Tensorial perturbations and stability of spherically symmetric d–dimensional black holes in string theory

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Leading α' corrections

- Effective action in the Einstein frame

$$\frac{1}{16\pi G}\int\sqrt{-g}\left[\mathcal{R}-\frac{4}{d-2}\left(\partial^{\mu}\phi\right)\partial_{\mu}\phi+\mathrm{e}^{\frac{4}{2-d}\phi}\frac{\lambda}{2}\mathcal{R}^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu\rho\sigma}\right]\mathrm{d}^{d}x,$$

$$\lambda=\frac{\alpha'}{2},\frac{\alpha'}{4}\ \text{(bosonic, heterotic).}$$

- Field equations

$$\mathcal{R}_{\mu\nu} + \lambda\mathrm{e}^{\frac{4}{2-d}\phi}\left(\mathcal{R}_{\mu\rho\sigma\tau}\mathcal{R}_{\nu}^{\ \rho\sigma\tau}-\frac{1}{2(d-2)}g_{\mu\nu}\mathcal{R}_{\rho\sigma\lambda\tau}\mathcal{R}^{\rho\sigma\lambda\tau}\right)=0;$$

$$\nabla^{2}\phi - \frac{\lambda}{4}\mathrm{e}^{\frac{4}{2-d}\phi}\left(\mathcal{R}_{\rho\sigma\lambda\tau}\mathcal{R}^{\rho\sigma\lambda\tau}\right)=0.$$

General perturbation setup

- Metric of the type

$$d\,s^2 = -f(r)\,d\,t^2 + g^{-1}(r)\,d\,r^2 + r^2\,d\,\Omega^2_{d-2}$$

(Einstein frame);

- Variation of the metric

$$h_{\mu\nu} = \delta g_{\mu\nu};$$

- Variation of the Riemann tensor:

$$\begin{aligned} \delta\mathcal{R}_{\rho\sigma\mu\nu} &= \frac{1}{2}\big(\mathcal{R}_{\mu\nu\rho}{}^{\lambda}h_{\lambda\sigma} - \mathcal{R}_{\mu\nu\sigma}{}^{\lambda}h_{\lambda\rho} \\ &- \nabla_\mu\nabla_\rho h_{\nu\sigma} + \nabla_\mu\nabla_\sigma h_{\nu\rho} - \nabla_\nu\nabla_\sigma h_{\mu\rho} + \nabla_\nu\nabla_\rho h_{\mu\sigma}\big)\,. \end{aligned}$$

Perturbations on the $(d-2)$-sphere

- General tensors of rank at least 2 on the $(d-2)$-sphere can be uniquely decomposed in their *tensorial, vectorial and scalar* components.
- One can in general consider perturbations to the metric and any other physical field of the system under consideration.

Tensorial perturbations of the metric

- We consider only the tensorial part of $h_{\mu\nu}$:

$$h_{ij} = 2r^2 H_T(r,t)\mathcal{T}_{ij}\left(\theta^i\right), \; h_{ia} = 0, h_{ab} = 0$$

with

$$\left(\gamma^{kl} D_k D_l + k_T\right)\mathcal{T}_{ij} = 0, \; D^i \mathcal{T}_{ij} = 0, \; g^{ij}\mathcal{T}_{ij} = 0.$$

- D_i: $(d-2)$-sphere covariant derivative, associated to the metric γ_{ij}.
- $\mathcal{T}_{ij}$ are the eigentensors of D^2 on S^{d-2}
- $-k_T = 2 - \ell\left(\ell + d - 3\right)$ are the eigenvalues of D^2 on S^{d-2}, where $\ell = 2, 3, 4, \ldots$.

Tensorial perturbations of fields

$$
\begin{aligned}
\delta\mathcal{R}_{ijkl} &= \left[(3g-1)\,H_T + rg\partial_r H_T\right]\left(g_{il}\mathcal{T}_{jk} - g_{ik}\mathcal{T}_{jl} - g_{jl}\mathcal{T}_{ik} + g_{jk}\mathcal{T}_{il}\right) \\
&+ r^2 H_T\left(D_iD_l\mathcal{T}_{jk} - D_iD_k\mathcal{T}_{jl} - D_jD_l\mathcal{T}_{ik} + D_jD_k\mathcal{T}_{il}\right); \\
\delta\mathcal{R}_{itjt} &= \left[-r^2\partial_t^2 H_T + \frac{1}{2}ff'r^2\partial_r H_T + ff'rH_T\right]\mathcal{T}_{ij}; \\
\delta\mathcal{R}_{itjr} &= \left(-r^2\partial_t\partial_r H_T - r\partial_t H_T + \frac{1}{2}r^2\frac{f'}{f}\partial_t H_T\right)\mathcal{T}_{ij}; \\
\delta\mathcal{R}_{irjr} &= \left(-r\frac{g'}{g}H_T - \frac{1}{2}r^2\frac{g'}{g}\partial_r H_T - 2r\partial_r H_T - r^2\partial_r^2 H_T\right)\mathcal{T}_{ij}.
\end{aligned}
$$

All other tensorial perturbations can be set to 0:

- $\phi(r)$ has no tensor modes on the sphere;
- $\delta A_t^n, \delta A_t^w$ do not matter, since $h_{t\mu} = 0$.

Perturbed graviton field equation

$$\delta\mathcal{R}_{ij} + \lambda \mathrm{e}^{\frac{4}{2-d}\phi}\left[\delta\left(\mathcal{R}_{i\rho\sigma\tau}\mathcal{R}_j^{\ \rho\sigma\tau}\right) - \frac{1}{2(d-2)}\mathcal{R}_{\rho\sigma\lambda\tau}\mathcal{R}^{\rho\sigma\lambda\tau}h_{ij} - \frac{1}{2(d-2)}g_{ij}\delta\left(\mathcal{R}_{\rho\sigma\lambda\tau}\mathcal{R}^{\rho\sigma\lambda\tau}\right)\right] + \frac{4}{d-2}\mathcal{R}_{ij}\delta\phi = 0$$

results in

$$\left(1-2\lambda\frac{f'}{r}\right)\frac{r^2}{f}\partial_t^2 H_T - \left(1-2\lambda\frac{g'}{r}\right)r^2 g\,\partial_r^2 H_T -$$
$$-\left[(d-2)rg + \frac{1}{2}r^2\left(f'+g'\right) + 4\lambda(d-4)\frac{g\,(1-g)}{r} - 4\lambda g g' - \lambda r\left(f'^2+g'^2\right)\right]\partial_r H_T +$$
$$+\left[\ell\,(\ell+d-3)\left(1+\frac{4\lambda}{r^2}\,(1-g)\right) + 2(d-2) - 2(d-3)g - r\left(f'+g'\right) +\right.$$
$$\left.+\,\lambda\left(8\frac{1-g}{r^2} + 2\,(d-3)\,\frac{(1-g)^2}{r^2} - \frac{r^2}{d-2}\left[f'' + \frac{1}{2}\left(\frac{f'g'}{g} - \frac{f'^2}{f}\right)\right]^2\right)\right] H_T = 0.$$

The Master Equation

The perturbation equation is of the form

$$\partial_t^2 H_T - F^2(r)\,\partial_r^2 H_T + P(r)\,\partial_r H_T + Q(r)\,H_T = 0$$

and it can be written as a "master equation"

$$\frac{\partial^2 \Phi}{\partial r_*^2} - \frac{\partial^2 \Phi}{\partial t^2} =: V_T \Phi.$$

- $\frac{d\,r_*}{d\,r} = \frac{1}{F(r)}$ ("tortoise" coordinate);
- $\Phi = k(r) H_T$ ("master" variable);
- $k(r) =$
$$\frac{1}{\sqrt[4]{fg}} \exp\left(\int \frac{(d-2)rg + \frac{1}{2}r^2(f'+g') + 4\lambda(d-4)\frac{g(1-g)}{r} - 4\lambda g g' - \lambda r\left(f'^2 + g'^2\right)}{2fg} dr \right);$$

The tensor potential

- V_T : potential for tensor-type gravitational perturbations. In classical EH gravity it is the same as the potential for scalar fields (Ishibashi, Kodama, 2000-2003);
- it is the potential for tensor–type gravitational perturbations of any kind of static, spherically symmetric $\mathcal{R}^2$ string–corrected black hole in d–dimensions:

The string-corrected tensor potential

$$\begin{aligned}
V_{\mathsf{T}}[f(r),g(r)] &= \frac{1}{r^4 fg}\Big(\ell(\ell+d-3)r^2f^2g + \frac{1}{4}(d-2)(d-4)r^2f^2g^2 \\
&+ \frac{1}{4}(d-6)r^3f^2gf' + r^3fg^2f' + \frac{1}{16}r^4f^2f'^2 + \frac{3}{16}r^4g^2f'^2 \\
&+ \frac{1}{4}(d-2)r^3f^2gg' - \frac{1}{8}r^4f(g+f)f'g' - \frac{1}{4}r^4fg(g-f)f''\Big) \\
&+ \frac{\lambda}{r^4fg}\big(4\ell(\ell+d-3)(1-g)gf^2 + 2(d-4)(d-5)(1-g)g^2f^2 \\
&+ (d-4)rf^2gf' + 2r\ell(\ell+d-3)f^2gf' + (d-3)(d-4)rf^2g^2f' \\
&+ \frac{1}{2}(d-6)r^2f^2gf'^2 + 2r^2fg^2f'^2 + (d-4)rf^2gg' - 5(d-4)rf^2g^2g' \\
&+ \left(d-\frac{7}{2}\right)r^2f^2gf'g' + \frac{1}{4}r^3f^2f'^2g' - \frac{1}{2}(d-1)r^2f^2gg'^2 - \frac{1}{2}r^3f^2f'g'^2 \\
&+ \frac{1}{4}r^3f^2g'^3 + (d-2)r^2f^2g^2f'' + \frac{1}{2}r^3f^2gg'f'' - 2r^2f^2g^2g'' \\
&+ \frac{1}{2}r^3f^2gf'g'' - r^3f^2gg'g''\Big)
\end{aligned}$$

Study of the stability

- That was the potential for tensor–type gravitational perturbations of any kind of static, spherically symmetric $\mathcal{R}^2$ string–corrected black hole in d–dimensions.
- Solutions of the form $\Phi(x,t) = e^{i\omega t}\phi(x)$;
- The master equation is then written in the Schrödinger form,

$$\left[-\frac{d^2}{dx^2} + V\right]\phi(x) =: A\phi(x) = \omega^2\phi(x);$$

- A solution to the field equation is then stable if the operator A has no negative eigenvalues (Gibbons, Hartnoll, 2002; Ishibashi, Kodama, 2003; Dotti, Gleiser, 2005).

"S-deformation" approach

Stability means positivity (for every possible ϕ) of the following inner product:

$$\begin{aligned}\langle \phi, A\phi \rangle &= \int_{-\infty}^{+\infty} \overline{\phi}(x) \left[-\frac{d^2}{dx^2} + V \right] \phi(x)\, dx \\ &= \int_{-\infty}^{+\infty} \left[\left| \frac{d\phi}{dx} \right|^2 + V \left| \phi \right|^2 \right] dx \\ &= \int_{-\infty}^{+\infty} \left[\left| D\phi \right|^2 + \widetilde{V} \left| \phi \right|^2 \right] dx\end{aligned}$$

with $D = \frac{d}{dx} + S$, $\widetilde{V} = V + \sqrt{fg}\frac{dS}{dr} - S^2$.

"S-deformation" approach (cont.)

- Taking $S = -\frac{\sqrt{fg}}{k}\frac{dk}{dr}$ we are left with

$$\langle \phi, A\phi \rangle = \int_{-\infty}^{+\infty} |D\phi|^2 \, dx + \int_{-\infty}^{+\infty} \frac{Q(r)}{\sqrt{fg}} |\phi|^2 \, dx,$$

with

$$Q = \frac{\ell(-3+d+\ell)f\left(r^2 + 4\lambda(1-g)\right) + r^3(g-f)f'}{r^3\left(r - 2\lambda f'\right)}$$

(after using equations of motion).

Stability condition

- The second term of $\langle \phi, A\phi \rangle$ can be written as

$$\int_{R_H}^{+\infty} Q(r) \frac{|\phi|^2}{\sqrt{fg}} dr.$$

- For $r > R_H$, $f(r), g(r) > 0$.
- This condition keeps valid with α' corrections as long as the black hole in consideration is *large*, i.e. $R_H \gg \sqrt{\lambda}$, which is true in string perturbation theory.
- This way the perturbative stability of a given black hole solution, with respect to tensor–type gravitational perturbations, follows if and only if one has $Q(r) > 0$ for $r \geq R_H$.

The Callan-Myers-Perry black hole

- Metric of the type
  $d\,s^2 = -f(r)\,d\,t^2 + g^{-1}(r)\,d\,r^2 + r^2\,d\,\Omega^2_{d-2}$;
- $f(r) = g(r) = g_0(r)\left[1 - \lambda \frac{(d-3)(d-4)}{2}\frac{R_H^{d-5}}{r^{d-1}}\frac{r^{d-1}-R_H^{d-1}}{r^{d-3}-R_H^{d-3}}\right]$;
- $\alpha' = 0$: Schwarzschild-Tangherlini solution;
- the only free parameter is the horizon radius R_H (secondary hair), which is not changed;
- dilaton vanishes classically and only gets α'-corrections (1988).

Leading α'-corrected dilaton

$$\begin{aligned}\varphi(r) &= \frac{\phi(r)}{\lambda} = \frac{(d-2)^2}{4R_H^2}\ln\left(1-\left(\frac{R_H}{r}\right)^{d-3}\right) - \frac{(d-3)(d-2)^2}{8(d-1)r^2}\left[(d-1)\right. \\ &+ +2\left(\frac{R_H}{r}\right)^{d-3} - 2\frac{d-1}{d-3}\left(\frac{r}{R_H}\right)^2 B\left(\left(\frac{R_H}{r}\right)^{d-3}; \frac{2}{d-3}, 0\right)\left.\right] < 0, \\ \varphi'(r) &= \frac{(d-3)(d-2)^2}{4}\frac{R_H^{d-3}}{r^{d-2}}\frac{1-\left(\frac{R_H}{r}\right)^{d-1}}{1-\left(\frac{R_H}{r}\right)^{d-3}} > 0\end{aligned}$$

with $B(x;\,a,b) = \int_0^x t^{a-1}\,(1-t)^{b-1}\,dt$ (Moura, 2010).
At the horizon,

$$\phi\left(R_H\right) = -\frac{\lambda}{R_H^2}\frac{(d-2)^2}{8(d-1)}\left(d^2-2d+2(d-1)\left(\psi^{(0)}\left(\frac{2}{d-3}\right)+\gamma\right)-3\right),$$

with

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)},\ \psi^{(n)}(z) = \frac{d^n\,\psi(z)}{d\,z^n},\ \gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right).$$

Dilatonic BH and compactified strings

- Metric in $d_s = 10$ (or 26) dimensions of the type

$$d\,s^2 = -f(r)\,d\,t^2 + g^{-1}(r)\,d\,r^2 + r^2\,d\,\Omega^2_{d-2} + h(\phi)\,g_{mn}(y)\,d\,y^m\,d\,y^n;$$

- Solution:

$$\begin{aligned} h(\phi) &= \left(1 - \frac{2}{d_s - 2}\phi\right)^2; \\ g(r) &= \left(1 - \left(\frac{R_H}{r}\right)^{d-3}\right)\left(1 - \frac{(d-3)(d-4)}{2}\,\frac{\lambda}{R_H^2}\left(\frac{R_H}{r}\right)^{d-3}\,\frac{1 - \left(\frac{R_H}{r}\right)^{d-1}}{1 - \left(\frac{R_H}{r}\right)^{d-3}}\right) \\ f(r) &= g(r) + 4\left(1 - \left(\frac{R_H}{r}\right)^{d-3}\right)\frac{d_s - d}{(d_s - 2)^2}\left(\phi - r\phi'\right). \end{aligned}$$

(Moura, 2011).

Stability of the dilatonic BH

$$\frac{Q}{F} = \left.\frac{Q}{F}\right|_0 + \lambda \left.\frac{Q}{F}\right|_1 .$$

$$r^4 \sqrt{fg}\, \left.\frac{Q}{F}\right|_1 = \frac{2}{r^2} \frac{\ell\,(\ell + d - 3)}{r} f_0^T \left(2 \frac{1 - f_0^T}{r} + f_0^{T\prime} \right) .$$

$$2 \frac{1 - f_0^T}{r} + f_0^{T\prime} = (d-3) \frac{R_H^{d-3}}{r^{d-2}} > 0.$$

$$r^2 \sqrt{fg}\, \left.\frac{Q}{F}\right|_0 = \frac{\ell\,(\ell + d - 3)}{r^2} f + \frac{(g - f)\, f'}{r} > 0.$$

The dilatonic black hole is stable under tensor perturbations in d dimensions.

From fundamental strings to black holes

- Consider an excited fundamental string with mass M in $d-$ dimensional flat spacetime, with momentum number n and winding w on an internal circle of radius R.
- String correspondence principle (Susskind, 1993)
- When the string interaction is strong enough, such string forms a black hole with with mass M and two electric charges that can be parameterized in terms of left and right-handed momenta (p_L, p_R):

$$p_{L,R} = \frac{n}{R} \mp \frac{wR}{\alpha'}.$$

Construction of the solution

- Idea: to add momentum and winding charges by "lifting" the metric to an additional dimension whose coordinate will be denoted by x.
- This means to produce a uniform black string.
- Take the additional dimension to be compact.
- Add the momentum charge: perform a boost in the x direction, which after Kaluza-Klein (KK) reduction would give one $U(1)$ charge.
- T-dualize in the x direction: a $(d+1)$-dimensional black string winding around the x circle. (Reducing to d dimensions would give a black hole with winding charge.)
- A second boost of the $(d+1)$-dimensional black string in the x direction gives back momentum charge.

Construction of the solution

- Then a reduction to d dimensions generates the black hole with generic momentum and winding charges generated by the boost parameters α_n, α_w.
- One works with the low energy effective action

$$I_{eff} = \frac{1}{16\,\pi\,G_d} \int \sqrt{-g}\, e^{-2\,\phi} \left(\mathcal{R} + 4\,(\nabla\phi)^2 - \frac{1}{12} H^2 \right)\, d^d x,$$

$H_{\alpha\beta\gamma} = 3\,\partial_{[\alpha} B_{\beta\gamma]}$,
and takes the Tangherlini black hole (other fields vanish). T-duality will turn on the additional fields.

The solution (Horowitz, Polchinski, 1997)

- Metric of the type
  $d\,s^2 = -f_0(r)\,d\,t^2 + g_0^{-1}(r)\,d\,r^2 + r^2\,d\,\Omega^2_{d-2};$
- $g_0(r) := 1 - \left(\frac{R_H}{r}\right)^{d-3};$
- $f_0(r) = \frac{g_0(r)}{\Delta(\alpha_n)\Delta(\alpha_w)},\ \Delta\,(x) := 1 + \left(\frac{R_H}{r}\right)^{d-3}\sinh^2 x$
- $\alpha_n, \alpha_w = 0:$ Tangherlini black hole.
- Dilaton: $\phi(r) = \phi_0 - \frac{1}{4}\log\Delta\,(\alpha_n) - \frac{1}{4}\log\Delta\,(\alpha_w)\,.$
- Two abelian gauge fields
  $A^n_t = \frac{1}{2}\left(\frac{R_H}{r}\right)^{d-3}\frac{\sinh 2\alpha_n}{\Delta(\alpha_n)}, A^w_t = \frac{1}{2}\left(\frac{R_H}{r}\right)^{d-3}\frac{\sinh 2\alpha_w}{\Delta(\alpha_w)}.$

The CMP black hole in the string frame

- $\phi(r) := \lambda\,\varphi(r)$ (same EOM as before to first order in λ);
- $g_{tt} = -g_0\;(1 + 2\,\lambda\,\mu(r))\,,\;\; g_{rr} = g_0^{-1}\;(1 + 2\,\lambda\,\epsilon(r))\,,$

$$\begin{aligned}
\epsilon(r) &= \frac{(d-3)R_H^{d-5}}{4\left(r^{d-3} - R_H^{d-3}\right)}\left[\frac{(d-2)(d-3)}{2} - \frac{2(2d-3)}{d-1}\right. \\
&+ \left.(d-2)\left(\psi^{(0)}\left(\frac{2}{d-3}\right) + \gamma\right) + d\left(\frac{R_H}{r}\right)^{d-1} + \frac{4R_H^2}{d-2}\varphi(r)\right] \\
\mu(r) &= -\epsilon(r) + \frac{2}{d-2}\left(\varphi(r) - r\,\varphi'(r)\right).
\end{aligned}$$

- This is the solution we will boost and T-dualize.

Transformations to the CMP solution

- Boost in the additional direction x with parameter α :

$$
\begin{aligned}
g^{\alpha}_{tt} &= \cosh^2(\alpha) g_{tt} + \sinh^2(\alpha), \\
g^{\alpha}_{xt} &= \sinh(\alpha)\cosh(\alpha)\left(g_{tt}+1\right), \\
g^{\alpha}_{xx} &= \sinh^2(\alpha) g_{tt} + \cosh^2(\alpha),
\end{aligned}
$$

- α'-corrected Buscher rules (Kaloper, Meissner (1997)):

$$
\begin{aligned}
g^{T}_{tt} &= g_{tt} - \frac{g^2_{xt}}{g_{xx}},\ g^{T}_{xx} = \frac{1}{g_{xx}}\left(1 + \frac{\lambda\,(g_{xx,r})^2}{g^2_{xx}\,g_{rr}} + \frac{\lambda\, g^{tt}\, g^2_{xx}\,(\partial_r V)^2}{g_{rr}}\right), \\
B^{T}_{xt} &= \frac{g_{xt}}{g_{xx}} - \frac{\lambda\,\partial_r V\, g_{xx,r}}{g_{rr}\, g_{xx}},\ \phi^{T} = \phi + \frac{1}{4}\ln\left(\frac{g^{T}_{xx}}{g_{xx}}\right), \\
V &\equiv \frac{g_{xt}}{g_{xx}}\,.
\end{aligned}
$$

- The rest of the metric components and other fields do not change.

First boost and T–duality

We first perform a boost on the CMP solution with α_w as a boost parameter which can be interpreted as related to the winding modes after a subsequent T-duality:

$$
\begin{aligned}
g_{tt}^{T,\alpha_w} &= -\frac{g_0}{\Delta(\alpha_w)}\Big[1+\frac{2\lambda\,\mu(r)\ \cosh^2\alpha_w}{\Delta(\alpha_w)}\Big],\\
g_{xx}^{T,\alpha_w} &= \frac{1}{\Delta(\alpha_w)}\Big[1+\frac{2\,\lambda\,\mu(r)\,g_0\ \sinh^2\alpha_w}{\Delta(\alpha_w)}-\frac{\lambda\,(d-3)^2 R_H^{2\,(d-3)}\ \sinh^2\alpha_w^2}{r^{2\,(d-2)}\Delta(\alpha_w)}\Big],\\
B_{xt}^{T,\alpha_w} &= \frac{1}{2}\left(\frac{R_H}{r}\right)^{d-3}\frac{\sinh 2\alpha_w}{\Delta\,(\alpha_w)}\Big[1-\frac{2\lambda\,\mu(r)R_H^{d-3}g_0}{r^{d-3}\Delta(\alpha_w)}-\frac{\lambda\,(d-3)^2\,g_0\,R_H^{d-3}}{r^{d-1}\,\Delta(\alpha_w)^2}\Big],\\
\phi^{T,\alpha_w} &= -\frac{1}{2}\ \ln(\Delta(\alpha_w))\\
&+ \lambda\Big[1+\varphi(r)+\frac{\mu(r)\,g_0\ \sinh^2\alpha_w}{\Delta(\alpha_w)}-\frac{(d-3)^2 R_H^{2(d-3)}\ \sinh^2\alpha_w}{4\,r^{2\,(d-2)}\Delta(\alpha_w)}\Big].
\end{aligned}
$$

The rest of the components remain unchanged.

Second boost and dimensional reduction

We perform the second boost in the x direction with a boost parameter α_n, and then we reduce to d dimensions (Giveon, Gorbonos (2006); Giveon, Gorbonos, Stern (2010)):

$$\begin{aligned}
A_t^n &= \frac{\sinh(\alpha_n)\,\cosh(\alpha_n)\left(g_{xx}^{T,\alpha_w}+g_{tt}^{T,\alpha_w}\right)}{\cosh^2(\alpha_n)\,g_{xx}^{T,\alpha_w}+\sinh^2(\alpha_n)\,g_{tt}^{T,\alpha_w}} \\
&= \frac{1}{2}\left(\frac{R_H}{r}\right)^{d-3}\frac{\sinh 2\alpha_n}{\Delta\,(\alpha_n)}\Big[1-\frac{2\,\lambda\,\mu(r)\,r^{d-3}\,g_0}{R_H^{d-3}\Delta(\alpha_n)}-\frac{\lambda\,(d-3)^2\,R_H^{d-3}\,g_0\,\sinh^2\alpha_w}{r^{d-1}\Delta(\alpha_n)\,\Delta(\alpha_w)}\Big], \\
A_t^w &= B_{xt}^{T,\alpha_w}, \\
e^{-2\,\phi} &= \sqrt{\Delta(\alpha_n)\,\Delta(\alpha_w)}\,\Bigg[1-2\,\lambda\,\varphi(r)-\lambda\mu(r)g_0\left(\frac{\sinh^2\alpha_n}{\Delta(\alpha_n)}+\frac{\sinh^2\alpha_w}{\Delta(\alpha_w)}\right) \\
&- \frac{\lambda}{R_H^2}\,\frac{(d-3)^2\,R_H^{2(d-2)}\,g_0\sinh^2\alpha_n\sinh^2\alpha_w}{2r^{2\,(d-2)}\Delta(\alpha_w)\Delta(\alpha_n)}\Bigg].
\end{aligned}$$

Final metric (string frame)

$$
\begin{aligned}
g_{tt} &= \frac{g_{xx}^{T,\alpha_w}\, g_{tt}^{T,\alpha_w}}{\cosh^2(\alpha_n)\, g_{xx}^{T,\alpha_w} + \sinh^2(\alpha_n)\, g_{tt}^{T,\alpha_w}} \\
&= -\frac{g_0}{\Delta(\alpha_n)\,\Delta(\alpha_w)} \left[1 + \frac{2\,\lambda\,\mu(r)}{\Delta(\alpha_n)\,\Delta(\alpha_w)} - \frac{2\,\lambda\,\mu(r)\, r^{2(d-3)}\, \sinh^2(\alpha_n)\, \sinh^2(\alpha_w)}{R_H^{2(d-3)}\Delta(\alpha_n)\,\Delta(\alpha_w)} \right. \\
&+ \left. 2\lambda\mu(r) \left(\frac{\sinh^2\alpha_n}{\Delta(\alpha_n)} + \frac{\sinh^2\alpha_w}{\Delta(\alpha_w)} \right) + \frac{(d-3)^2\,\lambda\, R_H^{2(d-3)} g_0 \sinh^2(\alpha_n) \sinh^2(\alpha_w)}{r^{2(d-2)}\Delta(\alpha_n)\,\Delta(\alpha_w)} \right], \\
g_{rr} &= g_0^{-1}\, (1 + 2\,\lambda\,\epsilon(r))\,.
\end{aligned}
$$

The other metric components remain unchanged.

Final metric (Einstein frame)

$$
\begin{aligned}
f(r) &= f_0^I(r)\left(1+\frac{\lambda}{R_H^2}f_c^I(r)\right),\ g(r)=f_0^I(r)\left(1+\frac{\lambda}{R_H^2}g_c^I(r)\right),\\
f_0^I &= \frac{f_0^T}{\sqrt{\Delta(\alpha_n)\Delta(\alpha_w)}},\\
f_c^I(r) &= \frac{1}{2\Delta(\alpha_n)\Delta(\alpha_w)}\Bigg(2\left(2-f_0^T\right)\left(\Delta(\alpha_n)\sinh^2(\alpha_w)+\Delta(\alpha_w)\sinh^2(\alpha_n)\right)\mu(r)\\
&+ 4\left(1-\left(\frac{R_H}{r}\right)^{2(d-3)}\sinh^2(\alpha_w)\sinh^2(\alpha_n)\right)\mu(r)\\
&+ (d-3)^2 f_0^T\left(\frac{R_H}{r}\right)^{2(d-2)}\sinh^2(\alpha_w)\sinh^2(\alpha_n)-4\Delta(\alpha_n)\Delta(\alpha_w)\varphi(r)\Bigg),\\
g_c^I(r) &= \frac{1}{2\Delta(\alpha_n)\Delta(\alpha_w)}\Bigg(2\left(\Delta(\alpha_n)\sinh^2(\alpha_w)+\Delta(\alpha_w)\sinh^2(\alpha_n)\right)\mu(r)f_0^T\\
&+ (d-3)^2 f_0^T\left(\frac{R_H}{r}\right)^{2(d-2)}\sinh^2(\alpha_w)\sinh^2(\alpha_n)+4\Delta(\alpha_n)\Delta(\alpha_w)\left(\varphi(r)-\epsilon(r)\right)\Bigg).
\end{aligned}
$$

Stability of the doubly charged BH

- Long calculation with simple final result:

$$g - f = \frac{4}{d-2} \lambda f_0^I(r) \left[(d-3)\, \varphi(r) + r\varphi'(r) \right].$$

- We just have to analyze $((d-3)\,\varphi(r) + r\varphi'(r))' = -\frac{(d-3)(d-2)^2}{4r} \left(\frac{R_H}{r}\right)^{2d-6} \frac{d-3-(d-1)\left(\frac{R_H}{r}\right)^2 + 2\left(\frac{R_H}{r}\right)^{d-1}}{\left(1-\left(\frac{R_H}{r}\right)^{d-3}\right)^2} < 0.$

- $g - f$ is a positive function which decreases to zero asymptotically.
- The doubly charged black hole is stable under tensor perturbations in d dimensions.

Some comments

- These results are to be compared with the corresponding ones in Lovelock theory, where several instabilities have been found (Dotti, Gleiser, 2005; Takahashi, Soda, 2010), depending on d.
- In Lovelock theories instabilities manifest themselves mainly on shorter scales, and there are domains of the parameters in which linear perturbation theory breaks down and is not applicable.
- Reason: Lovelock theories are seen as exact and not effective theories; the dependence of the solutions on the coupling constants goes beyond perturbation theory. The order at which they appear in the lagrangian does not matter for such dependence (often nonlinear).

Some comments (concl.)

- String–theoretical solutions are perturbative in α': their dependence on α' is of the same order in which α' appears on the lagrangian.
- This is why linear perturbation theory is fully applicable to these solutions we have studied.
- One must keep in mind that the stability we have shown is just perturbative.
- General question concerning perturbative string–theoretical black holes: do string α' corrections preserve the stability properties of the corresponding classical solutions?