On Solutions of Recurrences I

Marko Petkovšek

Faculty of Mathematics and Physics
University of Ljubljana

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Outline

Summation and recurrences

$$s_n = \sum_{k=0}^{n-1} t_k$$

$$\Updownarrow$$

$$s_{n+1} - s_n = t_n; \quad s_0 = 0$$

$$\Updownarrow$$

$$t_n s_{n+2} - (t_n + t_{n+1}) s_{n+1} + t_{n+1} s_n = 0; \quad s_0 = 0, s_1 = t_0$$

What is a solution?

Representation of sequences

- *recursive:* $a_0, a_1, \ldots, a_{d-1}$ given,

$$a_n \;=\; F(a_{n-1}, a_{n-2}, \ldots, a_0, n) \quad \text{for } n \geq d$$

- *explicit:* $a_n = f(n) \quad \text{for } n \geq 0$

- *by generating function:*

$$G_a(x) \;=\; \sum_{n=0}^{\infty} a_n x^n$$

What is a solution?

Example

- *recursive:* $a_0 = 0,\ a_1 = 1,$

$$a_n = 2a_{n-1} - a_{n-2} \quad \text{for } n \geq 2$$

- *explicit:* $a_n = n \quad \text{for } n \geq 0$

- *by generating function:*

$$G_a(x) = \frac{x}{(1-x)^2}$$

C-recursive sequences

Notation:

K ... algebraically closed field of characteristic 0

Definition

A sequence $\langle a_n \rangle_{n=0}^{\infty} \in K^{\mathbb{N}}$ is *C-recursive* if there are $d \in \mathbb{N}$ and constants $c_1, c_2, \ldots, c_d \in K$, $c_d \neq 0$, such that

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_d a_{n-d}$$

for all $n \geq d$.

C-recursive sequences

Let $\langle a_n \rangle_{n=0}^{\infty} \in K^{\mathbb{N}}$ and $G_a(x) = \sum_{n=0}^{\infty} a_n x^n$.

Theorem

The following are equivalent:

1. $\langle a_n \rangle_{n=0}^{\infty}$ *is C-recursive,*
2. $a_n = \sum_{i=1}^{r} P_i(n)\, \alpha_i^n$ *where* $P_i \in K[n]$ *and* $\alpha_i \in K$*,*
3. $G_a(x) = \dfrac{P(x)}{Q(x)}$ *where* $P, Q \in K[x]$, $\deg P < \deg Q$ *and* $Q(0) \neq 0$.

C-recursive sequences

Theorem

C-recursive sequences are closed under the following binary operations $(a, b) \mapsto c$*:*

1. *addition:* $c_n = a_n + b_n$
2. *Hadamard (termwise) multiplication:* $c_n = a_n b_n$
3. *Cauchy multiplication (convolution):* $c_n = \sum_{i=0}^{n} a_i b_{n-i}$
4. *interlacing:* $\langle c_0, c_1, c_2, c_3, \ldots \rangle = \langle a_0, b_0, a_1, b_1, \ldots \rangle$

Remark

These operations extend naturally to an arbitrary nonzero finite number of operands.

C-recursive sequences

Theorem

C-recursive sequences are closed under the following unary operations $a \mapsto c$*:*

1. *scalar multiplication:* $c_n = \lambda a_n \quad (\lambda \in K)$
2. *(left) shift:* $c_n = a_{n+1}$
3. *indefinite summation:* $c_n = \sum_{k=0}^{n} a_k$
4. *multisection:* $c_n = a_{kn+r} \quad (k \in \mathbb{N},\ 0 \le r < k)$

C-recursive sequences

Example

- $a_n = n + 1$ is C-recursive,
- $a_n^{-1} = \dfrac{1}{n+1}$ is not C-recursive.

Question: When are a and $1/a$ are both C-recursive?

C-recursive sequences

Definition

A sequence $a \in K^{\mathbb{N}}$ is *geometric* if $a_0 \neq 0$ and $\exists q \in K^*$:

$$a_n \;=\; q\, a_{n-1}$$

for all $n \geq 1$ (equivalently: $a_n = a_0\, q^n$ for all $n \geq 0$).

Observation

a geometric $\implies$ a^{-1} geometric

Theorem

(Larson & Taft, 1990) Sequences a and $1/a$ both C-recursive $\iff$ a is the interlacing of one or more geometric sequences.

P-recursive sequences

Definition

A sequence $\langle a_n \rangle_{n=0}^{\infty} \in K^{\mathbb{N}}$ is *P-recursive* if there are $d \in \mathbb{N}$ and polynomials $p_0, p_1, \ldots, p_d \in K[n]$, $p_d \neq 0$, such that

$$p_d(n) a_{n+d} + p_{d-1}(n) a_{n+d-1} + \cdots + p_0(n) a_n = 0$$

for all $n \geq 0$.

Definition

A f.p.s. $f(x) \in K[[x]]$ is *D-finite* if there are $d \in \mathbb{N}$ and polynomials $q_0, q_1, \ldots, q_d \in K[x]$, $q_d \neq 0$, such that

$$q_d(x) f^{(d)}(x) + q_{d-1}(x) f^{(d-1)}(x) + \cdots + q_0(x) f(x) = 0.$$

P-recursive sequences

Let $\langle a_n \rangle_{n=0}^{\infty} \in K^{\mathbb{N}}$ and $G_a(x) = \sum_{n=0}^{\infty} a_n x^n$.

Theorem

The following are equivalent:

1. $\langle a_n \rangle_{n=0}^{\infty}$ *is P-recursive,*
2. $G_a(x)$ *is D-finite.*

P-recursive sequences

Theorem

P-recursive sequences are closed under the following operations:

1. *addition*
2. *Hadamard multiplication*
3. *Cauchy multiplication*
4. *interlacing*
5. *scalar multiplication*
6. *shift*
7. *indefinite summation*
8. *multisection*

P-recursive sequences

Example

- $a_n \ = \ 2^n + 1$ is P-recursive (even C-recursive),
- $b_n \ := \ a_n^{-1} \ = \ \dfrac{1}{2^n + 1}$ is not P-recursive.

Sketch of proof:

$$G_b(x) \ = \ \sum_{n=0}^{\infty} b_n x^n \ = \ \sum_{n=0}^{\infty} \frac{x^n}{2^n + 1}$$

radius of convergence:

$$r \ = \ \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|b_n|}} \ = \ \lim_{n\to\infty} \sqrt[n]{2^n + 1} \ = \ 2$$

P-recursive sequences

$$G_b(2x) = \sum_{n=0}^{\infty} \frac{2^n}{2^n+1} x^n = \sum_{n=0}^{\infty} \left(1 - \frac{1}{2^n+1}\right) x^n$$

$$G_b(2x) = \frac{1}{1-x} - G_b(x) \qquad (1)$$

$x = 1$: $\frac{1}{1-x}$ singular, G_b regular $\Longrightarrow$ G_b singular at $x = 2$

$x = 2$: $\frac{1}{1-x}$ regular, G_b singular $\Longrightarrow$ G_b singular at $x = 4$

$x = 4$: $\frac{1}{1-x}$ regular, G_b singular $\Longrightarrow$ G_b singular at $x = 8$

$\cdots$

By induction on k: $G_b(x)$ singular at $x = 2^k$ for all $k \in \mathbb{N}$

$\implies$ G_b not D-finite

$\implies$ b not P-recursive $\square$

Question: When are a and $1/a$ are both P-recursive?

Definition

A sequence $a \in K^{\mathbb{N}}$ is *hypergeometric* if:

1. $\exists N \in \mathbb{N}$: $a_n \neq 0$ for all $n \geq N$,
2. $\exists p, q \in K[n] \setminus \{0\}$:

$$p(n)\, a_{n+1} + q(n)\, a_n \;=\; 0$$

for all $n \geq 0$.

P-recursive sequences

Observation

a hypergeometric $\implies$ a^{-1} *hypergeometric*

Theorem

(Singer, 1997) Sequences a and $1/a$ *both P-recursive* $\iff$ *a is the interlacing of one or more hypergeometric sequences.*

Notation:

$\mathcal{H}(K)$ ... hypergeometric sequences in $K^{\mathbb{N}}$

P-recursive sequences

Solving linear recurrences
with polynomial coefficients

Given: $d \in \mathbb{N}$ and $p_0, p_1, \ldots, p_d \in K[n]$, $p_d \neq 0$

Find: all *nice* solutions $a \in K^{\mathbb{N}}$ of

$$p_d(n)a_{n+d} + p_{d-1}(n)a_{n+d-1} + \cdots + p_0(n)a_n \ = \ 0$$

P-recursive sequences

P-recursive: $\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$

Liouvillian: $n!!$

d'Alembertian: $n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$

hypergeometric: $n!$

quasirational: $\frac{2^n}{n+1} + 1$

rational: $\frac{1}{n+1}$

C-recursive: $n2^n + 1$

polynomial: $n + 1$

geometric: 2^n

Recurrence operators

Notation:

$$E: K^{\mathbb{N}} \to K^{\mathbb{N}} \qquad \textit{(left) shift operator},$$
$$(Ea)_n = a_{n+1} \qquad \text{application of } E$$

$$(E^k a)_n = a_{n+k} \quad (k \in \mathbb{N})$$

Given $d \in \mathbb{N}$ and $p_0, p_1, \ldots, p_d \in K[n]$, $p_d \neq 0$:

$$L: K^{\mathbb{N}} \to K^{\mathbb{N}} \qquad \textit{linear recurrence operator},$$
$$L = \sum_{k=0}^{d} p_k E^k$$

$$(La)_n = \sum_{k=0}^{d} p_k(n)\, a_{n+k} \quad \text{application of } L$$

Recurrence operators

Notation:

$K[n]\langle E\rangle$ ... algebra of linear recurrence operators with polynomial coefficients

Commutation rule for *composition* of operators:

$$E \cdot p(n) = p(n+1)\, E$$

$$\sum_{k=0}^{d} p_k(n)E^k \cdot \sum_{j=0}^{e} q_j(n)E^j = \sum_{k=0}^{d}\sum_{j=0}^{e} p_k(n)q_j(n+k)E^{j+k}$$

Difference rings

Definition

A *difference ring* is a pair (K, σ) where:

- K is a commutative ring with multiplicative identity,
- $\sigma : \ K \to K$ is a ring automorphism.

If, in addition, K is a field then (K, σ) is a *difference field*.

Example

- $(K[x], \sigma)$ with $\sigma x = x + 1$, $\sigma|_K = \mathrm{id}_K$ is a difference ring.
- $(K(x), \sigma)$ with $\sigma x = x + 1, \sigma|_K = \mathrm{id}_K$ is a difference field.
- $(K^{\mathbb{N}}, E)$ where $E : \ \langle a_0, a_1, a_2, \ldots \rangle \mapsto \langle a_1, a_2, a_3 \ldots \rangle$ is *not* a difference ring.

Difference rings

For $a, b \in K^{\mathbb{N}}$ define:

$$a \sim b \quad \iff \quad \exists N \in \mathbb{N} : \forall n \geq N : \; a_n = b_n$$

Notation:

$\mathcal{S}(K) \; = \; K^{\mathbb{N}} / \sim$ ... ring of germs of sequences

$\sigma : \mathcal{S}(K) \to \mathcal{S}(K)$ ... unique automorphism of $\mathcal{S}(K)$ defined by $\sigma \circ \varphi = \varphi \circ E$

Observation

$(\mathcal{S}(K), \sigma)$ is a difference ring.

Example

- $a = 0$ in $\mathcal{S}(K) \iff a_n = 0$ for all large enough n
- $a = b$ in $\mathcal{S}(K) \iff a_n = b_n$ for all large enough n
- $K[n]$, $K(n)$, $\mathcal{H}(K)$ naturally embed into $\mathcal{S}(K)$

Henceforth we work in $(\mathcal{S}(K), \sigma)$.

A. Polynomial solutions

Given: $L \in K[n]\langle\sigma\rangle, \ \ L \neq 0$

Find: a basis of the space $\{y \in K[n]; \ \ Ly = 0\}$

Outline of algorithm

1. Find an upper bound for $\deg y$.
2. Use the method of undetermined coefficients.

B. Rational solutions

Given: $L \in K[n]\langle\sigma\rangle, \; L \neq 0$

Find: a basis of the space $\{y \in K(n); \; Ly = 0\}$

Outline of algorithm

1. Find a universal denominator for y.
2. Find polynomial solutions of the equation satisfied by the numerator of y.

C. Hypergeometric solutions

Given: $L = \sum_{k=0}^{d} p_k \, \sigma^k \in K[n]\langle\sigma\rangle, \quad L \neq 0$

Find: a generating set for $\mathrm{Lin}(\{y \in \mathcal{H}(K); \quad Ly = 0\})$

Outline of algorithm

1. Construct the Riccati equation for $r = \frac{\sigma y}{y} \in K(n)$:

$$\sum_{k=0}^{d} p_k \prod_{j=0}^{k-1} \sigma^j r \;=\; 0 \tag{2}$$

C. Hypergeometric solutions

2 Use the ansatz

$$r \;=\; z\,\frac{a}{b}\,\frac{\sigma c}{c}$$

with $z \in K^*$, $a, b, c \in K[n]$ monic,
a, c coprime, $b, \sigma c$ coprime, $a, \sigma^k b$ coprime for all $k \in \mathbb{N}$:

$$\sum_{k=0}^{d} z^k p_k \left(\prod_{j=0}^{k-1} \sigma^j a\right) \left(\prod_{j=k}^{d-1} \sigma^j b\right) \sigma^k c \;=\; 0 \qquad (3)$$

C. Hypergeometric solutions

3. Construct a finite set of candidates for (a, b, z) using the following consequences of (3):

   - $a \mid p_0$,
   - $b \mid \sigma^{1-d} p_d$,
   - $$\sum_{\substack{0 \le k \le d \\ \deg P_k = m}} \mathrm{lc}(P_k) z^k = 0$$

   where $P_k = p_k \left(\prod_{j=0}^{k-1} \sigma^j a \right) \left(\prod_{j=k}^{d-1} \sigma^j b \right)$,
   $m = \max_{0 \le k \le d} \deg P_k$.

C. Hypergeometric solutions

4. For each candidate triple (a, b, z), find polynomial solutions c of the equation

$$\sum_{k=0}^{d} z^k P_k \, \sigma^k c \; = \; 0.$$

Example

(AMM problem no. 10375) Solve

$$y_{n+2} - 2(2n+3)^2 y_{n+1} + 4(n+1)^2(2n+1)(2n+3)y_n = 0.$$

$$\begin{aligned} p_2(n) &= 1 \\ p_1(n) &= -2(2n+3)^2 \\ p_0(n) &= 4(n+1)^2(2n+1)(2n+3) \end{aligned}$$

C. Hypergeometric solutions

1. Riccati equation:

$$p_2(n)\, r(n+1) r(n) \; + \; p_1(n)\, r(n) \; + \; p_0(n) \; = \; 0$$

2. plug in the ansatz:

$$\begin{array}{lllll} & z^2 & p_2(n) & a(n+1)\, a(n) & c(n+2) \\ + & z & p_1(n) & a(n)\, b(n+1) & c(n+1) \\ + & & p_0(n) & b(n+1)\, b(n) & c(n) \end{array} \quad = \quad 0$$

C. Hypergeometric solutions

3. candidates for (a, b, z):
   - $a(n) \mid 4(n+1)^2(2n+1)(2n+3)$
   - $b(n) \mid 1$

   Take, e.g., $a(n) = (n+1)(n+\frac{1}{2})$ and $b(n) = 1$.
   - $z^2 - 8z + 16 \;=\; (z-4)^2 \;=\; 0$

   Take $z = 4$.

C. Hypergeometric solutions

4 equation for c:

$$(n+2)c(n+2) - (2n+3)c(n+1) + (n+1)c(n) = 0$$

Polynomial solution: $c(n) = 1$

$$r(n) = z\frac{a(n)}{b(n)}\frac{c(n+1)}{c(n)} = 4(n+1)\left(n+\frac{1}{2}\right)$$

$$\frac{y_{n+1}}{y_n} = r(n) = (2n+1)(2n+2) \implies y_n = C(2n)!$$