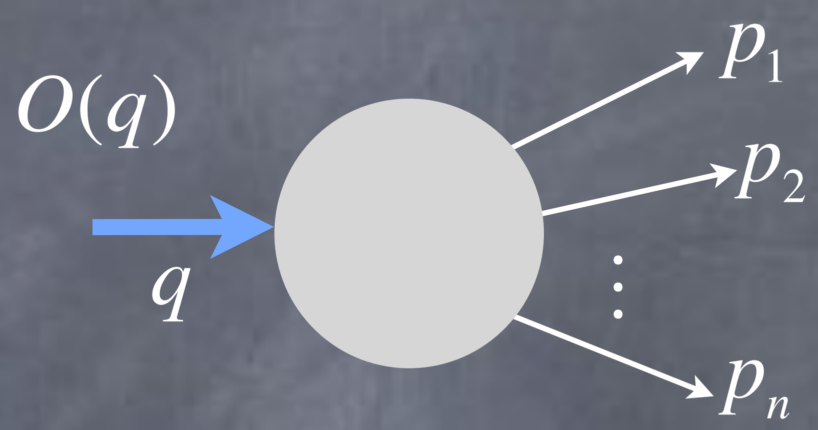
Motivation

- On-shell method to off-shell observables
- From N=4 SYM to QCD
- Strong coupling computations

What are form factors?

Partially on-shell, partially off-shell observables:

$$F = \int d^4x\, e^{iqx} \langle 0 | O(x) | \text{states} \rangle$$
$$= \langle 0 | O(q) | p_1 p_2 \cdots p_n \rangle$$



$$\boxed{q = \sum_i p_i, \;\; p_i^2 = 0, \; q^2 \neq 0}$$

Scattering amplitudes

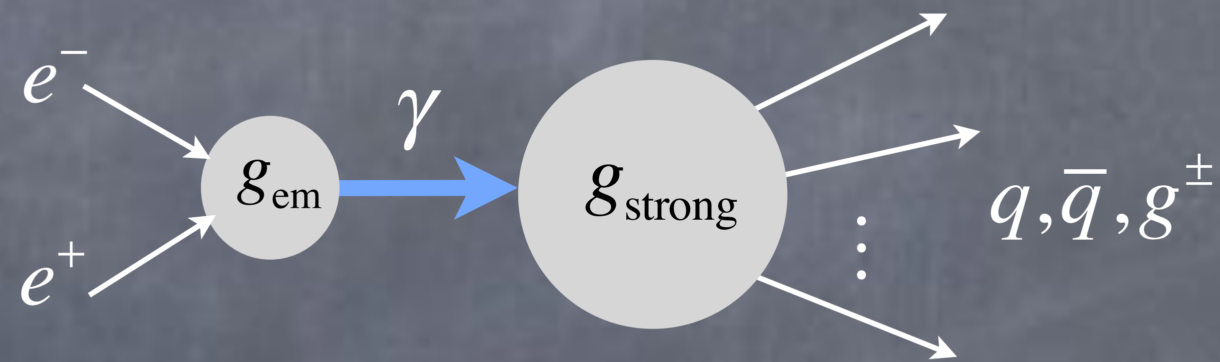
$\langle 0 | p_1 p_2 \cdots p_n \rangle$

Correlation functions

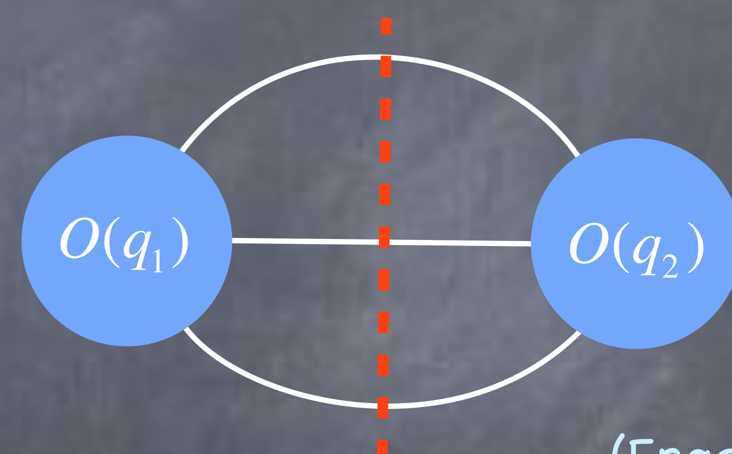
$\langle 0 | O(x_1) O(x_2) \cdots O(x_n) | 0 \rangle$

Examples

e+ e- -> hadrons

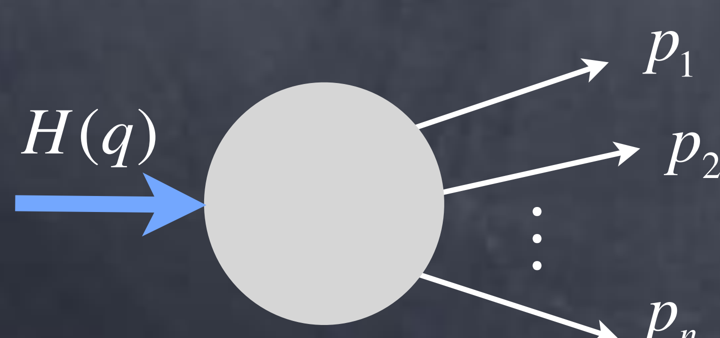


Block of correlation functions



(Engelund, Roiban)

Higgs-gluon amplitudes



$$\boxed{\pounds_{\text{eff}} = \lambda\, H\, \text{tr}(F^{\mu\nu} F_{\mu\nu})}$$

$$\longleftrightarrow \quad \langle 0 \,|\, \text{tr}(F^2)(q) \,|\, p_1 p_2 \cdots p_n \rangle$$

Outline

- What are form factors
- **Form factors at weak coupling**
  - surprising relation to Higgs amplitudes in QCD
- Form factors at strong coupling
- Outlook

(Brandhuber, Spence, Travaglini, GY)
(Brandhuber, Gurdogan, Mooney, Travaglini, GY)
(see also works by Bork, Kazakov, Vartanov)

Weak coupling

Parke-Taylor formula for MHV form factors

$$F_n(1^+,\cdots,i_\phi,\cdots j_\phi,\cdots n^+;\mathrm{tr}(\phi^2)(q)) = \delta^4(\Sigma_i p_i - q)\frac{\langle i\, j\rangle^2}{\langle 1\, 2\rangle\cdots\langle n\, 1\rangle}$$

Form factor/periodic Wilson line correspondence at one-loop

Super form factors

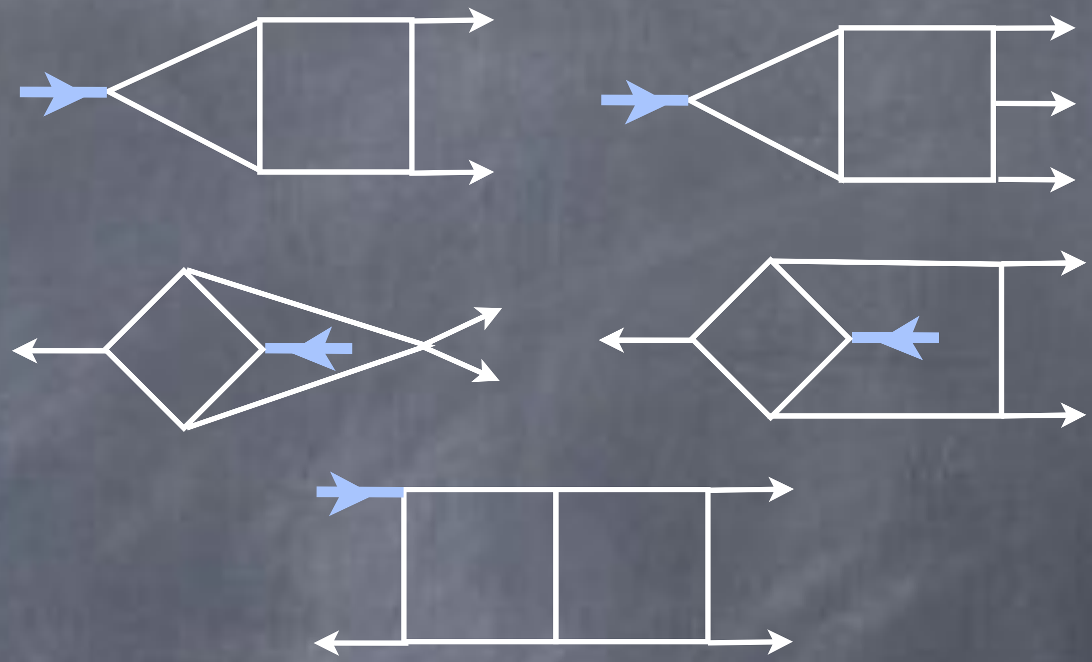
dual MHV rules

- unify not only super on-shell particles in N=4,
  but also the operators in the stress tensor supermultiplets

(Brandhuber, Travaglini, GY)

2-loop 3-pt form factor

Unitarity method



- in terms of simple basis integrals
- numerical results

"symbol" technique (Goncharov, Spradlin, Vergu Volovich)

- we obtain very compact analytic expression

$$R_3^{(2)} = -2\left[J_4\left(-\frac{uv}{w}\right)+J_4\left(-\frac{vw}{u}\right)+J_4\left(-\frac{wu}{v}\right)\right]-8\sum_i\left[Li_2\left(1-u_i^{-1}\right)+\frac{\log^4 u_i}{4!}\right]-2\left[\sum_i Li_2\left(1-u_i^{-1}\right)\right]^2-\frac{\log^4(uvw)}{4!}-\frac{23\zeta_4}{2}$$

$$u := \frac{s_{12}}{q^2}, \quad v := \frac{s_{23}}{q^2}, \quad w := \frac{s_{31}}{q^2} \qquad J_4(z) := Li_4(z)-\log(-z)Li_3(z)+\frac{\log^2(-z)}{2!}Li_2(z)-\frac{\log^3(-z)}{3!}Li_1(z)-\frac{\log^4(-z)}{48}$$

Surprising observation

(Brandhuber, Travaglini, GY)

2-loop 3-pt form factor in N=4 SYM	=	Leading transcendental piece of 2-loop Higgs amplitudes in QCD

$\langle 0 | \mathrm{tr}(F^2)(q) | g(p_1) g(p_2) g(p_3) \rangle$

$H(q) \rightarrow 3$ gluons

(Gehrmann, Jaquier, Glover, Koukoutsakis)

This implies some new hidden relations between QCD and N=4 SYM.
The knowledge of the N=4 may tell us more about QCD.

It is still not clear why.
Is this also true for higher points?
Is this also true for higher loops?
More data is needed.

(work in progress in N=4 side)

Motivation to study strong coupling

- a non-trivial part of strong coupling QCD amplitudes (at least for 3-point case? we need to go to AdS_5)
- helpful to guess the "right" variables at weak coupling (in particular for the computations with symbols)
- generalize on-shell method to off-shell observable at strong coupling (to correlation functions?)

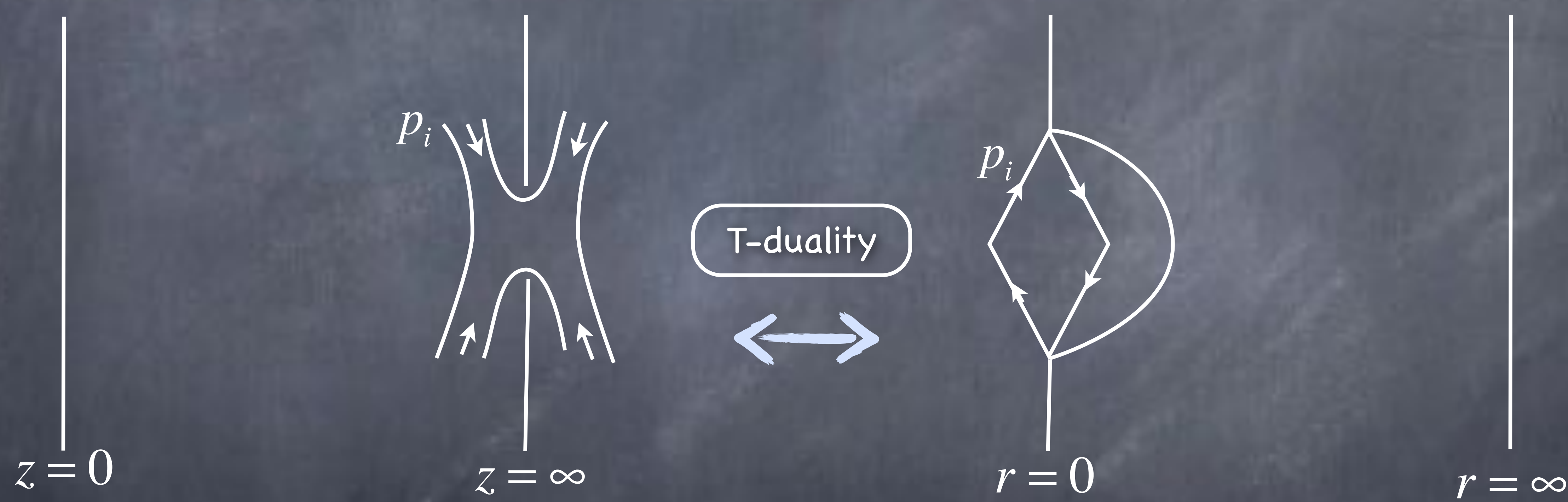
Outline

- What are form factors
- Form factors at weak coupling
  - surprising relation to Higgs amplitudes in QCD
- **Form factors at strong coupling**
- Outlook

(Alday, Maldacena)

Strong coupling picture

N=4 SYM ⟷ String in AdS_5



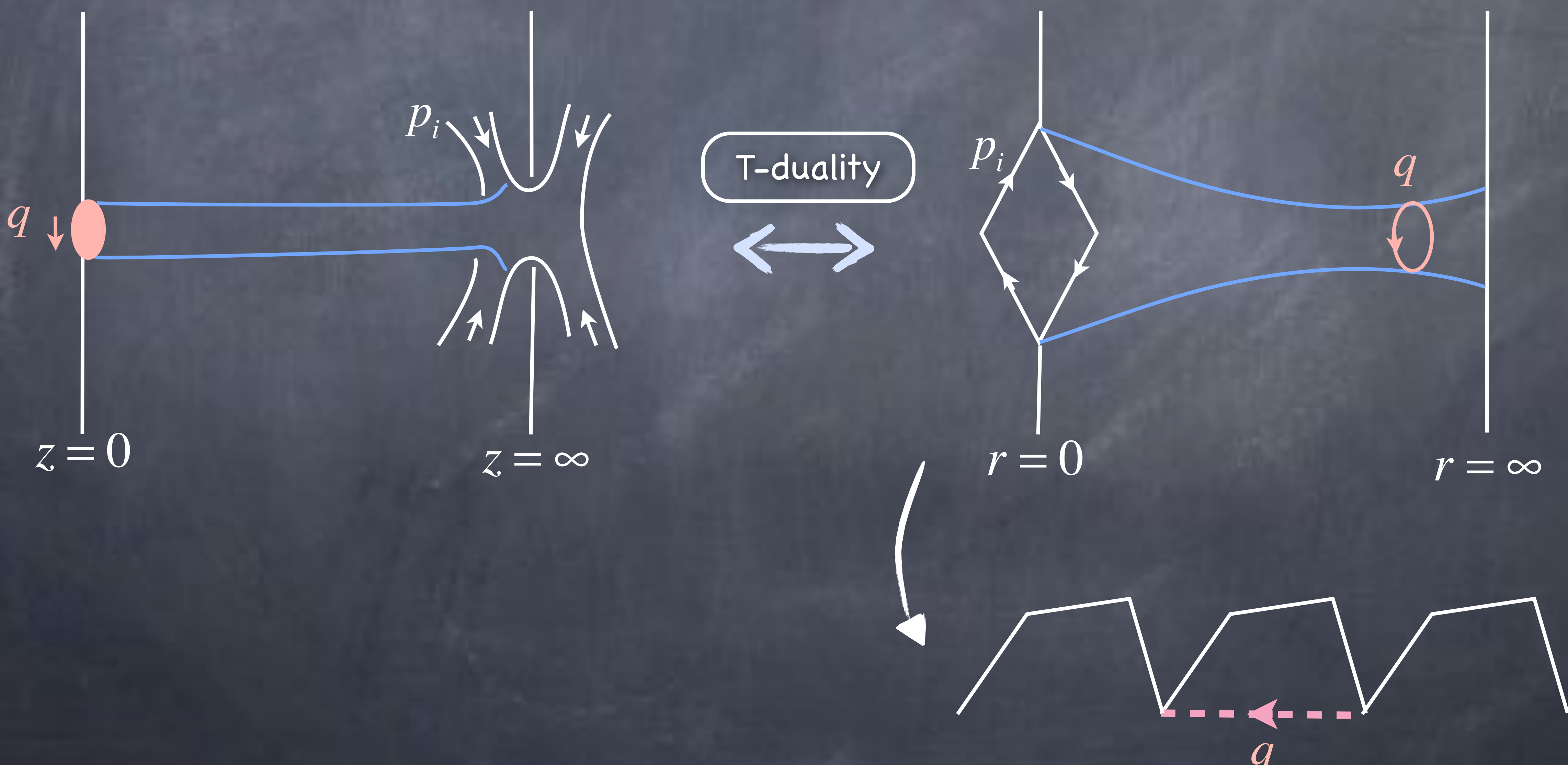
Amplitudes

Wilson loops

(Alday, Maldacena)
(Maldacena, Zhiboedov)

Strong coupling picture

N=4 SYM ⟷ String in AdS5



(see the series papers by Alday, Maldacena, Gaiotto, Sever, Vieira)

(Yang, n=4K case)

An Bird's-eye view

Amplitudes or Form factors

Area

Y system

AdS/CFT
T-duality

Integrability ζ

Small solutions

polynomial P(z)

Minimum surface problem (String equations)

Boundary conditions

Pohlmeyer reduction

Hitchin equations

$SU(4)$ with Z_4-projection

Small solutions

The underlying idea:
relate spacetime variables to the solutions on the worldsheet

Pohlmeyer reduction → we have new variables and new equations

$$\partial_z \psi = B_z \psi\,, \quad \partial_{\bar{z}} = B_{\bar{z}} \psi$$
$$\partial_z B_{\bar{z}} - \partial_{\bar{z}} B_z + [B_z, B_{\bar{z}}] = 0$$
$$\text{where} \;\; B := A + \Phi$$

$$X(z,\bar{z}) \leftrightarrow \psi(z,\bar{z})$$

$s_i(z,\bar{z})$

The most important variables are the small solutions:
for a given edge (cusp), there is a unique solution that decreases fastest when approaching that given edge.
On the other hand, spacetime variables $X(z,\bar{z})$ (which directly relate to boundary condition) are given as the coefficients of big solutions.

There is a dictionary between small solutions and spacetime variables.

$$X_i = \lambda_i \wedge \lambda_{i+1}\,, \quad X_i \cdot X_j = \langle \lambda_i, \lambda_{i+1}, \lambda_j, \lambda_{j+1} \rangle \rightarrow \langle s_i, s_{i+1}, s_j, s_{j+1} \rangle$$

Polynomial P(z)

$$P(z) = z^{n-4} + a_{n-2} z^{n-2} + \cdots + a_1 z + a_0$$

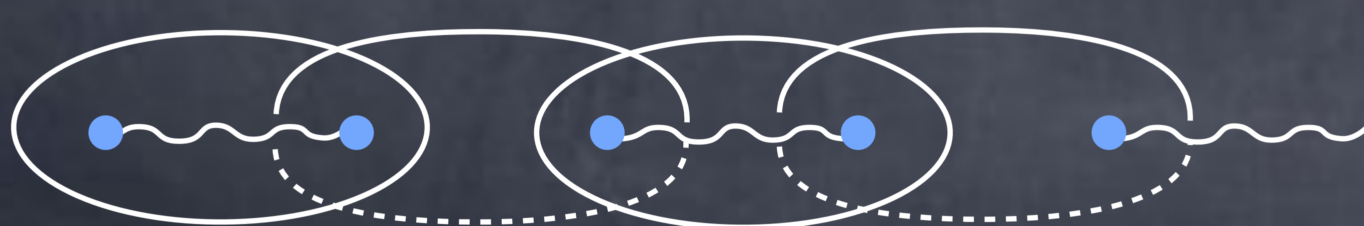
Information of boundary conditions (i.e. cross ratios) are (implicitly) given by the coefficients of the polynomial.

One may extract the information by computing the cycle integrals from the algebraic curve defined by:

$$\oint dw = \oint P(z)^{1/4}\, dz$$

$$x^4 = P(z)$$

For AdS3 case it is much simpler: $x^2 = p(z)$



For form factors, there will be not only zeros but poles!

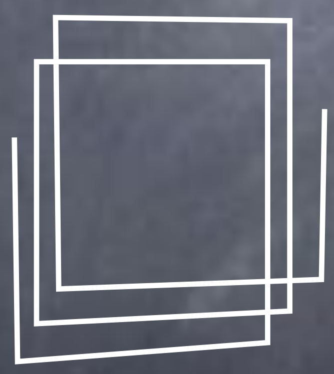
w-plane

The reason of introducing P(z): it is convenient to consider the small solution on w-plane, which is relation to z-plane by :

$$dw = P(z)^{1/4}\,dz, \quad P(z) = z^{n-4} + a_{n-2} z^{n-2} + \cdots + a_1 z + a_0$$

Small solutions have simpler structure on w-plane (since equations are simplified):

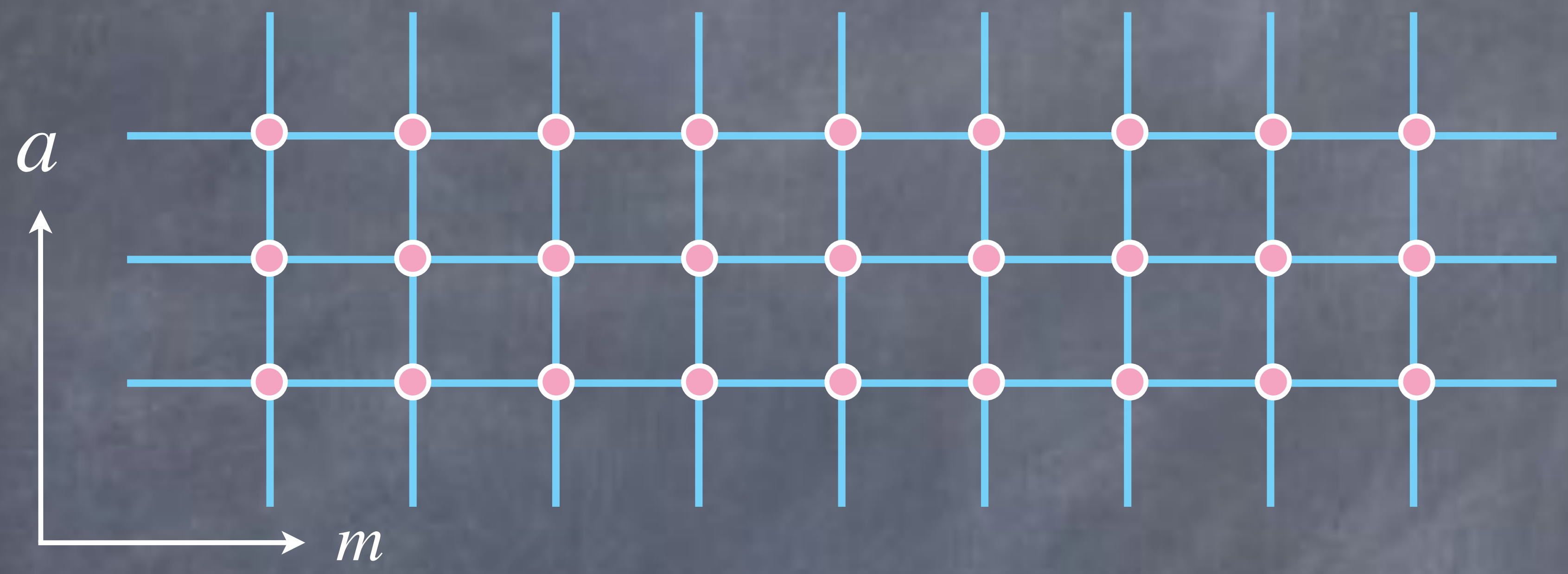
A single cover of z-plane



n/4 covers of w-plane,
each cover contains 4-cusp

This picture tells us how to impose the boundary condition!

Y system



Y functions: cross ratios defined by small solutions

$$Y_{a,m}(\zeta) \to \frac{\langle s_i, s_j \rangle \langle s_k, s_l \rangle}{\langle s_i, s_k \rangle \langle s_j, s_l \rangle}(\zeta)$$

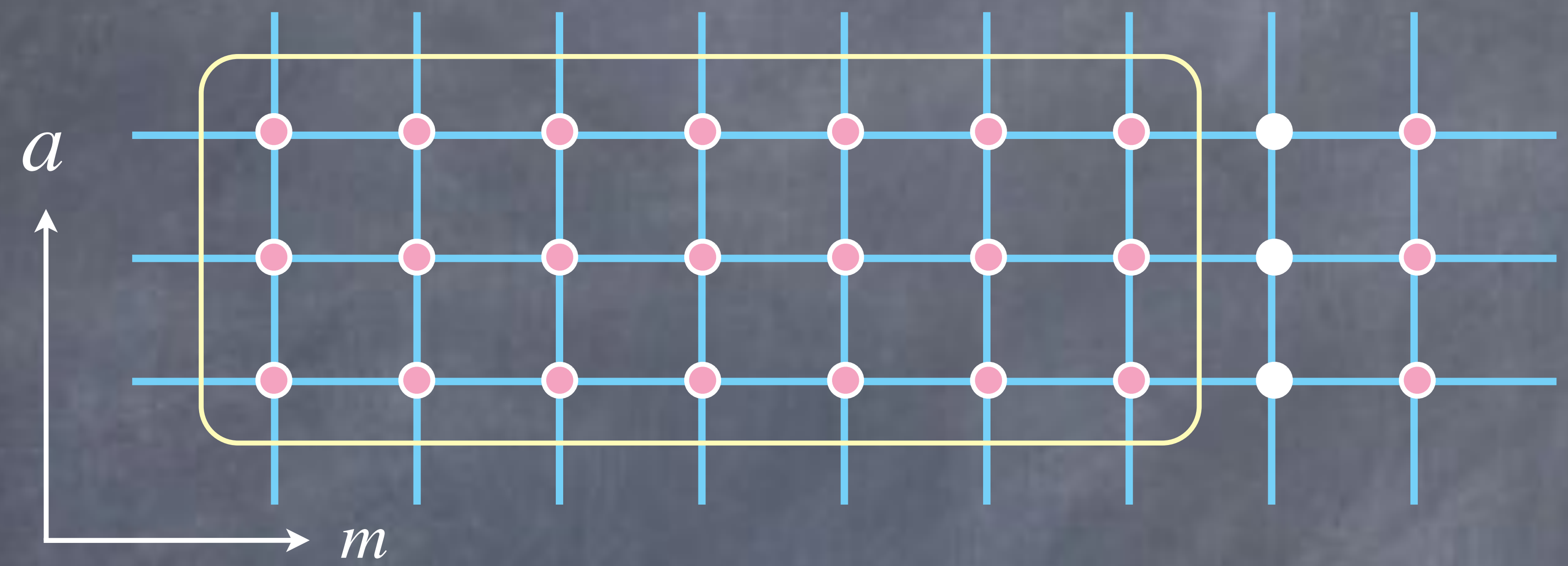
Y system: a finite set of difference equations of Y functions

Easy to solve numerically. Area can be extracted from the Y functions.

A natural truncation for amplitudes: $s_{i+n} \propto s_i$

$$\langle s_1, s_2 \rangle, \quad \langle s_1, s_3 \rangle, \quad \cdots \quad \langle s_1, s_n \rangle, \quad \langle s_1, s_{n+1} \rangle = 0$$

Y system



Y functions: cross ratios defined by small solutions

$$Y_{a,m}(\zeta) \to \frac{\langle s_i, s_j \rangle \langle s_k, s_l \rangle}{\langle s_i, s_k \rangle \langle s_j, s_l \rangle}(\zeta)$$

Y system: a finite set of difference equations of Y functions

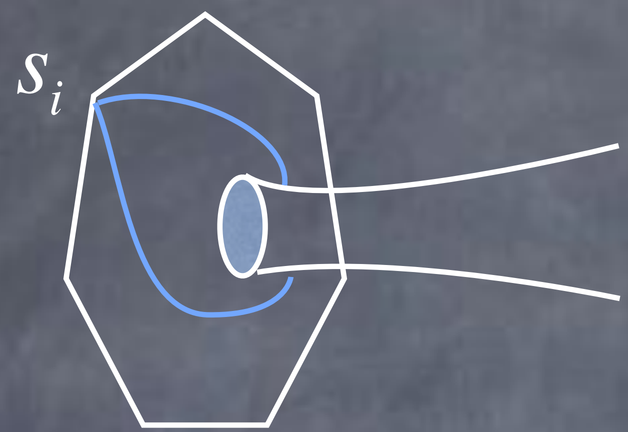
Easy to solve numerically. Area can be extracted from the Y functions.

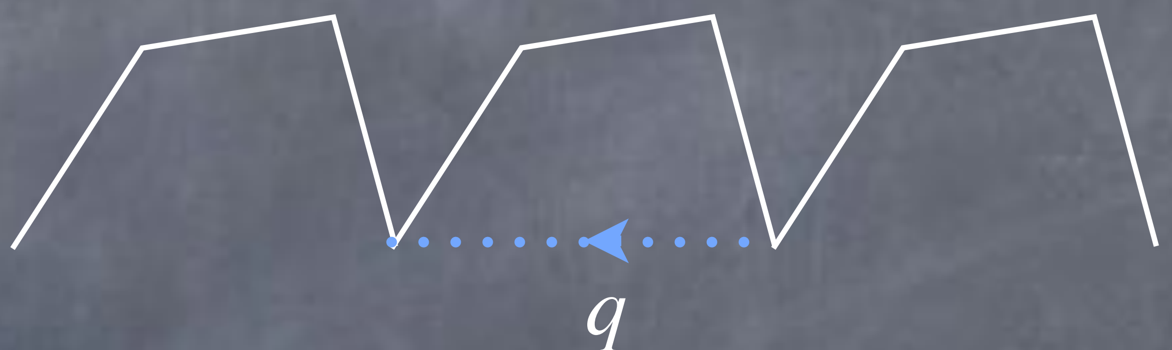
A natural truncation for amplitudes: $s_{i+n} \propto s_i$

$$\langle s_1, s_2 \rangle, \quad \langle s_1, s_3 \rangle, \quad \cdots \quad \langle s_1, s_n \rangle, \quad \langle s_1, s_{n+1} \rangle = 0$$

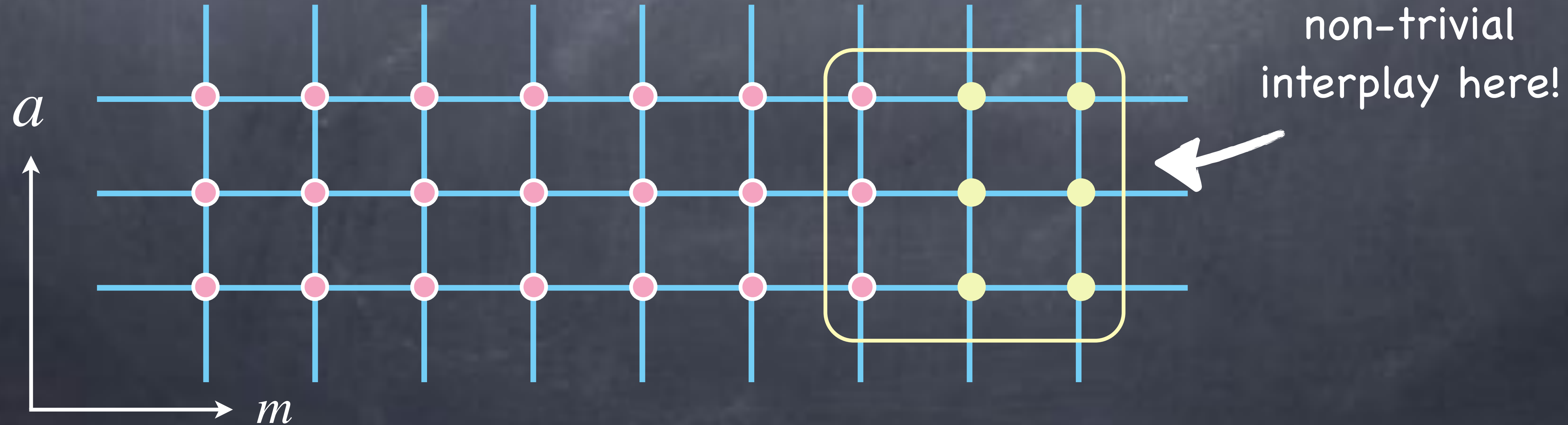
(Maldacena, Zhiboedov)

Form factor





S_{n+i} is no longer simply proportional to S_i.
One may think there are infinite number of cusps in the periodic picture, however, they are related by the monodromy.
This provides a truncation for the infinite long system.



Monodromy

Monodromy defined by boundary conditions with spacetime variables:

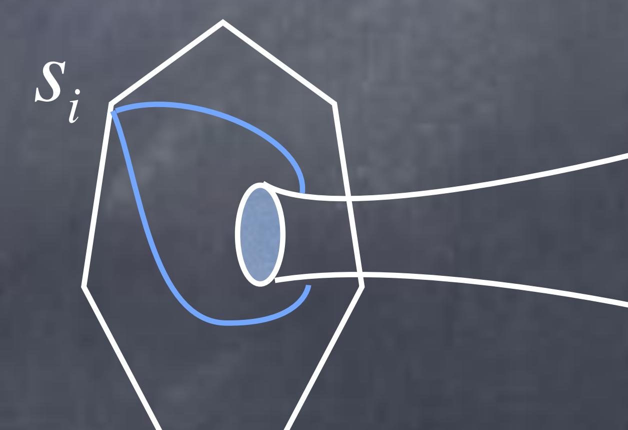
$$\lambda_{i+n} = \widehat{\Omega}\ \lambda_i$$

For periodic cases:

$$\widehat{\Omega} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ q & 0 & 1 & 0 \\ 0 & q & 0 & 1 \end{pmatrix}$$

Monodromy in terms of small solutions:

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix}(ze^{2\pi i},\zeta) = \Omega(\zeta)\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix}(z,\zeta)$$



not trivial to go from $\widehat{\Omega}$ to Ω

Truncation of Y system

From the monodromy and cyclic property:

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix}(ze^{2\pi i},\zeta) = \Omega(\zeta)\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix}(z,\zeta), \quad s_{n+i}(ze^{2\pi i},\zeta) = B_i(\zeta)s_i(z,\zeta) \mapsto \begin{pmatrix} s_{n+1} \\ s_{n+2} \\ s_{n+3} \\ s_{n+4} \end{pmatrix}(z,\zeta) = B(\zeta)\Omega^{-1}(\zeta)\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix}(z,\zeta)$$

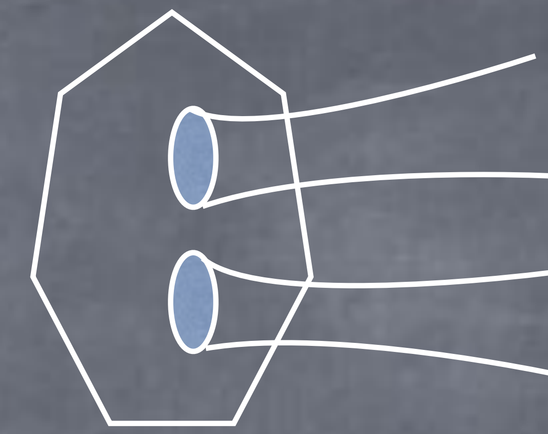
we can obtain a relation between S_{n+i} and S_i at the same point of the worldsheet, which provides a truncation for the form factor Y system.

For AdS_3 (SU(2)) the truncation is very simple for general monodromy.

(Maldacena, Zhiboedov)

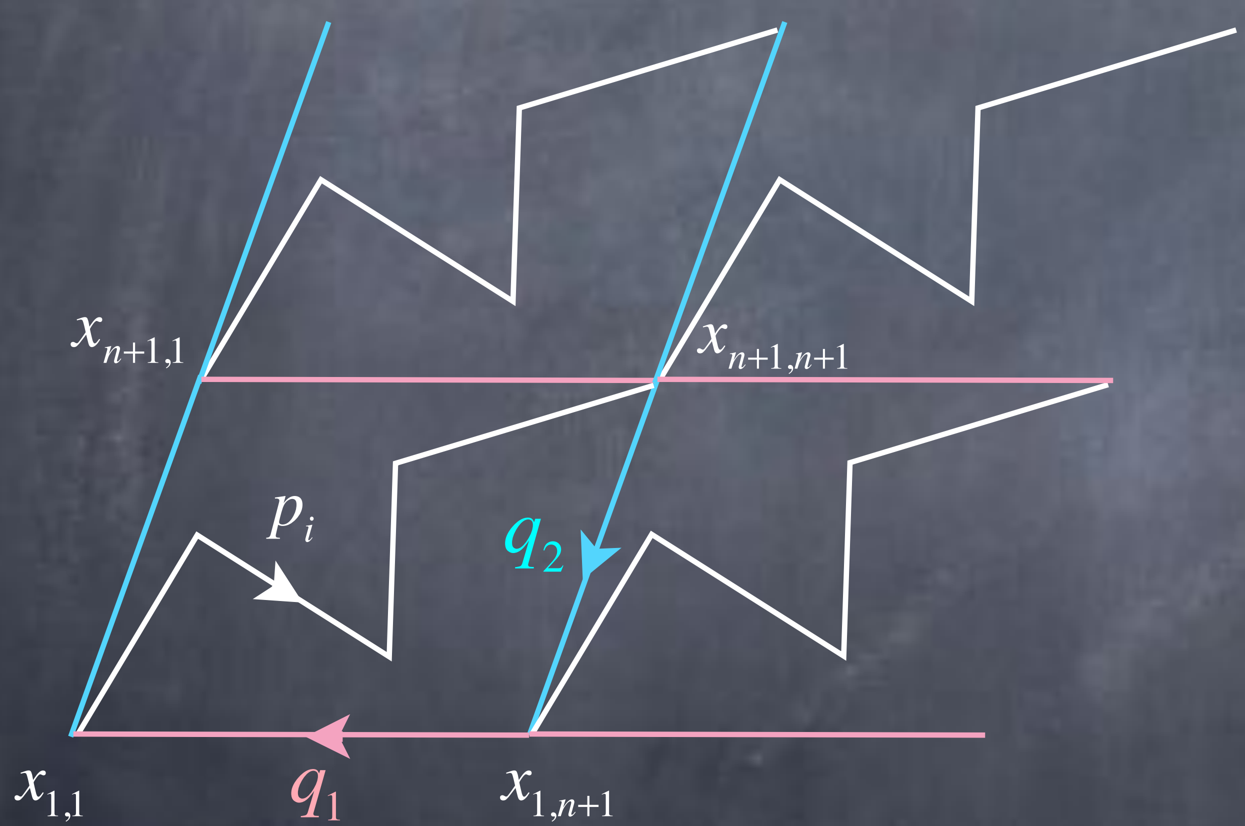
For AdS_5 (SU(4)) case, the main challenge is to truncate the integrable system in a conformal invariant way. Simplification is expected happen for the simpler periodic case.

Multi-operator insertion



Spacetime picture:

$$\lambda_{i+n,i} = \hat{\Omega}_1 \, \lambda_{i,i} \qquad \lambda_{i,n+i} = \hat{\Omega}_2 \, \lambda_{i,i}$$



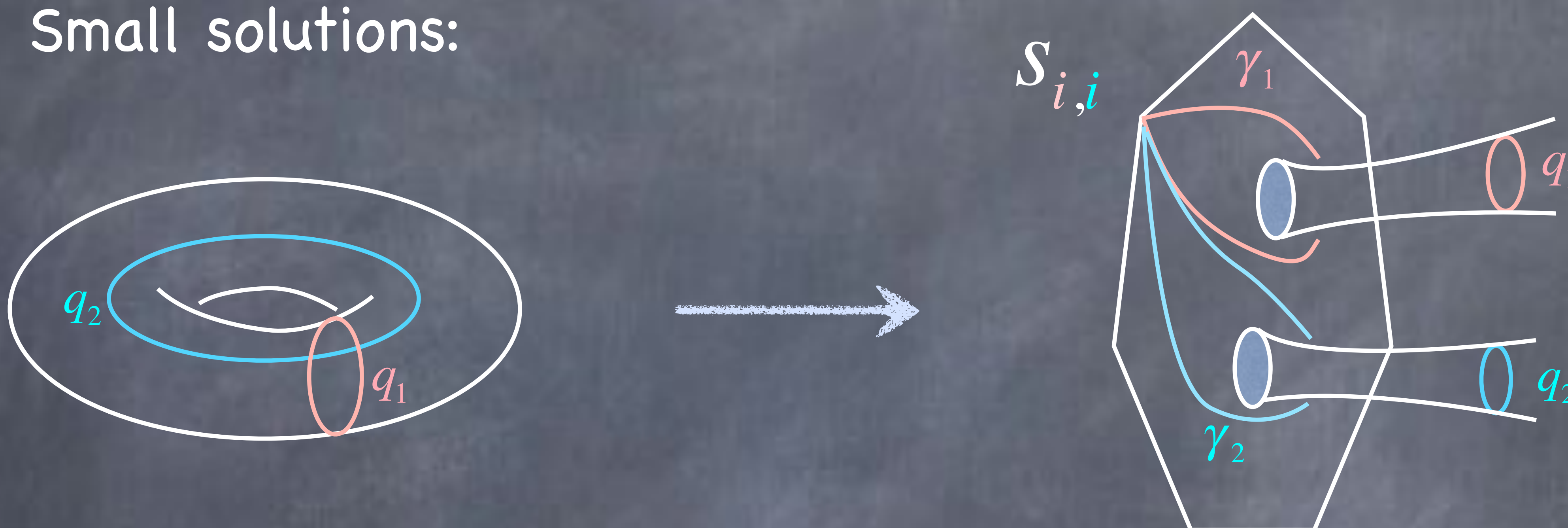
Wilson loop closed on a torus!

$$\hat{\Omega}_1 \cdot \hat{\Omega}_2 = \hat{\Omega}_2 \cdot \hat{\Omega}_1, \qquad \hat{\Omega}_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ q_i & 0 & 1 & 0 \\ 0 & q_i & 0 & 1 \end{pmatrix}$$

dual MHV rules works

Multi-operator insertion

Small solutions:



$$\begin{pmatrix} s_{n+1,1} \\ s_{n+2,2} \end{pmatrix}(ze^{\oint \gamma_1},\zeta) = \Omega_1(\zeta)\begin{pmatrix} s_{1,1} \\ s_{2,2} \end{pmatrix}(z,\zeta)$$

(AdS3)

$$\begin{pmatrix} s_{1,n+1} \\ s_{2,n+2} \end{pmatrix}(ze^{\oint \gamma_2},\zeta) = \Omega_2(\zeta)\begin{pmatrix} s_{1,1} \\ s_{2,2} \end{pmatrix}(z,\zeta)$$

We have a similar truncation as in the single insertion case, but with more Y functions.

Polynomial P(z)

The study of boundary condition shows that there is a pole for each insertion of operators.

For AdS3 case (Maldacena, Zhiboedov)

periodic monodromy	AdS3	AdS5
amplitudes	$p(z)=z^{n/2-2}+a_{n/2-4}z^{n/2-4}+\cdots+a_1z+a_0$	$P(z)=z^{n-4}+a_{n-2}z^{n-2}+\cdots+a_1z+a_0$
form factors	$p(z)=z^{n/2-2}+a_{n/2-4}z^{n/2-4}+\cdots+a_1z+a_0+\frac{a_{-1}}{z-z_0}$	$P(z)=z^{n-4}+a_{n-2}z^{n-2}+\cdots+a_1z+a_0+\frac{a_{-1}}{z-z_0}+\frac{a_{-2}}{(z-z_0)^2}$
multi-operators	$p(z)=z^{n/2-2}+a_{n/2-4}z^{n/2-4}+\cdots+a_1z+a_0+\sum_i\frac{a_{-1}^{(i)}}{z-z_0^{(i)}}$	$P(z)=z^{n-4}+a_{n-2}z^{n-2}+\cdots+a_1z+a_0+\sum_i\left(\frac{a_{-1}^{(i)}}{z-z_0^{(i)}}+\frac{a_{-2}^{(i)}}{(z-z_0^{(i)})^2}\right)$

One needs to study the non-trivial algebraic curves defined by: $x^4=P(z)$

Outlook

- more (technical) details to work out
- relation to QCD?
- amplitudes -> form factors -> correlations functions at strong coupling?
- apply monodromy picture at weak coupling -> hidden structure?

Thank you for your attention !