ONE-LOOP MODULAR INTEGRALS REVISITED

Work with
Ioannis Florakis
Boris Pioline

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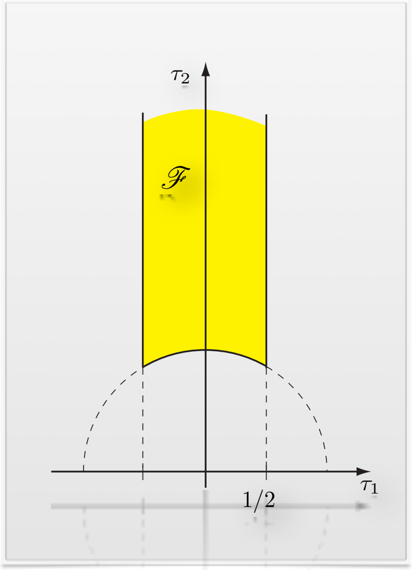
In (oriented) closed string theory one loop diagrams have the topology of a 2d torus. World-sheet reparametrisation invariance implies that the amplitude be invariant under the modular group SL(2,Z).

One-loop couplings in closed String Theory are then given by integrals over a fundamental domain

$$\int_{\mathscr{F}} d\mu \, \mathscr{A}(\tau_1, \tau_2; G, B, \ldots)$$



$$d\mu = \tau_2^{-2}\, d\tau_1\, d\tau_2$$

Evaluating (one-loop) modular integrals is often a daunting task due to the shape of the fundamental domain.

Four main classes of modular integrals

1. $$\int_{\mathscr{F}} d\mu \, \Phi(\tau)$$

2. $$\int_{\mathscr{F}} d\mu \, \Gamma_{d,d}(G, B; \tau_1, \tau_2)$$

3. $$\int_{\mathscr{F}} d\mu \, \Gamma_{d+k,d}(G, B, Y; \tau_1, \tau_2) \, \Phi(\tau)$$

4. $$\int_{\mathscr{F}} d\mu \, \mathscr{Z}(\tau_1, \tau_2)$$

Methods to compute the integrals

2. $$\int_{\mathscr{F}} d\mu\, \varGamma_{d,d}(G,B;\tau_1,\tau_2)$$

3. $$\int_{\mathscr{F}} d\mu\, \varGamma_{d+k,d}(G,B,Y;\tau_1,\tau_2)\, \varPhi(\tau)$$

rely on the unfolding (orbit) method, first introduces in the Physics' literature by McClain, Roth, O'Brien and Tan in 1987, but brought to its full glory by

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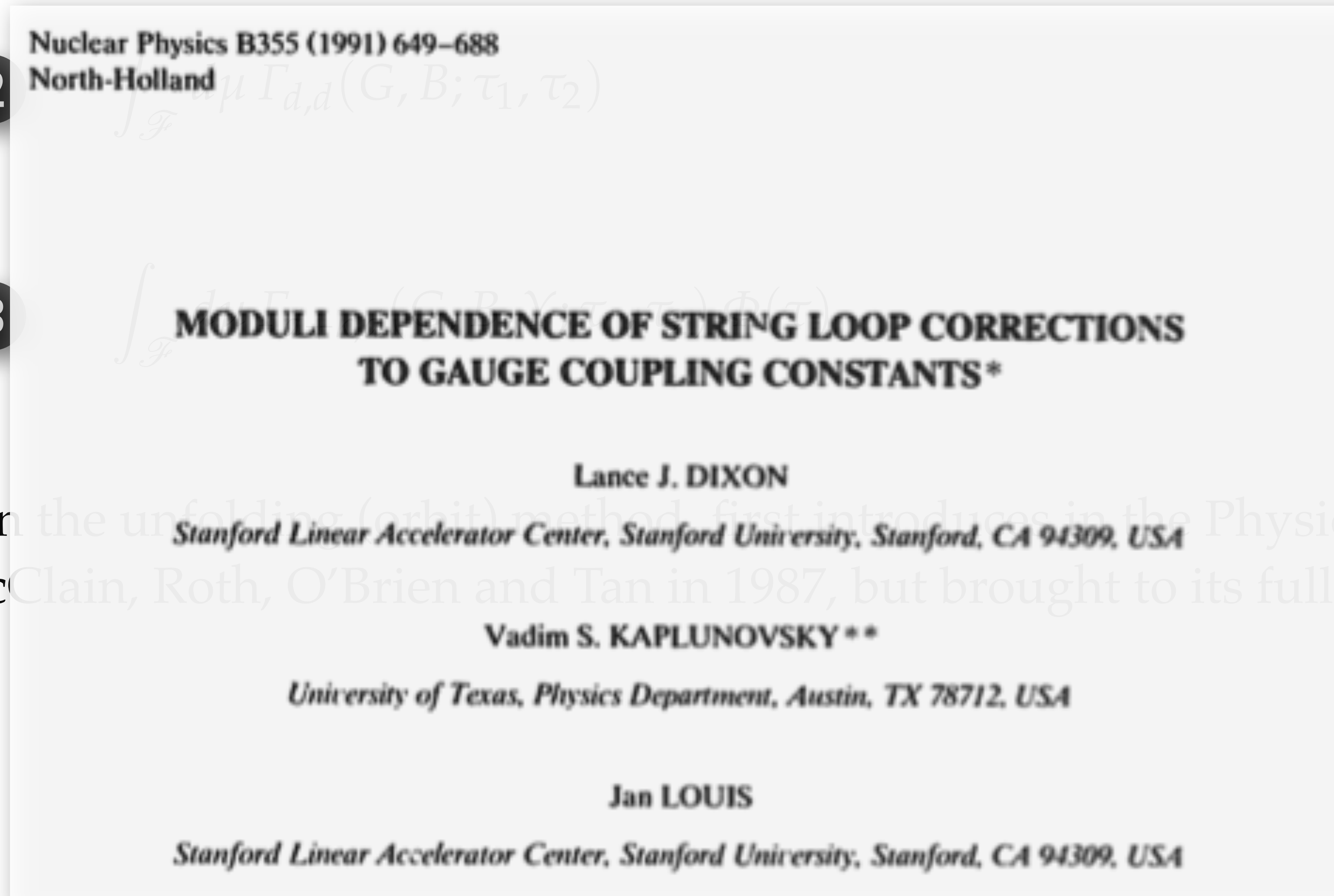
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MODULI DEPENDENCE OF STRING LOOP CORRECTIONS TO GAUGE COUPLING CONSTANTS*

Lance J. DIXON

Stanford Linear Accelerator Center, Stanford University, Stanford, CA 94309, USA

Vadim S. KAPLUNOVSKY**

University of Texas, Physics Department, Austin, TX 78712, USA

Jan LOUIS

Stanford Linear Accelerator Center, Stanford University, Stanford, CA 94309, USA

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Such integrals arise in a variety of examples in string theory

- ✦ Gauge thresholds, $R^2\, F^{2h-2}$ in heterotic on K3xT^2;
- ✦ F^4 couplings in heterotic on T^d;
- ✦ R^4 couplings in type II on T^d;
- ✦ R^2 couplings in type II on K3xT^2;
- ✦ .....
- ✦ Quantum corrections in F-theory [see talk by Shiu]

Such amplitudes are highly protected by supersymmetry and thus can be used to test string dualities.

$$\int_{\mathscr{F}} d\mu \, \Gamma_{d,d}(G,B;\tau_1,\tau_2)\, \Phi(\tau)$$

The main observation of the ***orbit method*** is that (in a given Weyl chamber) the Narain lattice can be written as

[Dixon, Kaplunovsky, Louis, 1991]

$$\Gamma_{d,d} = \sum_{A\in \mathrm{Mat}_{2\times d}(\mathbb{Z})} \Gamma(A\cdot (1\ \tau)^t)$$

$$A' = A\,V \qquad\qquad V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$$

give the same contribution to the integral since they are related by

$$\tau \to \tau' = \frac{a\tau+b}{c\tau+d}$$

Therefore

$$\int_{\mathscr{F}} d\mu\, \Gamma(A' = A\, V)\, \Phi = \int_{V\cdot\mathscr{F}} d\mu\, \Gamma(A)\, \Phi$$

and we can restrict the sum over representatives of SL(2,Z) orbits.

This implies that only selected momenta and windings survive, thus breaking explicitly the T duality symmetry!

Let me remind that

$$\mathscr{F} = \Gamma \backslash \mathscr{H}$$

where $\mathscr{H}$ is the upper complex plane with hyperbolic metric, and

$$\Gamma = \mathrm{SL}(2;\mathbb{Z})$$

An illustrative one-dimensional example

$$\begin{aligned}\mathcal{I}_1 &= \int_{\mathcal{F}} d\mu\, \Gamma_{1,1}(R;\tau)\, j(\tau) \\ &= R \int_{\mathcal{F}} d\mu\, j(\tau) + R \sum_{N>0} \int_0^\infty d\tau_2 \int_{-1/2}^{1/2} d\tau_1 \tau_2^{-2}\, e^{-\pi N^2 R/\tau_2}\, j(\tau) \\ &= -8\pi R\end{aligned}$$

$$\begin{aligned}j(q) &= q^{-1} + O(q) \\ \Gamma_{1,1}(R;\tau) &= R \sum_{\tilde{m},n} \exp\left(-\pi\, R^2 \frac{|\tilde{m}+n\tau|^2}{\tau_2}\right) \\ &= R \sum_{N\geq 0} \sum_{(c,d)=1} \exp\left(-\pi\, N^2\, R^2 \frac{|c\tau+d|^2}{\tau_2}\right)\end{aligned}$$

An illustrative one-dimensional example

$$\begin{aligned}\mathcal{I}_1 &= \int_{\mathcal{F}} d\mu\, \Gamma_{1,1}(R;\tau)\, j(\tau)\\ &= R\int_{\mathcal{F}} d\mu\, j(\tau) + R\sum_{N>0}\int_0^\infty d\tau_2 \int_{-1/2}^{1/2} d\tau_1 \tau_2^{-2}\, e^{-\pi N^2 R/\tau_2}\, j(\tau)\\ &= -8\pi R\end{aligned}$$

$$\begin{aligned} j(q) &= q^{-1} + O(q)\\ \Gamma_{1,1}(R;\tau) &= R\sum_{\tilde m,n}\exp\left(-\pi\, R^2\frac{|\tilde m + n\tau|^2}{\tau_2}\right)\\ &= R\sum_{N\geq 0}\sum_{(c,d)=1}\exp\left(-\pi\, N^2\, R^2\frac{|c\tau+d|^2}{\tau_2}\right)\end{aligned}$$

Wrong answer!

$$\mathcal{I}_1 = -4\pi\left(R+R^{-1}\right) - 4\pi|R-R^{-1}|$$

The orbit method rely on the choice of a Weyl chamber so that the result is (in general) not valid throughout the Narain moduli space.

Possible singularities of the amplitude are (in general) difficult to analyse.

Still, this approach gives naturally the asymptotic (*e.g.* large volume) behaviour of the amplitude.

$$
\begin{aligned}
\int_{\mathscr{F}} d\mu\, \Gamma_{2,2}(T,U)\, \frac{\hat{E}_2\, E_4\, E_6}{\Delta} \simeq \mathrm{Re} \Bigg[& -24 \sum_{k>0} \left(11\, \mathrm{Li}_1(e^{2\pi i k T}) - \frac{30}{\pi T_2\, U_2} \mathcal{P}(kT) \right) \\
& - 24 \sum_{\ell>0} \left(11\, \mathrm{Li}_1(e^{2\pi i \ell U}) - \frac{30}{\pi T_2\, U_2} \mathcal{P}(\ell U) \right) \\
& + \sum_{k>0,\ell>0} \left(\tilde{c}(k\ell)\, \mathrm{Li}_1(e^{2\pi i (kT+\ell U)}) - \frac{3\, c(k\ell)}{\pi T_2\, U_2} \mathcal{P}(kT+\ell U) \right) \\
& + \mathrm{Li}_1(e^{2\pi i (T_1 - U_1 + i|T_2 - U_2|)}) - \frac{3}{\pi T_2\, U_2} \mathcal{P}\left(T_1 - U_1 + i|T_2 - U_2|\right) \Bigg] \\
& + \frac{60\, \zeta(3)}{\pi^2\, T_2 U_2} + 22 \log \left(\frac{8\pi e^{1-\gamma}}{\sqrt{27}}\, T_2 U_2 \right) \\
& + \left(\frac{4\pi}{3} \frac{U_2^2}{T_2} - \frac{22\pi}{3} U_2 - 4\pi T_2 \right) \Theta(T_2 - U_2) \\
& + \left(\frac{4\pi}{3} \frac{T_2^2}{U_2} - \frac{22\pi}{3} T_2 - 4\pi U_2 \right) \Theta(U_2 - T_2)
\end{aligned}
$$

where

$$\frac{E_4\, E_6}{\Delta} = \sum_{n=-1}^{\infty} c(n)\, q^n$$

$$\frac{E_2\, E_4\, E_6}{\Delta} = \sum_{n=-1}^{\infty} \tilde{c}(n)\, q^n$$

$$\mathcal{P}(z) = y\, \mathrm{Li}_2(e^{2\pi i z}) + \frac{1}{2\pi}\, \mathrm{Li}_3(e^{2\pi i z})$$

We propose a new way to compute modular integrals that fully respects the duality symmetries.

The result is cast in a way that neatly reflects the contribution of (each?) BPS state to the amplitude of interest

The result is chamber independent (i.e. valid at any point in moduli space) as a direct consequence of our approach

The singularity structure of the amplitude is crystal-clear in this approach

The large-volume expansion can be obtained straightforwardly and the monodromies of generalised prepotentials easily obtained

$$\int_{\mathscr{F}} d\mu \, \Gamma_{d+k,d}(G, B, Y; \tau_1, \tau_2) \, \Phi(\tau)$$

The typical integral is

$$\int_{\mathscr{F}} d\mu\, \Gamma_{d+k,d}(G,B,Y)\, \Phi(\tau)$$

with $\Phi(\tau)$ a (almost) holomorphic weak modular form of negative weight w=-$k/2$ (=0 mod 4) and simple pole at the cusp

ex.

$$\Phi = \frac{\hat{E}_2^n\, E_4^m\, E_6^k}{\Delta}$$

$\Phi(\tau)$ itself admits a Poincaré series representation

$$\Phi(\tau) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \psi(\tau)\Big|_w \gamma \equiv \sum_{(c,d)=1} (c\tau+d)^{-w}\, \psi\left(\frac{a\tau+b}{c\tau+d}\right)$$

IDEA: use $\Phi(\tau)$ to unfold $\mathcal{F}$

$$\int_0^\infty \frac{d\tau_2}{\tau_2^2} \int_{-1/2}^{1/2} d\tau_1\, \Gamma_{d+k,d}(G,B,Y)\, \psi(\tau)$$

So simple?

The Poincaré series ought to be absolutely convergent for the unfolding to be justified

$$\psi(\tau) \ll \tau_2^{1-w/2} \qquad \text{as} \qquad \tau_2 \to 0$$

Unfortunately, the natural choice $\psi(\tau) = q^{-1}$ does not yield an absolutely convergent series

There are several ways to regularise the series:

- introduce an explicit convergence factor at the expense of modular invariance;
- analytically continue to a non-holomorphic modular form (hard!)

Let us consider the (Niebur) Poincaré series

$$\mathcal{F}(s,w) = \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (4\pi\tau_2)^{-w/2} M_{-\frac{w}{2}, s-\frac{1}{2}}(4\pi\tau_2)\, e^{-2\pi i \tau_1} \Big|_w \gamma$$

Whittaker M-function

that is indeed absolutely convergent for $w<0$.

The Niebur Poincaré series satisfies the differential equation

$$\left[\Delta_w + \frac{1}{2}s(1-s) + \frac{1}{8}w(w+2)\right] \mathcal{F}(s,w) = 0$$

Δ_w is the (weight-w) hyperbolic Laplacian

and for s=1-$w/2$ reduces to

$$\left(\Delta_w + \frac{1}{2}w\right) \mathcal{F}(s,w) = 0$$

the same equation satisfied by weak holomorphic modular forms of weight w!

Any (weak) holomorphic modular form can be written as a linear combination of NP series with definite coefficients given by the principal part of its Laurent series.

$$\Phi = \frac{\hat{E}_2^n \, E_4^m \, E_6^k}{\Delta}$$

If $$\Phi_w^-(\tau) = q^{-1} \sum_{\ell=0}^{n} C_\ell \, \tau_2^{\ell-n}$$

then $$\boxed{\Phi_w(\tau) = \sum_{\ell=0}^{n} d_\ell \, \mathcal{F}(1 - \tfrac{w}{2} + \ell, w)}$$

$$\begin{aligned} d_n &= C_0 / A_{n,0} \\ d_{n-\ell} &= \frac{C_\ell - \sum_{p=n-\ell+1}^{n} A_{p,\ell} \, d_p}{A_{n-\ell,\ell}} \end{aligned}$$

$$A_{p,\ell} = \frac{\Gamma(2p+2-w)\,\Gamma(p-w+n-\ell+1)}{\Gamma(p+1-w)\,\Gamma(n-p+1)\,\Gamma(\ell+p-n+1)}\,(-4\pi)^{\ell-n}$$

$$j(q) = q^{-1} + O(q)$$

$$= d_1\,\mathcal{F}(1,0) + d_0$$

$$\mathcal{F}(1,0) \sim q^{-1} + 24$$

the solution is: d_1=1, d_0=-24

$$j(q) = \mathcal{F}(1,0) - 24$$

Similarly

$$\frac{\hat{E}_2\,E_4\,E_6}{\Delta} = q^{-1}\left(1 + \frac{3}{\pi\tau_2}\right) - 264 + \frac{720}{\pi\tau_2} + O(q)$$

$$= d_2\,\mathcal{F}(2,0) + d_1\,\mathcal{F}(1,0) + d_0$$

$$\mathcal{F}(2,0) \sim 6\,q^{-1} + \frac{3}{\pi\tau_2}\,q^{-1} + \frac{720}{\pi\tau_2}$$

the solution is: d_2=1, d_1=-5, d_0=-144

$$\frac{\hat{E}_2\,E_4\,E_6}{\Delta} = \mathcal{F}(2,0) - 5\,\mathcal{F}(1,0) - 144$$

w	$\mathscr{F}(1-\frac{w}{2}, w)$
0	$j+24$
-2	$3!\, E_4\, E_6\, \Delta^{-1}$
-4	$5!\, E_4^2\, \Delta^{-1}$
-6	$7!\, E_6\, \Delta^{-1}$
-8	$9!\, E_4\, \Delta^{-1}$

$w = 0$
$\frac{\hat{E}_2 E_4 E_6}{\Delta} = \mathcal{F}(2,1,0) - 5\,\mathcal{F}(1,1,0) - 144$
$\frac{\hat{E}_2^2 E_4^2}{\Delta} = \frac{1}{5}\,\mathcal{F}(3,1,0) - 4\mathcal{F}(2,1,0) + 13\,\mathcal{F}(1,1,0) + 144$
$\frac{\hat{E}_2^3 E_6}{\Delta} = \frac{3}{175}\,\mathcal{F}(4,1,0) - \frac{3}{5}\,\mathcal{F}(3,1,0) + \frac{33}{5}\,\mathcal{F}(2,1,0) - 17\,\mathcal{F}(1,1,0) - 144$
$\frac{\hat{E}_2^4 E_4}{\Delta} = \frac{1}{1225}\,\mathcal{F}(5,1,0) - \frac{6}{175}\,\mathcal{F}(4,1,0) + \frac{18}{35}\,\mathcal{F}(3,1,0) - \frac{16}{5}\,\mathcal{F}(2,1,0) + \frac{29}{5}\,\mathcal{F}(1,1,0) + \frac{144}{5}$
$\frac{\hat{E}_2^6}{\Delta} = \frac{1}{1926925}\mathcal{F}(7,1,0) - \frac{3}{2695}\mathcal{F}(5,1,0) + \frac{6}{175}\mathcal{F}(4,1,0) - \frac{3}{7}\mathcal{F}(3,1,0) + \frac{12}{5}\mathcal{F}(2,1,0) - \frac{29}{7}\mathcal{F}(1,1,0) - \frac{144}{7}$
$w = -2$
$\frac{\hat{E}_2 E_4^2}{\Delta} = \frac{1}{40}\mathcal{F}(3,1,-2) - \frac{1}{3}\mathcal{F}(2,1,-2)$
$\frac{\hat{E}_2^2 E_6}{\Delta} = \frac{1}{525}\mathcal{F}(4,1,-2) - \frac{1}{20}\mathcal{F}(3,1,-2) + \frac{11}{30}\mathcal{F}(2,1,-2)$
$\frac{\hat{E}_2^3 E_4}{\Delta} = \frac{1}{11760}\mathcal{F}(5,1,-2) - \frac{1}{350}\mathcal{F}(4,1,-2) + \frac{9}{280}\mathcal{F}(3,1,-2) - \frac{2}{15}\mathcal{F}(2,1,-2)$
$\frac{\hat{E}_2^5}{\Delta} = \frac{1}{19819800}\mathcal{F}(7,1,-2) - \frac{1}{12936}\mathcal{F}(5,1,-2) + \frac{1}{525}\mathcal{F}(4,1,-2) - \frac{1}{56}\mathcal{F}(3,1,-2) + \frac{1}{15}\mathcal{F}(2,1,-2)$
$w = -4$
$\frac{\hat{E}_2 E_6}{\Delta} = \frac{1}{2520}\mathcal{F}(4,1,-4) - \frac{1}{120}\mathcal{F}(3,1,-4)$
$\frac{\hat{E}_2^2 E_4}{\Delta} = \frac{1}{70560}\mathcal{F}(5,1,-4) - \frac{1}{2520}\mathcal{F}(4,1,-4) + \frac{1}{280}\mathcal{F}(3,1,-4)$
$\frac{\hat{E}_2^4}{\Delta} = \frac{1}{148648500}\mathcal{F}(7,1,-4) - \frac{1}{129360}\mathcal{F}(5,1,-4) + \frac{1}{6300}\mathcal{F}(4,1,-4) - \frac{1}{840}\mathcal{F}(3,1,-4)$
$w = -6$
$\frac{\hat{E}_2 E_4}{\Delta} = \frac{1}{241920}\mathcal{F}(5,1,-6) - \frac{1}{10080}\mathcal{F}(4,1,-6)$
$\frac{\hat{E}_2^3}{\Delta} = \frac{1}{792792000}\mathcal{F}(7,1,-6) - \frac{1}{887040}\mathcal{F}(5,1,-6) + \frac{1}{50400}\mathcal{F}(4,1,-6)$
$w = -8$
$\frac{\hat{E}_2^2}{\Delta} = \frac{1}{2854051200}\mathcal{F}(7,1,-8) - \frac{1}{3991680}\mathcal{F}(5,1,-8)$
$w = -10$
$\frac{\hat{E}_2}{\Delta} = \frac{1}{13!}\mathcal{F}(7,1,-10)$

Table 3. List of all weak almost holomorphic modular forms of negative weight with a simple pole at $q = 0$, as linear combination of Niebur-Poincaré series $\mathcal{F}(1 - \frac{w}{2} + n, 1, w)$ (the holomorphic ones appear in the first column of table 1).

As a result, one has just to consider integrals

$$\mathcal{I}_{d+k,d}(s) = \int_{\mathcal{F}} d\mu \, \Gamma_{d+k,d}(G,B,Y) \, \mathcal{F}(s,-\tfrac{k}{2})$$

that can be computed by unfolding the fundamental domain against the Niebur Poincaré series, thus obtaining

$$\boxed{\mathcal{I}_{d+k,d}(s) = (4\pi)^{s+\frac{k}{4}} \int_0^\infty dt\, t^{\frac{d}{2}+\frac{k}{4}+s-2} \, {}_1F_1(s-\tfrac{k}{4};2s;4\pi t) \sum_{\text{BPS}} e^{-\pi t (p_{\text{L}}^2+p_{\text{R}}^2+4)/2}}$$

$$\sum_{\text{BPS}} \equiv \sum_{p_{\text{L}},p_{\text{R}}} \delta(p_{\text{L}}^2 - p_{\text{R}}^2 - 4)$$

1st POSSIBILITY: INTEGRATE AND THEN SUM

$$\mathcal{I}_{d+k,d}(s) = (4\pi)^{1-\frac{d}{2}}\,\Gamma(s+\tfrac{2d+k}{4}-1) \times \sum_{\mathrm{BPS}} {}_2F_1\left(s-\frac{k}{4}, s+\frac{2d+k}{4}-1; 2s; \frac{4}{p_{\mathrm{L}}^2}\right)\left(\frac{p_{\mathrm{L}}^2}{4}\right)^{1-s-\frac{2d+k}{4}}$$

For $s>1$ the BPS sum is absolutely convergent and for $s=1+n+k/4$ the hypergeometric function reduces to a combination of elementary functions!

[Bruinier, 2002]

1st POSSIBILITY: INTEGRATE AND THEN SUM

for example, for d=2 and vanishing Wilson lines

$$\mathcal{I}_{2,2}(1+n) = \frac{(2n+1)!}{n!} \sum_{\mathrm{BPS}} \left(\frac{p_{\mathrm{L}}^2}{4\kappa}\right)^n \sum_{m=0}^{n} \binom{n}{m}^2 \left\{ \left(\frac{p_{\mathrm{R}}^2}{p_{\mathrm{L}}^2}\right)^m \left[H_m - \log\left(\frac{p_{\mathrm{R}}^2}{p_{\mathrm{L}}^2}\right)\right]\right.$$

$$\left. - \sum_{\ell=0}^{m} (-1)^\ell \binom{m}{\ell} \left(\frac{p_{\mathrm{L}}^2}{4\kappa}\right)^{-\ell} H_{m-\ell} - (-1)^m \sum_{\ell=m+1}^{2n} \frac{\Gamma(n-m)\, m!}{\ell!} \left(\frac{p_{\mathrm{L}}^2}{4\kappa}\right)^{-\ell} \right\}$$

The result is clearly invariant under T-duality, Weyl-chamber independent, and moreover the singularity structure of the amplitude (corresponding to p_{R}=0) is nicely encoded in the final expression.

$$H_N = \sum_{k=1}^{N} k^{-1}$$

In this 2-dimensional example one finds $\quad \mathcal{I}_{2,2}(1+n) \sim -\frac{(2n+1)!}{n!} \log|j(U)-j(T)|^4$

BACK TO THE PROTOTYPE EXAMPLE

$$\int_{\mathscr{F}} d\mu\, \Gamma_{1,1}(R)\, j(\tau) = \int_{\mathcal{F}} d\mu\, \Gamma_{1,1}(R)\, \mathcal{F}(1,0) - 24 \int_{\mathcal{F}} d\mu\, \Gamma_{1,1}(R)$$

$$= 2\pi \sum_{mn=1} {}_2F_1\left(1,\, \tfrac{1}{2};\, 2;\, \frac{4}{p_{\mathrm{L}}^2}\right) \sqrt{\frac{4}{p_{\mathrm{L}}^2}} - 8\pi(R + R^{-1})$$

$$= -4\pi \left[R + R^{-1} + \left|R - R^{-1}\right|\right]$$

$${}_2F_1(1,\, \tfrac{1}{2};\, 2;\, 4\, p_{\mathrm{L}}^{-2}) = \tfrac{1}{2} p_{\mathrm{L}}^2 \left(1 - \sqrt{\frac{p_{\mathrm{R}}^2}{p_{\mathrm{L}}^2}}\right)$$

BACK TO THE PROTOTYPE EXAMPLE

$$\int_{\mathscr{F}} d\mu\, \Gamma_{2,2}(T,U)\, \frac{\hat{E}_2\, E_4\, E_6}{\Delta} = \int_{\mathscr{F}} d\mu\, \Gamma_{2,2}(T,U)\, \left(\mathcal{F}(2,0) - 5\,\mathcal{F}(1,0) - 144\right)$$

$$= \mathcal{I}_{2,2}(2) - 5\,\mathcal{I}_{2,2}(1) - 144\,\mathcal{I}_{\rm DKL}$$

$$= \sum_{\rm BPS} \left[-12 - (1+3p_{\rm R}^2)\, \log\left(\frac{p_{\rm R}^2}{p_{\rm L}^2}\right)\right] - 144\,\mathcal{I}_{\rm DKL}$$

$$\mathcal{I}_{\rm DKL} \sim \log\left[T_2\, U_2\, |\eta(T)\,\eta(U)|^4\right]$$

$$\begin{aligned}\int_{\mathscr{F}} d\mu\, \Gamma_{2,2}(T,U)\, \frac{\hat{E}_2\, E_4\, E_6}{\Delta} \simeq \mathrm{Re}\Bigg[& -24 \sum_{k>0} \left(11\, \mathrm{Li}_1(e^{2\pi i k T}) - \frac{30}{\pi T_2\, U_2} \mathcal{P}(kT) \right) \\ & - 24 \sum_{\ell>0} \left(11\, \mathrm{Li}_1(e^{2\pi i \ell U}) - \frac{30}{\pi T_2\, U_2} \mathcal{P}(\ell U) \right) \\ & + \sum_{k>0,\ell>0} \left(\tilde{c}(k\ell)\, \mathrm{Li}_1(e^{2\pi i (kT+\ell U)}) - \frac{3\, c(k\ell)}{\pi T_2\, U_2} \mathcal{P}(kT+\ell U) \right) \\ & + \mathrm{Li}_1(e^{2\pi i (T_1 - U_1 + i|T_2 - U_2|)}) - \frac{3}{\pi T_2\, U_2} \mathcal{P}\left(T_1 - U_1 + i|T_2 - U_2|\right) \Bigg] \\ & + \frac{60\, \zeta(3)}{\pi^2\, T_2 U_2} + 22 \log\left(\frac{8\pi e^{1-\gamma}}{\sqrt{27}}\, T_2 U_2 \right) \\ & + \left(\frac{4\pi}{3} \frac{U_2^2}{T_2} - \frac{22\pi}{3} U_2 - 4\pi T_2 \right) \Theta(T_2 - U_2) \\ & + \left(\frac{4\pi}{3} \frac{T_2^2}{U_2} - \frac{22\pi}{3} T_2 - 4\pi U_2 \right) \Theta(U_2 - T_2)\end{aligned}$$

$\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string on $T^2 \times T^4/\mathbb{Z}_2$ with Wilson lines in the unbroken E_8

$$\Delta_{\mathrm{E}_7} = -\frac{1}{12} \int_{\mathcal{F}} d\mu \, \Gamma_{2,10} \, \frac{\hat{E}_2 \, E_6 - E_4^2}{\Delta}$$

Notice that

$$\frac{\hat{E}_2 \, E_6 - E_4^2}{\Delta} = \frac{2}{7!} \mathcal{F}(4,-4) - \frac{2}{5!} \, \mathcal{F}(3,-4)$$

and thus

$$\begin{aligned} \Delta_{\mathrm{E}_7} &= -\frac{1}{720} \left[\frac{1}{42} \, \mathcal{I}_{10,2}(4) - \mathcal{I}_{10,2}(3) \right] \\ &= \sum_{\mathrm{BPS}} \left[1 + \frac{p_{\mathrm{R}}^2}{4} \log\left(\frac{p_{\mathrm{R}}^2}{p_{\mathrm{L}}^2} \right) - \frac{2}{p_{\mathrm{L}}^2} - \frac{8}{3\, p_{\mathrm{L}}^4} - \frac{16}{3\, p_{\mathrm{L}}^6} - \frac{64}{5\, p_{\mathrm{L}}^8} \right] . \end{aligned}$$

no singularity at $p_R = 0$!

2nd POSSIBILITY: SOLVE THE BPS CONSTRAINT FIRST

(at least for the two-dimensional lattice)

The BPS constraint has solutions

$$m_1 n^1 + m_2 n^2 = 1 \qquad \Rightarrow \qquad m_1 = \tilde{m}_1 + M\, n^2\,, \quad m_2 = \tilde{m}_2 - M\, n^1$$

After Poisson resumming over M

$$\mathcal{I}_{2,2}(n) = \sum_{M\in\mathbb{Z}} \sum_{(n^1,n^2)=1} \sqrt{T_2\, \tilde{U}_2}\, e^{2i\pi M(T_1-\tilde{U}_1)}$$
$$\times \int_0^\infty \frac{d\tau_2}{\tau_2^{3/2}}\, M_{0,n+\frac{1}{2}}(4\pi\tau_2)\ \exp\left[-\pi\tau_2\left(\frac{T_2}{\tilde{U}_2}+\frac{\tilde{U}_2}{T_2}\right)-\frac{\pi M^2\, T_2\,\tilde{U}_2}{\tau_2}\right]$$

Invariance under $\mathrm{SL}(2,\mathrm{Z})_U$ is still manifest because $\tilde{U} \equiv \tilde{U}_1 + i\tilde{U}_2 = \dfrac{\tilde{m}_1\, U - \tilde{m}_2}{n^2\, U + n^1}$

Fourier expansion with respect to T

2nd POSSIBILITY: SOLVE THE BPS CONSTRAINT FIRST

(at least for the two-dimensional lattice)

Computing the integral

Bessel K-function

$$\boxed{\mathcal{I}_{2,2}(1+n) = 4 \sum_{M\in\mathbb{Z}} \sqrt{\frac{T_2}{|M|}}\, e^{2\pi i M T_1}\, K_{n+\frac{1}{2}}(2\pi |M| T_2)\, H_M \cdot \mathcal{F}(1+n, 0; U)}$$

NP series

Hecke operator

In this case, T-duality is broken, though the subgroup SL(2,Z)$_U$ is still manifest.

The Fourier expansion gives automatically the large volume behaviour of the amplitude

2nd POSSIBILITY: SOLVE THE BPS CONSTRAINT FIRST

(at least for the two-dimensional lattice)

Moreover,

$$\mathcal{I}_{2,2}(1+n) = 4 \sum_{M\in\mathbb{Z}} \sqrt{\frac{T_2}{|M|}}\, e^{2\pi i M T_1}\, K_{n+\frac{1}{2}}(2\pi|M|T_2)\, H_M \cdot \mathcal{F}(1+n,0;U)$$

$$D_w = \frac{i}{\pi}\left(\partial_z - \frac{i\,w}{2\,y}\right)$$

$$= \mathrm{Re}\left[\frac{(-D_T\, D_U)^n}{n!}\, f_n(T,U)\right]$$

$$D_w \cdot \mathcal{F}(s,w) \propto \mathcal{F}(s,w-2)$$

$$K_{n+\frac{1}{2}} \propto D^n\, q_T$$

where $f_n(T,U)$ are generalised prepotentials. In particular, $F = STU + f_1(T,U)$

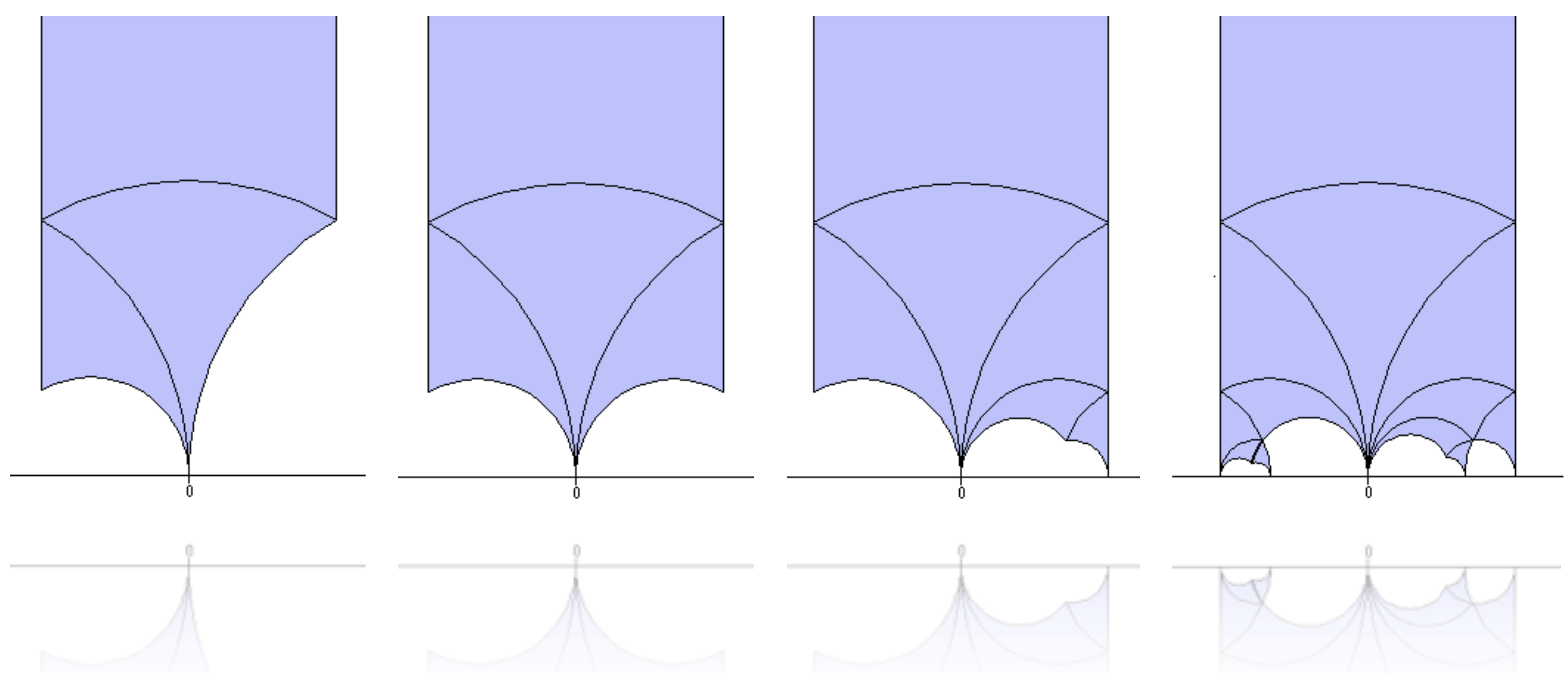
$$f_n(T,U) = G_{2n+2}(U) + \sum_{M>0} \frac{q_T^M}{(2M)^{2n+1}}\, H_M \cdot \mathcal{F}(1+n,-2n;U)$$

with G_{2n+2} being the Eichler integral of the holomorphic Eisenstein series E_{2n+2}.

EXTENSION TO CONGRUENCE SUBGROUPS

$$\mathcal{A} = \int_{\mathcal{F}} d\mu \left(0\,\underset{0}{\blacksquare} + 1\,\underset{0}{\blacksquare} + 0\,\underset{1}{\blacksquare} + 1\,\underset{1}{\blacksquare} \right)$$

$$= \int_{\mathcal{F}} d\mu \;\; 0\,\underset{0}{\blacksquare} + \int_{\mathcal{F}_2} d\mu \;\; 1\,\underset{0}{\blacksquare}$$



This approach allows also to compute more involved integrals where the Narain sum includes non-trivial insertions of left/right momenta

$$\int_{\mathscr{F}} d\mu\, \tau_2^{\delta} \sum_{p_{\rm L}, p_{\rm R}} p_{\rm L}^{a_1} \cdots p_{\rm L}^{a_\alpha}\, p_{\rm R}^{b_1} \cdots p_{\rm R}^{b_\beta}\, q^{\frac{1}{4} p_{\rm L}^2} \bar{q}^{\frac{1}{4} p_{\rm R}^2}\, \mathcal{F}(s,w) \sim$$

$$(4\pi\kappa)^{1-\delta}\, \Gamma\left(s + \frac{|w|}{2} + \delta - 1\right)$$

$$\times \sum_{\rm BPS} p_{\rm L}^{a_1} \cdots p_{\rm L}^{a_\alpha}\, p_{\rm R}^{b_1} \cdots p_{\rm R}^{b_\beta}\, {}_2F_1\left(s - \frac{|w|}{2}, s + \frac{|w|}{2} + \delta - 1; 2s; \frac{4\kappa}{p_{\rm L}^2}\right) \left(\frac{p_L^2}{4\kappa}\right)^{1-s-\frac{|w|}{2}-\delta}$$

as well as integrals without lattice insertions

$$\int_{\mathscr{F}} d\mu \left(\sqrt{\tau_2}\,\eta\,\bar{\eta}\right)^3 \frac{\hat{E}_2\,E_4\left(\hat{E}_2\,E_4 - 2E_6\right)}{\Delta} = -20\sqrt{2}$$

[this integral enters in gauge threshold corrections of heterotic strings on ALE spaces with NS5 branes]

[Carlevaro, Israel, 2012]

Amplitudes of the form

$$\mathcal{A} = \int_{\mathcal{F}} d\mu \, \Gamma_{d+k,d}(G, B, Y) \, \Phi(\tau)$$

are special, since they receive contributions only from BPS states.

In N=2 heterotic vacua, half-BPS states are given by

$$\tfrac{1}{4} \, p_{\mathrm{L}}^2 + N_{\mathrm{L}} - 1 = \tfrac{1}{4} \, p_{\mathrm{R}}^2$$

A SUGGESTIVE SPECULATION (!?!)

After performing the unfolding the amplitude can be cast in a Schwinger-like representation where only a subsector of BPS states contributes

$$\mathcal{A} = \sum_{\rm BPS} \int_0^\infty \frac{dt}{t}\, \mathcal{M}_{s,-\frac{k}{2}}(t)\, e^{-\pi t\, M^2}$$

$$M^2 = \tfrac{1}{2}(p_{\rm L}^2 + p_{\rm R}^2)$$

$$p_{\rm L}^2 - p_{\rm R}^2 = 1$$

Is String Theory nothing but a Field Theory (though with infinite states)?

c.f. Stieberger-Taylor: four gluon/graviton scattering amplitudes are a double Mellin transform of the analogous field-theory amplitudes!

SUMMARY

I have described a new approach to the computation of – a class of – one-loop modular integrals in string theory where the Narain lattice is a spectator in the unfolding procedure

The result can be expressed ***either*** as a BPS sum that is manifestly invariant under the T-duality group and valid throughout moduli space,

or as a Fourier-series expansion that is suitable for extracting the large-moduli behaviour and, for instance, the quantum corrected prepotential.

THANK YOU