RISC, Hagenberg, Austria

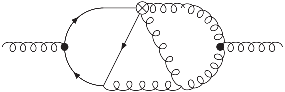
LHCPhenoNet School: Integration, Summation and Special Functions in QFT

Difference field algorithms for Feynman integrals

Carsten Schneider
Research Institute for Symbolic Computation (RISC)
J. Kepler University Linz, Austria

July 12, 2012

Consider a massive 3–loop ladder graph (Ablinger, Blümlein,Hasselhuhn,Klein, CS,Wißbrock, 2012)



$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}$$

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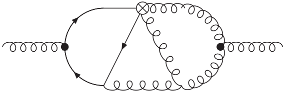

$$= F_{-3}(n)\varepsilon^{-3} + F_{-2}(n)\varepsilon^{-2} + F_{-1}(n)\varepsilon^{-1} + \boxed{F_0(n)}$$

Simplify

$$||$$

$$\sum_{j=0}^{n-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+n-3}\sum_{s=1}^{-l+n-q-3}\sum_{r=0}^{-l+n-q-s-3}(-1)^{-j+k-l+n-q-3}\times$$

$$\times\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{n-1}{j+2}\binom{-j+n-3}{q}\binom{-l+n-q-3}{s}\binom{-l+n-q-s-3}{r}r!(-l+n-q-r-s-3)!(s-1)!}{(-l+n-q-2)!(-j+n-1)(n-q-r-s-2)(q+s+1)}$$

$$\Big[4S_1(-j+n-1)-4S_1(-j+n-2)-2S_1(k)$$

$$-(S_1(-l+n-q-2)+S_1(-l+n-q-r-s-3)-2S_1(r+s))$$

$$+2S_1(s-1)-2S_1(r+s)\Big]$$ + **3 further 6–fold sums**

$$\boxed{F_0(n)} =$$

$$\frac{7}{12}S_1(n)^4 + \frac{(17n+5)S_1(n)^3}{3n(n+1)} + \Big(\frac{35n^2-2n-5}{2n^2(n+1)^2} + \frac{13S_2(n)}{2} + \frac{5(-1)^n}{2n^2}\Big)S_1(n)^2$$
$$+ \Big(-\frac{4(13n+5)}{n^2(n+1)^2} + \Big(\frac{4(-1)^n(2n+1)}{n(n+1)} - \frac{13}{n}\Big)S_2(n) + \Big(\frac{29}{3} - (-1)^n\Big)S_3(n)$$
$$+ \big(2+2(-1)^n\big)S_{2,1}(n) - 28S_{-2,1}(n) + \frac{20(-1)^n}{n^2(n+1)}\Big)S_1(n) + \Big(\frac{3}{4} + (-1)^n\Big)S_2(n)^2$$
$$- 2(-1)^n S_{-2}(n)^2 + S_{-3}(n)\Big(\frac{2(3n-5)}{n(n+1)} + \big(26+4(-1)^n\big)S_1(n) + \frac{4(-1)^n}{n+1}\Big)$$
$$+ \Big(\frac{(-1)^n(5-3n)}{2n^2(n+1)} - \frac{5}{2n^2}\Big)S_2(n) + S_{-2}(n)\Big(10S_1(n)^2 + \Big(\frac{8(-1)^n(2n+1)}{n(n+1)}$$
$$+ \frac{4(3n-1)}{n(n+1)}\Big)S_1(n) + \frac{8(-1)^n(3n+1)}{n(n+1)^2} + \big(-22+6(-1)^n\big)S_2(n) - \frac{16}{n(n+1)}\Big)$$
$$+ \Big(\frac{(-1)^n(9n+5)}{n(n+1)} - \frac{29}{3n}\Big)S_3(n) + \Big(\frac{19}{2} - 2(-1)^n\Big)S_4(n) + \big(-6+5(-1)^n\big)S_{-4}(n)$$
$$+ \Big(-\frac{2(-1)^n(9n+5)}{n(n+1)} - \frac{2}{n}\Big)S_{2,1}(n) + \big(20+2(-1)^n\big)S_{2,-2}(n) + \big(-17+13(-1)^n\big)S_{3,1}(n)$$
$$- \frac{8(-1)^n(2n+1)+4(9n+1)}{n(n+1)}S_{-2,1}(n) - \big(24+4(-1)^n\big)S_{-3,1}(n) + \big(3-5(-1)^n\big)S_{2,1,1}(n)$$
$$+ 32S_{-2,1,1}(n) + \left(\frac{3}{2}S_1(n)^2 - \frac{3S_1(n)}{n} + \frac{3}{2}(-1)^n S_{-2}(n)\right)\zeta(2)$$

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Simplify

$$\sum_{k=1}^{n} S_1(k)$$

where $S_1(k) = \sum_{i=1}^{k} \frac{1}{i}$

Telescoping

GIVEN $f(k) = S_1(k)$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Telescoping

GIVEN $f(k) = S_1(k)$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

We compute

$$g(k) = (S_1(k) - 1)k.$$

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GIVEN $f(k) = S_1(k)$.
FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Summing this equation over k from 1 to n gives

$$\boxed{\sum_{k=1}^{n} S_1(k)} = \boxed{g(n+1) - g(1)}$$

$$= (S_1(n+1) - 1)(n+1).$$

Telescoping in the given difference field

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

Telescoping in the given difference field

FIND a closed form for

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A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1, \qquad\qquad \mathcal{S}\, k = k + 1,$$

Telescoping in the given difference field

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)(h)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1, \qquad \mathcal{S}\, k = k + 1,$$

$$\sigma(h) = h + \frac{1}{k+1}, \qquad \mathcal{S}\, S_1(k) = S_1(k) + \frac{1}{k+1}.$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = h.}$$

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FIND $g \in \mathbb{F}$:

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We compute

$$g = (h-1)k \in \mathbb{F}.$$

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This gives

$$\boxed{g(k+1) - g(k) = S_1(k)}$$

with

$$g(k) = (S_1(k) - 1)k.$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

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We compute

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This gives

$$\boxed{g(k+1) - g(k) = S_1(k)}$$

with

$$g(k) = (S_1(k) - 1)k.$$

Hence,

$$(S_1(n+1) - 1)(n+1) = \sum_{k=1}^{n} S_1(k).$$

The basic summation algorithm

(a simplified version of Karr's algorithm, 1981)

CONSTRUCT a difference field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}$$

- with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

CONSTRUCT a difference field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)$$

- with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$
$$\sigma(t_1) = a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \qquad\qquad f_1 \in \mathbb{K}$$

CONSTRUCT a difference field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)$$

- with an automorphism

$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{K} \\ \sigma(t_1) &= a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \qquad f_1 \in \mathbb{K} \\ \sigma(t_2) &= a_2\, t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, \qquad f_2 \in \mathbb{K}(t_1) \end{aligned}$$

CONSTRUCT a difference field $(\mathbb{F},\sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$$

- with an automorphism

$$\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(t_1) &= a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, && f_1 \in \mathbb{K} \\
\sigma(t_2) &= a_2\, t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, && f_2 \in \mathbb{K}(t_1) \\
&\vdots \\
\sigma(t_e) &= a_e\, t_e + f_e, \quad a_e \in \mathbb{K}(t_1,\dots,t_{e-1})^*, && f_e \in \mathbb{K}(t_1,\dots,t_{e-1})
\end{aligned}$$

CONSTRUCT a difference field $(\mathbb{F}, \sigma)$:

- a rational function field

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\end{aligned}$$

GIVEN $f \in \mathbb{F}$;
FIND $g \in \mathbb{F}$ such that

$$\sigma(g) - g = f.$$

CONSTRUCT a $\Pi\Sigma$-field $(\mathbb{F},\sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$$

- with an automorphism

$$\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(t_1) &= a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, && f_1 \in \mathbb{K} \\
\sigma(t_2) &= a_2\, t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, && f_2 \in \mathbb{K}(t_1) \\
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\sigma(t_e) &= a_e\, t_e + f_e, \quad a_e \in \mathbb{K}(t_1,\dots,t_{e-1})^*, && f_e \in \mathbb{K}(t_1,\dots,t_{e-1})
\end{aligned}$$

such that

$$\mathrm{const}_\sigma\mathbb{F} = \{c \in \mathbb{K}(t_1)(t_2)\dots(t_e) | \sigma(c) = c\} = \mathbb{K}.$$

GIVEN $f \in \mathbb{F}$;
FIND $g \in \mathbb{F}$ such that

$$\sigma(g) - g = f.$$

Telescoping in the given difference field

FIND a closed form for

$$\sum_{k=1}^{n} S_1(k).$$

A $\Pi\Sigma^*$-field for the summand

$$\boxed{\text{const}_\sigma \mathbb{F} = \mathbb{Q}}$$

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)(h)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1, \qquad \mathcal{S}\, k = k + 1,$$

$$\sigma(h) = h + \frac{1}{k+1}, \qquad \mathcal{S}\, S_1(k) = S_1(k) + \frac{1}{k+1}.$$

FIND $g \in \mathbb{Q}(k)(h)$:

$$\sigma(g) - g = h.$$

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$$\sigma(g) - g = h.$$

Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]^*$:

$d = 1$

$$\forall g \in \mathbb{Q}(k)(h): \quad \sigma(g) - g = h \quad \Rightarrow \quad g\,d \in \mathbb{Q}(k)[h].$$

FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\sigma(\frac{g'}{d}) - \frac{g'}{d} = h.$$

FIND $g \in \mathbb{Q}(k)(h)$:

$$\sigma(g) - g = h.$$

Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]^*$:

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FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\sigma\Big(\frac{g'}{d}\Big) - \frac{g'}{d} = h.$$

Degree bound: COMPUTE $b \geq 0$:

$$b = 2$$

$$\forall g \in \mathbb{Q}(k)[h] \quad \sigma(g) - g = h \quad \Rightarrow \quad \deg(g) \leq b.$$

FIND $g \in \mathbb{Q}(k)(h)$:

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$b = 2$

Polynomial Solution: FIND

$$g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h].$$

g=hk-k

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\sigma(g) - g = h$$

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\left[\sigma(g_2)\left(h+\tfrac{1}{k+1}\right)^2+\sigma(g_1 h+g_0)\right]$$
$$-\left[g_2\, h^2+g_1 h+g_0\right]=h$$

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h+\tfrac{1}{k+1}\big)^2+\sigma(g_1 h+g_0)\Big] - \big[g_2\,h^2+g_1 h+g_0\big] = h$$

coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

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$g_2 = c \in \mathbb{Q}$

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coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\,h+g_0) - (g_1\,h+g_0) = h - c\Big[\tfrac{2h(k+1)+1}{(k+1)^2}\Big]$$

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h+\tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\,h^2 + g_1 h + g_0\big] = h$$

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coeff. comp.

$$\sigma(g_1) - g_1 = 1 - c\,\frac{2}{k+1}$$

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \Big[g_2\,h^2 + g_1 h + g_0\Big] = h$$

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$c = 0,\quad g_1 = k + d,\quad d \in \mathbb{Q}$

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coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

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$$\sigma(g_1\,h + g_0) - (g_1\,h + g_0) = h - c\Big[\tfrac{2h(k+1)+1}{(k+1)^2}\Big]$$

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$$\boxed{\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{k+1}}$$

$c = 0,\quad g_1 = k + d,\ d \in \mathbb{Q}$

$$\boxed{\sigma(g_0) - g_0 = -1 - d\frac{1}{k+1}}$$

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\left[\sigma(g_2)\left(h+\tfrac{1}{k+1}\right)^2+\sigma(g_1h+g_0)\right] - \left[g_2\,h^2+g_1h+g_0\right] = h$$

coeff. comp.

g=hk-k

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\,h+g_0) - (g_1\,h+g_0) = h - c\left[\tfrac{2h(k+1)+1}{(k+1)^2}\right]$$

coeff. comp.

$$\boxed{\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{k+1}}$$

$c = 0$, $g_1 = k + d$, $d \in \mathbb{Q}$

$$\boxed{\sigma(g_0) - g_0 = -1 - d\frac{1}{k+1}}$$

$\leftarrow$ $g_0 = -k$, $d = 0$

Difference equations in difference fields

Let $(\mathbb{F}, \sigma)$ be a $\Pi\Sigma$-field with constant field $\mathbb{K}$

Telescoping

- Given $f \in \mathbb{F}$.
- Find $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = f.}$$

Difference equations in difference fields

Let $(\mathbb{F}, \sigma)$ be a $\Pi\Sigma$-field with constant field $\mathbb{K}$

Telescoping

- Given $f \in \mathbb{F}$.
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$$\boxed{\sigma(g) - g = f.}$$

$\downarrow$ $\uparrow$

Parameterized Telescoping

- Given $f_0, \dots, f_d \in \mathbb{F}$.
- Find all $c_0, \dots, c_d \in \mathbb{K}$, $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = c_0\, f_0 + \dots + c_d\, f_d.}$$

Difference equations in difference fields

Let $(\mathbb{F},\sigma)$ be a $\Pi\Sigma$-field with constant field $\mathbb{K}$

Telescoping

- Given $f\in\mathbb{F}$.
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Parameterized Telescoping

- Given $f_0,\dots,f_d\in\mathbb{F}$.
- Find all $c_0,\dots,c_d\in\mathbb{K}$, $g\in\mathbb{F}$:

$$\boxed{\sigma(g)-g=c_0\,f_0+\dots+c_d\,f_d.}$$

$$\downarrow \qquad\qquad \uparrow$$

Parameterized first order difference equation

- Given $f_0,\dots,f_d\in\mathbb{F}$, $a_0,a_1\in\mathbb{F}$.
- Find all $c_0,\dots,c_d\in\mathbb{K}$, $g\in\mathbb{F}$:

$$\boxed{a_1\,\sigma(g)+a_0\,g=c_0\,f_0+\dots+c_d\,f_d.}$$

Constructing $\Pi\Sigma$-fields

CONSTRUCT a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}$$

- with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

such that

$$\text{const}_\sigma \mathbb{F} = \{c \in \mathbb{K} \qquad |\sigma(c) = c\} = \mathbb{K}.$$

CONSTRUCT a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)$$

- with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$
$$\sigma(t_1) = a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, \qquad\qquad f_1 \in \mathbb{K}$$

such that

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\sigma(t_e) &= a_e\, t_e + f_e, \quad a_e \in \mathbb{K}(t_1,\dots,t_{e-1})^*, && f_e \in \mathbb{K}(t_1,\dots,t_{e-1})
\end{aligned}$$

such that

$$\text{const}_\sigma\mathbb{F} = \{c \in \mathbb{K}(t_1)(t_2)\dots(t_e) | \sigma(c) = c\} = \mathbb{K}.$$

CONSTRUCT a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:

- a rational function field

$$\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$$

- with an automorphism

$$\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(t_1) &= a_1\, t_1 + f_1, \quad a_1 \in \mathbb{K}^*, & f_1 \in \mathbb{K} \\
\sigma(t_2) &= a_2\, t_2 + f_2, \quad a_2 \in \mathbb{K}(t_1)^*, & f_2 \in \mathbb{K}(t_1) \\
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GIVEN $f \in \mathbb{F}$;
FIND $g \in \mathbb{F}$ such that

$$\sigma(g) - g = f.$$

Construction of Σ^*-extensions

- Let $(\mathbb{F}, \sigma)$ be a difference field with constant field

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2. $\boxed{\exists g \in \mathbb{F} : \sigma(g) = g + f}$ No need for a Σ^*-extension!

Symbolic summation in $\Pi\Sigma$-fields

A difference field approach (M. Karr, 1981)

GIVEN a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ with $f \in \mathbb{F}$.

FIND $g \in \mathbb{F}$:

$$\sigma(g) - g = f.$$

A symbolic summation approach (see, e.g., J. Symb. Comput 2008; arXiv:0808.2543)

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appropriate = sum representations with optimal nesting depth

Example:

$$\sum_{k=1}^{n}\frac{\displaystyle\sum_{j=1}^{k}\frac{\displaystyle\sum_{i=1}^{j}\frac{1}{i}}{j}}{k}=\frac{1}{6}\left(\sum_{i=1}^{n}\frac{1}{i}\right)^3+\frac{1}{2}\left(\sum_{i=1}^{n}\frac{1}{i^2}\right)\left(\sum_{i=1}^{n}\frac{1}{i}\right)+\frac{1}{3}\sum_{i=1}^{n}\frac{1}{i^3}$$

depth 3 depth 1

$$\sum_{k=1}^{n}\frac{\displaystyle\sum_{j=1}^{k}\frac{\displaystyle\sum_{i=1}^{j}\frac{1}{i}}{j}}{k}$$

$$\sum_{k=1}^{n} \frac{\sum_{j=1}^{k} \frac{\sum_{i=1}^{j} \frac{1}{i}}{j}}{k}$$

h

$\Pi\Sigma$-field $(\mathbb{Q}(k)(h), \sigma)$ with

$$\sigma(k) = k + 1$$

$$\sigma(h) = h + \frac{1}{k+1}$$

$$\sum_{k=1}^{n}\frac{\displaystyle\sum_{j=1}^{k}\frac{\displaystyle\sum_{i=1}^{j}\frac{1}{i}}{j}}{k} \quad \longrightarrow \quad s$$

$\Pi\Sigma$-field $(\mathbb{Q}(k)(h)(s),\sigma)$ with

$$\sigma(k)=k+1$$
$$\sigma(h)=h+\frac{1}{k+1}$$
$$\sigma(s)=s+\frac{\sigma(h)}{k+1}$$

$$\sum_{k=1}^{n}\frac{\sum_{j=1}^{k}\frac{\sum_{i=1}^{j}\frac{1}{i}}{j}}{k}$$

$$t$$

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No simplification 

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$$\frac{1}{2}(h^2+h_2)$$

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$$\sum_{k=1}^{n} \frac{\sum_{j=1}^{k} \frac{\sum_{i=1}^{j} \frac{1}{i}}{j}}{k} \qquad \frac{1}{6}h^3 + \frac{1}{2}h_2^2 h + \frac{1}{3}h_3$$

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Example

$$\sum_{k=1}^{n} \frac{\sum_{j=1}^{k} \frac{\sum_{i=1}^{j} \frac{1}{i}}{j}}{k} = \frac{1}{6}\left(\sum_{i=1}^{n}\frac{1}{i}\right)^3 + \frac{1}{2}\left(\sum_{i=1}^{n}\frac{1}{i^2}\right)\left(\sum_{i=1}^{n}\frac{1}{i}\right) + \frac{1}{3}\sum_{i=1}^{n}\frac{1}{i^3}$$

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Apéry's proof (1979) of the irrationality of $\zeta(3)$ relies on the following fact:

$$a(n) = \sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2$$

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$$b(n) = \sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2 \left(S_3(n) + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n+m}{m} \binom{n}{m}} \right)$$

satisfy both the recurrence relation

$$(n+1)^3 A(n) - (2n+3)\left(17n^2 + 51n + 39\right) A(n+1) + (n+2)^3 A(n+2) = 0.$$

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Van der Poorten (1979) points out that Henri Cohen and Don Zagier showed this fact by

"some rather complicated but ingenious explanations"

based on the creative telescoping method.

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$a(n)$*-case:* trivial exercise by Zeilberger's algorithm (1991)

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$b(n)$*-case:* skilful application of computer algebra

1. Generalization of the Cohen/Zagier method in the WZ-setting (Zeilberger, 1993)
2. Multi-summation + holonomic closure properties (Chyzak/Salvy, 1998)

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$b(n)$-case: plain sailing (and not plane sailing) by Sigma

Appendix

Construction of Π-extensions

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