Multiple zeta values and modular forms in quantum field theory

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In the first talk, I shall consider multiple zeta values (MZVs) and alternating Euler sums, exposing some of the wonderful mathematical structure of these objects and indicating where they arise in quantum field theory (QFT). In the second, I shall consider modular forms whose L-functions give remarkable evaluations of massive Feynman integrals in QFT.

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1 Multiple zeta values

1.1 Zeta values

For integer $s > 1$, the zeta values

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

divide themselves into two radically different classes. At *even* integers we have

$$\begin{aligned}
\zeta(2) &= \frac{\pi^2}{6} \\
\zeta(4) &= \frac{\pi^4}{90} \\
\zeta(6) &= \frac{\pi^6}{945} \\
\zeta(8) &= \frac{\pi^8}{9450} \\
\zeta(10) &= \frac{\pi^{10}}{93555}
\end{aligned}$$

and hence integer relations such as

$$5\zeta(4) - 2\zeta^2(2) = 0. \tag{1}$$

Yet no such relations have been found for *odd* arguments.

To prove (1), consider the wonderful formula

$$\frac{\cos(z)}{\sin(z)} = \sum_{n=-\infty}^{\infty} \frac{1}{z - n\pi}$$

in which the cotangent function is given by the sum of its pole terms, each with unit residue. Multiplying by z, to remove the singularity at $z = 0$, and then combining the terms with positive and negative n, we obtain

$$\frac{z\cos(z)}{\sin(z)} = 1 - 2z^2 \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2 - z^2}.$$

Expanding about $z = 0$ we obtain

$$\frac{1 - z^2/2! + z^4/4! + O(z^6)}{1 - z^2/3! + z^4/5! + O(z^6)} = 1 - 2\,\zeta(2)\frac{z^2}{\pi^2} - 2\,\zeta(4)\frac{z^4}{\pi^4} + O(z^6)$$

and easily prove that $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$.

1.2 Double sums

For integers $a > 1$ and $b > 0$, let

$$\zeta(a,b) = \sum_{m>n>0} \frac{1}{m^a n^b}$$

which is a multiple zeta value (MZV) with weight $a + b$ and depth 2. Then, when a and b are both greater than 1, the double sum in the product

$$\zeta(a)\zeta(b) = \sum_{m>0} \frac{1}{m^a} \sum_{n>0} \frac{1}{n^b}$$

can be split into 3 terms, with $m > n > 0$, $m = n > 0$ and $n > m > 0$. Hence

$$\zeta(a)\zeta(b) = \zeta(a,b) + \zeta(a+b) + \zeta(b,a). \tag{2}$$

There are further algebraic relations. Consider the sums

$$T(a,b,c) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{(j+k)^a j^b k^c}.$$

Multiplying the numerator by $(j+k) - j - k = 0$ we obtain

$$0 = T(a-1,b,c) - T(a,b-1,c) - T(a,b,c-1)$$

and hence by repeated application of

$$T(a,b,c) = T(a+1,b-1,c) + T(a+1,b,c-1)$$

we may reduce these Tornheim double sums to MZVs. For example

$$T(1,1,1) = 2\zeta(2,1).$$

We also have

$$T(1,1,1) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{(j+k)jk} = \sum_{j=1}^{\infty} \frac{1}{j^2} \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{j+k}\right).$$

But now the inner sum has only j terms and hence

$$T(1,1,1) = \sum_{j=1}^{\infty} \frac{1}{j^2} \sum_{n=1}^{j} \frac{1}{n} = \zeta(2,1) + \zeta(3).$$

Comparing the two results for $T(1,1,1)$, we find that

$$\zeta(2,1) = \zeta(3).$$

More generally, for $a > 1$, Euler found that

$$\zeta(a,1) = \frac{a}{2}\zeta(a+1) - \frac{1}{2}\sum_{b=2}^{a-1}\zeta(a+1-b)\zeta(b). \tag{3}$$

Moreover, Euler found the evaluation of all MZVs with odd weight and depth 2. For odd $a > 1$ and even $b > 0$ we have

$$\begin{aligned}\zeta(a,b) \quad &= \quad -\frac{1+C(a,b,a+b)}{2}\zeta(a+b) \\ &\quad + \sum_{k=1}^{(a+b-3)/2} C(a,b,2k+1)\zeta(a+b-2k-1)\zeta(2k+1)\end{aligned} \tag{4}$$

where

$$C(a,b,c) = \binom{c-1}{a-1} + \binom{c-1}{b-1}.$$

For example, we obtain

$$\begin{aligned}\zeta(3,2) &= -\frac{11}{2}\zeta(5) + \frac{\pi^2}{2}\zeta(3) \\ \zeta(2,3) &= \zeta(2)\zeta(3) - \zeta(5) - \zeta(3,2) \\ &= \frac{9}{2}\zeta(5) - \frac{\pi^2}{3}\zeta(3)\end{aligned}$$

using (4) and (2).

With weight $w = a + b < 8$ there is only one double sum $\zeta(a, b)$ not covered by Euler's explicit formulas, namely

$$\zeta(4, 2) = \zeta^2(3) - \frac{4}{3}\zeta(6)$$

with an evaluation whose proof will be considered later.

To obtain such evaluations by empirical methods, you may use the `EZFace` interface
`http://oldweb.cecm.sfu.ca/cgi-bin/EZFace/zetaform.cgi`
which supports the `lindep` function of `Pari-GP`. For example, the input
`lindep([z(4,2),z(3)^2,z(6)])`
produces the output
`3., -3., 4.`
corresponding to the integer relation

$$3\zeta(4, 2) - 3\zeta^2(3) + 4\zeta(6) = 0.$$

At weight $w = 8$, it appears that $\zeta(5, 3)$ cannot be reduced to zeta values and their products, though we have no way of proving that such a reduction cannot exist. [We cannot even prove that $\zeta(3)/\pi^3$ is irrational.] I shall take $\zeta(5, 3)$ as an (empirically) irreducible MZV of weight 8 and depth 2. Then all other double sums of weight 8 may be reduced to $\zeta(5, 3)$ and zeta values. For example,

$$20\zeta(6, 2) = 40\zeta(5)\zeta(3) - 8\zeta(5, 3) - 49\zeta(8).$$

It is proven that the number of irreducible double sums of even weight $w = 2n$ is no greater than $\lceil n/3 \rceil - 1$. Up to weight $w = 12$, we may take the irreducible double sums to be $\zeta(5,3)$, $\zeta(7,3)$ and $\zeta(9,3)$. Later we shall see that the proven reduction

$$\zeta(7,5) = \frac{14}{9}\zeta(9,3) + \frac{28}{3}\zeta(7)\zeta(5) - \frac{24257\pi^{12}}{2298646350} \tag{5}$$

sets us a puzzle. There is only one irreducible MZV with weight 12 and depth 2.

1.3 Triple sums

The first MZV of depth 3 that has not been reduced to MZVs of lesser depth (and their products) occurs at weight 11. It is proven that

$$\zeta(a,b,c) = \sum_{l>m>n>0} \frac{1}{l^a m^b n^c}$$

is reducible when the weight $w = a + b + c$ is even or less than 11. I conjectured that all MZVs of depth 3 are expressible in terms of $\mathbf{Q}$-linear combinations of the set

$$\mathcal{B}_3 = \{\zeta(2a+1, 2b+1, 2c+1) | a \geq b \geq c,\ a > c\}$$

together with double sums, $\zeta(a,b)$, single sums, $\zeta(c)$, and their products. This was borne out by the investigations by Borwein and Girgensohn in
`http://www.combinatorics.org/Volume_3/PDF/v3i1r23.pdf`
and more recently by Blümlein, Broadhurst and Vermaseren in
`http://arxiv.org/PS_cache/arxiv/pdf/0907/0907.2557v2.pdf`
with the associated MZV `DataMine`
`http://www.nikhef.nl/~form/datamine/`
providing strong evidence for many of the claims made in this talk.

My conjecture implies that the number of irreducible MZVs of weight $w = 2n+3$ and depth 3 is $\lceil n^2/12 \rceil - 1$, with the sequence

$$1, 2, 2, 4, 5, 6, 8, 10, 11, 14, 16, 18, 21, 24, 26, 30$$

giving the numbers for odd weights from 11 to 41.

1.4 A quadruple sum

The mystery of MZVs really begins here. At weight 12 there first appears a quadruple sum that has not been reduced to MZVs with depths less than 4. In the `DataMine` we take this to be

$$\zeta(6,4,1,1) = \sum_{k>l>m>n>0} \frac{1}{k^6 l^4 m n}$$

and prove, by exhaustion, that the following methods are insufficient to reduce it.

1.5 Shuffles, stuffles and duality

For integers $s_j > 0$ and $s_1 > 1$, the MZV

$$\zeta(s_1, s_2, \ldots, s_k) = \sum_{n_1>n_2>\ldots>n_k>0} \frac{1}{n_1^{s_1} n_2^{s_2} \ldots n_k^{s_k}}$$

may be encoded by a word of length $w = \sum_{j=1}^{k} s_j$ in the two letter alphabet (A, B), as follows. We write A, $s_1 - 1$ times, then B, then A, $s_2 - 1$ times, then B, and so on, until we end with B. For example

$$\begin{aligned}\zeta(5,3) &= Z(AAAABAAB)\\ \zeta(6,4,1,1) &= Z(AAAAABAAABBB)\end{aligned}$$

where the function Z takes a word as it argument and evaluates to the corresponding MZV. Note that the word must begin with A and end with B. The weight of the MZV is the length of the word and the depth is the number of B's in the word.

We may evaluate the MZV from an iterated integral defined by its word. For example

$$\zeta(2,1) = Z(ABB) = \int_0^1 \frac{\mathrm{d}x_1}{x_1} \int_0^{x_1} \frac{\mathrm{d}x_2}{1-x_2} \int_0^{x_2} \frac{\mathrm{d}x_3}{1-x_3} \tag{6}$$

where we use the differential form $\mathrm{d}x/x$ whenever we see the letter A and the differential form $\mathrm{d}x/(1-x)$ whenever we see the letter B. Then the equality of the nested sum $\zeta(2,1)$ with the iterated integral $Z(ABB)$ follows from binomial expansion of $1/(1-x_2)$ and $1/(1-x_3)$ in (6).

The shuffle algebra of MZVs is the identity

$$Z(U)Z(V) = \sum_{W\in\mathcal{S}(U,V)} Z(W) \tag{7}$$

where $\mathcal{S}(U,V)$ is the set of words obtained by all permutations of the letters of UV that preserve the order of letters in U and the order of letters in V. For example, suppose that $U = ab$ and $V = xy$. Then $\mathcal{S}(U,V)$ consists of the words

$$\mathcal{S}(ab,xy) = \{abxy, axby, xaby, axyb, xayb, xyab\}\,.$$

The only legal two-letter word is AB. Hence setting $a = x = A$ and $b = y = B$ we obtain

$$Z(AB)Z(AB) = 2Z(ABAB) + 4Z(AABB)$$

which shows that

$$\zeta^2(2) = 2\zeta(2,2) + 4\zeta(3,1).$$

We also have the "stuffle" identity

$$\zeta(2)\zeta(2) = \zeta(2,2) + \zeta(4) + \zeta(2,2)$$

from shuffling the arguments in a product of zetas and adding in the extra "stuff" that originates when summation variables are equal. Hence we conclude that $\zeta(3,1) = \frac{1}{4}\zeta(4)$. The evaluation $\zeta(2,2) = \frac{3}{4}\zeta(4)$ requires the extra piece of information $\zeta^2(2) = \frac{5}{2}\zeta(4)$ obtained from expanding the cotangent function.

Like the shuffle algebra, the stuffle algebra can be used to express any product of MZVs as a sum of MZVs. For example

$$\zeta(3,1)\zeta(2) = \zeta(3,1,2) + \zeta(3,3) + \zeta(3,2,1) + \zeta(5,1) + \zeta(2,3,1).$$

By combining shuffles, stuffles and reductions of $\zeta(2)$, $\zeta(4)$ and $\zeta(6)$ to powers of π^2 we may prove that

$$Z(AAABAB) = \zeta(4,2) = \zeta^2(3) - \frac{4}{3}\zeta(6).$$

Moreover, we obtain the same value for the depth-4 MZV

$$Z(ABABBB) = \zeta(2,2,1,1)$$

since $Z(W) = Z(\widetilde{W})$, where the dual $\widetilde{W}$ of a word W is obtained by writing it backwards and then exchanging A and B. This duality was observed by Zagier. It follows from the transformation $x \to 1-x$ in the iterated integral, which exchanges the differential forms $\mathrm{d}x/x$ and $\mathrm{d}x/(1-x)$ and reverses the ordering of the integrations. Hence

$$\zeta(2,3,1) = Z(ABAABB) = Z(AABBAB) = \zeta(3,1,2).$$

Thus we arrive at a well-defined question: for a given weight $w > 2$ and a given depth $d > 0$, what is rank-deficiency $D_{w,d}$ of all the algebraic relations that follow from the shuffle and stuffle algebras algebras of MZVs, combined with duality and the reduction of even zeta values to powers of π^2? Note that $D_{w,d}$ is an upper limit for the number of irreducible MZVs at this weight and depth. There may conceivably (but rather improbably) be fewer, since we cannot rule out the possibility of additional integer relations. [We cannot even prove that $\zeta(3)/\pi^3$ is irrational.]

In 1996, Dirk Kreimer and I conjectured that the answer to this question is given by the generating function

$$\prod_{w>2}\prod_{d>0}(1-x^w y^d)^{D_{w,d}} \stackrel{?}{=} 1 - \frac{x^3 y}{1-x^2} + \frac{x^{12}y^2(1-y^2)}{(1-x^4)(1-x^6)} \tag{8}$$

which produces the following table of values, with underlined values verified by Jos Vermaseren.

To explain how I guessed the final term in the generating function (8), I shall need to discuss alternating Euler sums.

1.6 MZVs in QFT

The counterterms in the renormalization of the coupling in ϕ^4 theory, at L loops, may involve MZVs with weights up to $2L-3$. Those associated with subdivergence-free diagrams may be obtained from finite massless 2-point diagrams with one less loop.

The first irreducible MZV of depth 2, namely $\zeta(5,3)$, occurs in a counterterm coming from the most symmetric 6-loop diagram for the ϕ^4 coupling, in which each of the 4 vertices connected to an external line is connected to each of the 3 other vertices, giving 12 internal propagators (or edges, as mathematicians prefer to call them). It hence diverges, at large loop momenta, in the manner of $\int \mathrm{d}^{24}k/k^{24}$. Its contribution to the β-function of ϕ^4-theory is scheme-independent and may be computed to high accuracy by using Gegenbauer polynomial expansions in x-space, which give the counterterm as a 4-fold sum that is far from obviously a MZV. Accelerated convergence of truncations of this sum gave an empirical $\mathbf{Q}$-linear of combination of $\zeta(5)\zeta(3)$ with

$$\zeta(5,3)-\frac{29}{12}\zeta(8)$$

and the latter combination was found to occur in another 6-loop counterterm. I shall attempt to demystify the multiple of $\zeta(8)$ after discussing alternating Euler sums.

At 7 loops, Dirk Kreimer and I found the combination

$$\zeta(3,5,3)-\zeta(3)\zeta(5,3)$$

in 3 different counterterms, where it occurs in combination with rational multiples of $\zeta(11)$ and $\zeta^2(3)\zeta(5)$.

w /d	1	2	3	4	5	6	7	8	9	10
3	$\underline{1}$									
4										
5	$\underline{1}$									
6										
7	$\underline{1}$									
8		$\underline{1}$								
9	$\underline{1}$									
10		$\underline{1}$								
11	$\underline{1}$		$\underline{1}$							
12		$\underline{1}$		$\underline{1}$						
13	$\underline{1}$		$\underline{2}$							
14		$\underline{2}$		$\underline{1}$						
15	$\underline{1}$		$\underline{2}$		$\underline{1}$					
16		$\underline{2}$		$\underline{3}$						
17	$\underline{1}$		$\underline{4}$		$\underline{2}$					
18		$\underline{2}$		$\underline{5}$		$\underline{1}$				
19	$\underline{1}$		$\underline{5}$		$\underline{5}$					
20		$\underline{3}$		$\underline{7}$		$\underline{3}$				
21	$\underline{1}$		$\underline{6}$		$\underline{9}$		$\underline{1}$			
22		$\underline{3}$		$\underline{11}$		$\underline{7}$				
23	$\underline{1}$		$\underline{8}$		$\underline{15}$		$\underline{4}$			
24		$\underline{3}$		$\underline{16}$		$\underline{14}$		$\underline{1}$		
25	$\underline{1}$		$\underline{10}$		$\underline{23}$		$\underline{11}$			
26		$\underline{4}$		$\underline{20}$		$\underline{27}$		$\underline{5}$		
27	$\underline{1}$		$\underline{11}$		$\underline{36}$		$\underline{23}$		2	
28		$\underline{4}$		$\underline{27}$		$\underline{45}$		16		
29	$\underline{1}$		$\underline{14}$		$\underline{50}$		$\underline{48}$		7	
30		$\underline{4}$		$\underline{35}$		$\underline{73}$		37		2

Table 1: Number of basis elements for MZVs as a function of weight and depth in a minimal depth representation. Underlined are the values we have verified with our programs.

2 Alternating Euler sums

My second topic is closely related to the first, namely alternating sums of the form

$$\sum_{n_1>n_2>\ldots>n_k>0}^{\infty} \frac{\varepsilon_1^{n_1}\ldots\varepsilon_k^{n_k}}{n_1^{s_1}\ldots n_k^{s_k}}$$

with positive integers s_j and signs $\varepsilon_j = \pm 1$. We may compactly indicate the presence of an alternating sign, when $\varepsilon_j = -1$, by placing a bar over the corresponding integer exponent s_j. Thus we write

$$\begin{aligned}\zeta(\overline{3},\overline{1}) &= \sum_{m>n>0} \frac{(-1)^{m+n}}{m^3 n}\\ \zeta(3,\overline{6},3,\overline{6},3) &= \sum_{j>k>l>m>n>0} \frac{(-1)^{k+m}}{j^3 k^6 l^3 m^6 n^3}\end{aligned}$$

using the same symbol ζ as we did for the MZVs. Such sums may be studied using `EZFace` and the `DataMine`.

2.1 Three-letter alphabet

Alternating sums have a stuffle algebra, from their representation as nested sums, and a shuffle algebra, from their representation as iterated integrals. In the integral representation we need a third letter, C, in our alphabet, corresponding to the differential form $\mathrm{d}x/(1+x)$. Consider

$$Z(ABC) = \int_0^1 \frac{\mathrm{d}x}{x} \int_0^x \frac{\mathrm{d}y}{1-y} \int_0^y \frac{\mathrm{d}z}{1+z}.$$

The z-integral gives $\log(1+y) = -\sum_{j>0}(-y)^j/j$ and hence

$$Z(ABC) = -\sum_{j>0}\int_0^1 \frac{\mathrm{d}x}{x}\int_0^x \frac{\mathrm{d}y}{1-y}\frac{(-y)^j}{j}.$$

Expanding $1/(1-y) = \sum_{k>0} y^{k-1}$ and integrating over y we obtain

$$Z(ABC) = -\sum_{k>0}\sum_{j>0}\int_0^1 \frac{\mathrm{d}x}{x}\frac{x^{j+k}}{j+k}\frac{(-1)^j}{j}$$

and the final integration gives

$$Z(ABC) = -\sum_{k>0}\sum_{j>0}\frac{1}{(j+k)^2}\frac{(-1)^j}{j}.$$

Finally, the substitution $k = m - j$ gives

$$Z(ABC) = -\sum_{m>j>0}\frac{(-1)^j}{m^2 j} = -\zeta(2,\overline{1}).$$

It takes a bit of practice to translate between words and sums. Here's another example:

$$Z(ACCAC) = (-1)^3\sum_{l>0}\sum_{k>0}\sum_{j>0}\frac{(-1)^l}{(j+k+l)^2}\frac{(-1)^k}{j+k}\frac{(-1)^j}{j^2}$$

gives

$$Z(ACCAC) = -\sum_{m>n>j>0}\frac{(-1)^m}{m^2 n j^2} = -\zeta(\overline{2},1,2)$$

after the substitutions $l = m - n$ and $k = n - j$.

Going from sums to words is quite tricky. For example, try to find the word W and the sign $\varepsilon(W)$ such that

$$\zeta(3, \overline{6}, 3, \overline{6}, 3) = \varepsilon(W) Z(W).$$

Note that $\varepsilon(W)$ is $+1$ or -1 according as whether there is an odd or even number of letters C in the word W. The word W begins $AABAAAAACAA\ldots$. The next letter is either B or C, but which is it?

2.2 Shuffles and stuffles

The 6 shuffles in

$$\mathcal{S}(ab, xy) = \{abxy, axby, xaby, axyb, xayb, xyab\}$$

give 6 different words, with $a = A$, $b = B$, $x = y = C$:

$$\begin{aligned} Z(AB)Z(CC) &= Z(ABCC) + Z(ACBC) + Z(CABC) \\ &\quad + Z(ACCB) + Z(CACB) + Z(CCAB) \end{aligned}$$

which translates to

$$\zeta(2)\zeta(\overline{1}, 1) = \zeta(2, \overline{1}, 1) + \zeta(\overline{2}, \overline{1}, \overline{1}) + \zeta(\overline{1}, \overline{2}, \overline{1}) + \zeta(\overline{2}, 1, \overline{1}) + \zeta(\overline{1}, 2, \overline{1}) + \zeta(\overline{1}, 1, \overline{2}).$$

The stuffles for this product are

$$\zeta(2)\zeta(\overline{1}, 1) = \zeta(2, \overline{1}, 1) + \zeta(\overline{3}, 1) + \zeta(\overline{1}, 2, 1) + \zeta(\overline{1}, 3) + \zeta(\overline{1}, 1, 2).$$

2.3 Transforming words

The transformation $x = (1-y)/(1+y)$ gives

$$\begin{aligned} \mathrm{d}\log(x) &= \mathrm{d}\log(1-y) - \mathrm{d}\log(1+y) \\ \mathrm{d}\log(1-x) &= \mathrm{d}\log(y) - \mathrm{d}\log(1+y) \\ \mathrm{d}\log(1+x) &= -\mathrm{d}\log(1+y) \end{aligned}$$

and maps $x = 0$ and $x = 1$ to $y = 1$ and $y = 0$. Thus, if we take a word W, write it backwards, and make the transformations

$$\begin{aligned} A &\to (B+C) \\ B &\to (A-C) \end{aligned}$$

we may obtain an expression for $Z(W)$ by expanding the brackets.

For example the transformation

$$AB \to (A-C)(B+C) = AB + AC - CB - CC$$

gives

$$Z(AB) = Z(AB) + Z(AC) - Z(CB) - Z(CC).$$

Combining this with the shuffle

$$Z(C)Z(C) = Z(CC) + Z(CC)$$

we obtain

$$0 = Z(AC) - Z(CB) - \frac{1}{2}Z(C)Z(C) = -\zeta(\overline{2}) + \zeta(\overline{1},\overline{1}) - \frac{1}{2}\zeta(\overline{1})\zeta(\overline{1}).$$

Combining this with the stuffle

$$\zeta(\overline{1})\zeta(\overline{1}) = \zeta(\overline{1},\overline{1}) + \zeta(2) + \zeta(\overline{1},\overline{1})$$

we obtain

$$\zeta(\overline{2}) = -\frac{1}{2}\zeta(2)$$

which is also obtainable as follows.

2.4 Doubling relations

For $a > 1$ we have

$$\zeta(a) + \zeta(\overline{a}) = \sum_{n>0} \frac{1+(-1)^n}{n^a} = \sum_{k>0} \frac{2}{(2k)^a} = 2^{1-a}\zeta(a)$$

by the substitution $n = 2k$. Hence

$$\zeta(\overline{a}) = (2^{1-a} - 1)\zeta(a).$$

At $a = 2$, we obtain $\zeta(\overline{2}) = -\zeta(2)/2$, as above. Note also that $\zeta(\overline{1}) = -\log(2)$.

We may take any MZV and convert it into a combination of MZVs and alternating sums, by doubling the summation variables. For example, we obtain

$$\begin{aligned} 2^{2-a-b}\zeta(a,b) &= \sum_{m>n>0} \frac{2}{(2m)^a}\frac{2}{(2n)^b} \\ &= \sum_{j>k>0} \frac{1+(-1)^j}{j^a}\frac{1+(-1)^k}{k^a} \\ &= \zeta(a,b) + \zeta(\overline{a},b) + \zeta(a,\overline{b}) + \zeta(\overline{a},\overline{b}) \end{aligned}$$

by the transformations $j = 2m$ and $k = 2n$.

More complicated doubling relations were used in constructing the `DataMine`. With these, it was possible to avoid using the time-consuming transformations $A \to (B + C)$ and $B \to (A - C)$ as algebraic input. It was verified that the output, obtained by shuffling, stuffling and doubling, satisfied the relations that follow from word transformation.

2.5 Conjectured enumeration of irreducibles

Before considering the enumeration of irreducible MZVs, in the (A, B) alphabet, I already had a rather simple conjecture for the generator of the number, $E_{w,d}$, of irreducible sums of weight w and depth d in the (A, B, C) alphabet, namely

$$\prod_{w>2}\prod_{d>0}(1 - x^w y^d)^{E_{w,d}} \stackrel{?}{=} 1 - \frac{x^3 y}{(1 - xy)(1 - x^2)}. \tag{9}$$

If this be true, it is easy to obtain $E_{w,d}$ by Möbius transformation of the binomial coefficients in Pascal's triangle. Let

$$T(a, b) = \frac{1}{a + b}\sum_{c|a,b}\mu(c)\,\frac{(a/c + b/c)!}{(a/c)!(b/c)!} \tag{10}$$

where the sum is over all positive integers c that divide both a and b and the Möbius function is defined by

$$\mu(c) = \begin{cases} 1 & \text{when } c = 1 \\ 0 & \text{when } c \text{ is divisible by the square of a prime} \\ (-1)^k & \text{when } c \text{ is the product of } k \text{ distinct primes.} \end{cases} \tag{11}$$

When w and d have the same parity, and $w > d$, one obtains from (9)

$$E_{w,d} = T\left(\frac{w-d}{2}, d\right)\,. \tag{12}$$

The `DataMine` now provides extensive evidence to support this conjecture. It was verified at depth 6 up to weight 12, solving the algebraic input in rational arithmetic, and then up to weight 18, using arithmetic modulo a 31-bit prime. At depth 5, the corresponding weights are 17 and 21. At depth 4, they are 22 and 30.

2.6 Pushdown

Now consider the integers $M_{w,d}$ generated by an even simpler process:

$$\prod_{w>2}\prod_{d>0}(1-x^w y^d)^{M_{w,d}} = 1 - \frac{x^3 y}{1-x^2}. \tag{13}$$

But what is the question, to which this is the answer?

I conjectured that $M_{w,d}$ is the number of irreducible sums of weight w and depth d in the (A, B, C) alphabet that suffice for the evaluation of MZVs in the (A, B) alphabet.

As already hinted, the first place that this conjecture becomes non-trivial is at weight 12, where the enumerations $M_{12,4} = 0$ and $M_{12,2} = 2$ are to be contrasted with the enumerations $D_{12,4} = 1$ and $D_{12,2} = 1$ of irreducible MZVs. The conjecture requires that

$$\zeta(6,4,1,1) = \sum_{k>l>m>n>0} \frac{1}{k^6 l^4 m n}$$

be reducible to sums of lesser depth, if we include an alternating double sum in the basis.

In 1996, I found such a "pushdown" empirically, using the integer-relation search routine `PSLQ`. It took another decade to prove such an integer relation, by the laborious process of solving all the known algebraic relations in the (A, B, C) alphabet at weight 12 and depths up to 4. Jos Vermaseren derived this proven identity from the `DataMine`:

$$\begin{aligned}\zeta(6,4,1,1) \quad = \quad & -\frac{64}{27}A(7,5) - \frac{7967}{1944}\zeta(9,3) + \frac{1}{12}\zeta^4(3) + \frac{11431}{1296}\zeta(7)\zeta(5) \\ & -\frac{799}{72}\zeta(9)\zeta(3) + 3\zeta(2)\zeta(7,3) + \frac{7}{2}\zeta(2)\zeta^2(5) + 10\zeta(2)\zeta(7)\zeta(3) \\ & +\frac{3}{5}\zeta^2(2)\zeta(5,3) - \frac{1}{5}\zeta^2(2)\zeta(5)\zeta(3) - \frac{18}{35}\zeta^3(2)\zeta^2(3) - \frac{5607853}{6081075}\zeta^6(2)\end{aligned}$$

where

$$A(7,5) = Z(AAAAAA(B-C)AAAAB) = \zeta(7,5) + \zeta(\overline{7},\overline{5}).$$

It is now proven that all MZVs of weight up to 12 are reducible to $\mathbf{Q}$-linear combinations of $\zeta(5,3)$, $\zeta(7,3)$, $\zeta(3,5,3)$, $\zeta(9,3)$, $\zeta(\overline{7},\overline{5})$, single zeta values, and products of these terms.

I can now explain the rather simple-minded procedure that Dirk Kreimer and I used in 1996 to arrive at the conjecture

$$\prod_{w>2}\prod_{d>0}(1-x^w y^d)^{D_{w,d}} \stackrel{?}{=} 1 - \frac{x^3 y}{1-x^2} + \frac{x^{12}y^2(1-y^2)}{(1-x^4)(1-x^6)}$$

for the number $D_{w,d}$ of irreducible sums in the (A, B) alphabet of pure MZVs. We added the third term to the much simpler conjectured generator for the much complicated question answered by $M_{w,d}$, namely the number of irreducibles in the (A, B, C) alphabet that suffice for reductions of MZVs. The numerator, $x^{12}y^2(1-y^2)$, of this term was determined by the single pushdown observed at weight 12, from an MZV of depth 4 to an alternating sum of depth 2. The denominator, $(1-x^x)(1-x^6)$, was chosen to agree with the empirical number $D_{2n,2} = \lceil n/3 \rceil - 1$ of double non-alternating irreducible sums of weight $2n$. Then we assumed that the enumeration of all other pushdowns would be generated by filtration. It was possible to check this, in a few cases, using PSLQ in 1996.

The list of explicit pushdowns that have now been obtained, in accord with the conjecture, has grown since then.

At weights 15, 16, 17, we have found pushdowns from MZVs to these alternating sums: $\zeta(\overline{6},3,\overline{6})$, $\zeta(\overline{13},\overline{3})$, $\zeta(\overline{6},5,\overline{6})$.
At weight 18, there were pushdowns to $\zeta(\overline{15},\overline{3})$ and $\zeta(6,\overline{5},\overline{4},3)$.
At weight 19, to $\zeta(\overline{8},3,\overline{8})$ and $\zeta(\overline{6},7,\overline{6})$.
At weight 20, to $\zeta(\overline{17},\overline{3})$, $\zeta(8,\overline{5},\overline{4},3)$ and $\zeta(6,\overline{5},\overline{6},3)$.

Our most ambitious efforts were at weight 21, where 3 MZVs of depth 5 are pushed down to the alternating sums $\zeta(\overline{8},5,\overline{8})$, $\zeta(\overline{6},9,\overline{6})$and $\zeta(\overline{8},3,\overline{10})$. Moreover the first pushdown from an MZV of depth 7 to an alternating sum of depth 5 is predicted at weight 21. A demanding `PSLQ` computation gave a relation of the form

$$\zeta(6,2,3,3,5,1,1) = -\frac{326}{81}\zeta(3,\overline{6},3,\overline{6},3) + \ldots \qquad (14)$$

where the remaining 150 terms are formed by MZVs with depth no greater than 5, and their products. At such weight and depth, it becomes rather non-trivial to decide on a single alternating sum that might replace a MZV of greater depth. It took several attempts to discover that the alternating sum

$$\zeta(3,\overline{6},3,\overline{6},3) = \sum_{j>k>l>m>n>0} \frac{(-1)^{k+m}}{(jk^2lm^2n)^3}$$

is an "honorary MZV" that performs this task.

2.7 Suppression of π in massless diagrams

Now I can demystify, somewhat, the combination

$$\zeta(5,3) - \frac{29}{12}\zeta(8)$$

that occurs in scheme-independent counterterms of ϕ^4 theory at 6 loops. Dirk Kreimer and I discovered that the combinations

$$N(a,b) = \zeta(\overline{a},b) - \zeta(\overline{b},a),$$

with distinct odd integers a and b, simplify the results for counterterms. In particular, the use of

$$N(3,5) = \frac{27}{80}\left(\zeta(5,3) - \frac{29}{12}\zeta(8)\right) + \frac{45}{64}\zeta(3)\zeta(5)$$

removes all powers of π from both subdivergence-free diagrams that contribute to the 6-loop β-function. In each case, the contribution is a $\mathbf{Z}$-linear combination of $N(3,5)$ and $\zeta(3)\zeta(5)$.

At higher loop numbers, Oliver Schnetz has found that $N(3,7)$ suppresses the appearance π^{10}. However, at 8 loops he found that $N(3,9)$ and $N(5,7)$ are not sufficient to remove π^{12}. Like the maths, the physics becomes different at weight 12.

2.8 Magnetic moment of the electron

The magnetic moment of an electron, with charge $-e$ and mass m, is slightly greater than the Bohr magneton

$$\frac{e\hbar}{2m} = 9.274 \times 10^{-24}\ \mathrm{J\ T^{-1}}$$

which was the value predicted by Dirac. Here I included $\hbar = h/(2\pi)$, which we usually set to unity in QFT.

Using perturbation theory, we may expand in powers of the fine structure constant:

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} = \frac{1}{137.035999\ldots}.$$

In QFT, we usually set $\varepsilon_0 = 1$ and $c = 1$ and expand in powers of $\alpha/\pi = e^2/(4\pi^2)$, obtaining a perturbation expansion

$$\frac{\text{magnetic moment}}{\text{Bohr magneton}} = 1 + A_1\frac{\alpha}{\pi} + A_2\left(\frac{\alpha}{\pi}\right)^2 + A_3\left(\frac{\alpha}{\pi}\right)^3 + \ldots$$

which is known up to 3 loops.

In 1947, Schwinger found the first correction term $A_1 = \frac{1}{2}$. In 1950, Karplus and Kroll claimed the value

$$28\zeta(3) - 54\zeta(2)\log(2) + \frac{125}{6}\zeta(2) - \frac{2687}{288} \ = \ -2.972604271\ldots$$

for the coefficient of the next correction. It turned out that they had made a mistake in this rather difficult calculation. The correct result

$$A_2 \ = \ \frac{3}{4}\zeta(3) - 3\zeta(2)\log(2) + \frac{1}{2}\zeta(2) + \frac{197}{144} \ = \ -0.3284789655\ldots$$

was not obtained until 1957. Not until 1996 was the next coefficient

$$
\begin{aligned}
A_3 &= -\frac{215}{24}\zeta(5) + \frac{83}{12}\zeta(3)\zeta(2) - \frac{13}{8}\zeta(4) - \frac{50}{3}\zeta(\overline{3},\overline{1}) \\
&\quad + \frac{139}{18}\zeta(3) - \frac{596}{3}\zeta(2)\log(2) + \frac{17101}{135}\zeta(2) + \frac{28259}{5184} \qquad (15)\\
&= 1.181241456\ldots
\end{aligned}
$$

found, by Stefano Laporta and Ettore Remiddi. The irrational numbers appearing on the second line are those already seen in A_2. On the first line we see zeta values and a new number, namely the alternating double sum

$$
\zeta(\overline{3},\overline{1}) = \sum_{m>n>0} \frac{(-1)^{m+n}}{m^3 n} \approx -0.117875999650509326841013950834137618715 2\ldots
$$

I visited Stefano and Ettore in Bologna when they were working on this formidable calculation and recommended to them a method of integration by parts, in D dimensions, that I had found useful for related calculations in the quantum field theory of electrons and photons. Here $D = 4 - 2\varepsilon$ is eventually set to 4, the number of dimensions of space-time. But it turns out to be easier if we keep it as a variable until the final stage of the calculation. Then if we find parts of the result that are singular at $\varepsilon = 0$ we need not worry: all that matters is that the complete result is finite. Based on my D-dimensional experience, I expected their final result to look simplest when written in terms of $\zeta(\overline{3},\overline{1})$.

The D-dimensional calculation that informed this intuition involved three-loop massive diagrams contributing to charge renormalization in QED. These yielded Saalschützian F_{32}

hypergeometric series, with parameters differing from $\frac{1}{2}$ by multiples of ε, namely

$$W(a_1,a_2;a_3,a_4)=\sum_{n=0}^{\infty}\frac{(\frac{1}{2}-a_1\varepsilon)_n(\frac{1}{2}-a_2\varepsilon)_n}{(\frac{1}{2}+a_3\varepsilon)_{n+1}(\frac{1}{2}+a_4\varepsilon)_{n+1}}$$

with $(\alpha)_n\equiv\Gamma(\alpha+n)/\Gamma(\alpha)$. In particular, I needed the expansions of $W(1,1;1,0)$ and $W(1,0;1,1)$ in ε. The result for the most difficult three-loop diagram had the value $\pi^2\log(2)-\frac{3}{2}\zeta(3)$ at $\varepsilon=0$. Noting that this also occurs in the two-loop contribution to the magnetic moment, I expanded the charge-renormalization result to $O(\varepsilon)$, where I found only $\zeta(\overline{3},\overline{1})$ and $\zeta(4)$. I thus hazarded the guess that these two sums would exhaust the weight-4 contributions to the magnetic moment at 3 loops, which happily is the case.

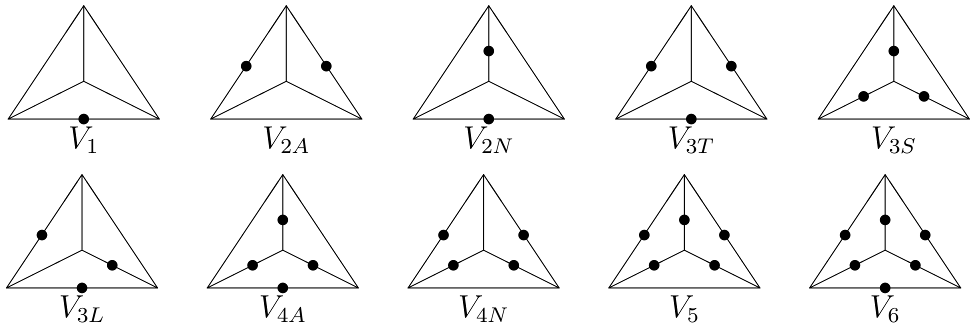
One may also write (15) in terms of a polylog that is not evaluated on the unit circle, such as

$$\mathrm{Li}_4(1/2)=\sum_{n=1}^{\infty}\frac{1}{n^4}\left(\frac{1}{2}\right)^n=-\frac{1}{24}\log^4(2)+\frac{1}{4}\zeta(2)\log^2(2)+\frac{1}{4}\zeta(4)-\frac{1}{2}\zeta(\overline{3},\overline{1}),$$

but then the result for A_3 will acquire extra terms, involving powers of $\log^2(2)$.

3 Polylogs of the sixth root of unity

Colourings of the tetrahedron by mass:



with finite parts guessed in `Eur.Phys.J. C8 (1999) 311-333`, as follows:

$$\begin{aligned}
\overline{V}_j &= \lim_{\varepsilon\to 0}\left(V_j - \frac{6\zeta(3)}{3\varepsilon}\right)\\
&= 6\zeta(3) + z_j\,\zeta(4) + u_j\,U_{3,1} + s_j\,\mathrm{Cl}_2^2(\pi/3) + v_j\,V_{3,1}
\end{aligned}$$

with rational coefficients of 4 quadrilogs: $\zeta(4) = \pi^4/90$, the alternating double sum

$$U_{3,1} = \zeta(\overline{3},\overline{1}) = \sum_{m>n>0}\frac{(-1)^{m+n}}{m^3 n},$$

the square of $\mathrm{Cl}_2(\pi/3) = \sum_{n>0}\sin(n\pi/3)/n^2$, and the double sum

$$V_{3,1} = \sum_{m>n>0}\frac{(-1)^m\cos(2\pi n/3)}{m^3 n}.$$

Rational fits to high-precision evaluations:

V_j	z_j	u_j	s_j	v_j	$\overline{V}_j$
V_1	3				10.4593111200909802869464400586922036529141
V_{2A}	-5				1.8007252504018747548184104863628604307161
V_{2N}	$-\frac{13}{2}$	-8			1.1202483970392420822725165482242095262757
V_{3T}	-9				$-$2.5285676844426780112456042998018111803828
V_{3S}	$-\frac{11}{2}$		-4		$-$2.8608622241393273502727845677732419175614
V_{3L}	$-\frac{15}{4}$		-6		$-$3.0270094939876520197863747017589572861507
V_{4A}	$-\frac{77}{12}$		-6		$-$5.9132047838840205304957178925354050268834
V_{4N}	-14	-16			$-$6.0541678585902197393693995691614487948131
V_5	$-\frac{469}{27}$		$\frac{8}{3}$	-16	$-$8.2168598175087380629133983386010858249695
V_6	-13	-8	-4		$-$10.0352784797687891719147006851589002386503

where only V_1, V_{2A}, V_{3T} and V_{4N} were known before, with V_{4N} obtained from my study of QED in `Z.Phys. C54 (1992) 599-606`, which gave

$$\Lambda^{\overline{\rm MS}}_{\rm QED} \approx \frac{m\,{\rm e}^{3\pi/2\alpha}}{(3\pi/\alpha)^{9/8}} \left(1 - \frac{175}{64}\frac{\alpha}{\pi} + \left\{-\frac{63}{64}\zeta(3) + \frac{1}{2}\pi^2\log 2 - \frac{23}{48}\pi^2 + \frac{492473}{73728}\right\}\frac{\alpha^2}{\pi^2}\right),$$

for the integration constant of the four-loop $\overline{\rm MS}$ β–function of single-favour QED.

Colleagues in condensed-matter physics asked me to evaluate the equal-mass tetrahedron in 3 dimensions, and I obliged with the guess

$$C^{\rm Tet} = 2^{5/2}\left({\rm Cl}_2(4\theta) - {\rm Cl}_2(2\theta)\right),\ \text{with } 3\sin\theta = 1,$$

given in `Eur.Phys.J. C8 (1999) 363-366`.

I was eventually able to prove this, by dint of considering a more general case, $C(a,b)$, with masses a and b on opposite edges of the tetrahedron and unit masses on the other 4 edges. After much work, I derived the PDE

$$\frac{b\sqrt{4-a^2-b^2}}{4}\frac{\partial}{\partial a}\left(\frac{a\sqrt{4-a^2-b^2}}{4}C(a,b)\right) =$$

$$-\log\left(\frac{a+2}{a+b+2}\right) - \frac{b}{a+2}\log\left(\frac{a+b+2}{b+2}\right) - \frac{2b}{a^2-4}\log\left(\frac{a+2}{4}\right)$$

which was eventually solved, to give

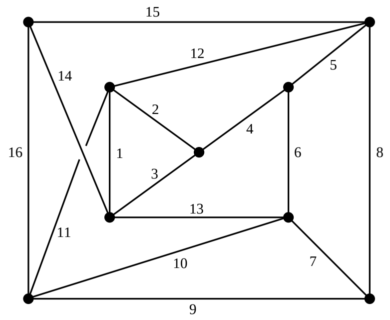
$$\begin{aligned}\frac{1}{8}ab\gamma\, C(a,b) &= {\rm Cl}_2(4\phi) + {\rm Cl}_2(2\phi_a+2\phi_b-2\phi) + {\rm Cl}_2(2\phi_a-2\phi) + {\rm Cl}_2(2\phi_b-2\phi)\\ &- {\rm Cl}_2(2\phi_a+2\phi_b-4\phi) - {\rm Cl}_2(2\phi_a) - {\rm Cl}_2(2\phi_b) - {\rm Cl}_2(2\phi)\end{aligned}$$

with

$$\gamma = \sqrt{4-a^2-b^2} = a\tan\phi_a = b\tan\phi_b = (a+b+2)\tan\phi.$$

4 Multi-loop sunrise diagrams and modular forms

Polylogs are clearly too small a domain for the glories of QFT.



Francis Brown and Oliver Schnetz identified an 8-loop ϕ^4 diagram for which a K3 surface, obtained from the symmetric square of an elliptic curve with complex multiplication by $\mathbf{Q}(\sqrt{-7})$, seems to create an insuperable obstacle to evaluation in terms of polylogs. I was able to identify the modular form whose Fourier coefficients enumerate the zeros of the variety obtained from the denominator of an integrand, namely

$$\begin{aligned}\Big[\eta(q)\eta(q^7)\Big]^3 &= q\prod_{j>0}(1-q^j)^3(1-q^{7j})^3 = \sum_{n>0} a_n q^n \\ &= q - 3q^2 + 5q^4 - 7q^7 - 3q^8 + 9q^9 - 6q^{11} + 3q^{12} - 6q^{13} + \cdots\end{aligned}$$

where

$$\frac{\eta(q)}{q^{1/24}} = \prod_{j>0}(1-q^j) = \sum_{n\in\mathbf{Z}}(-1)^n q^{n(3n+1)/2}$$

provides us with the modular form of weight 12:

$$\Delta(z) = [\eta(\exp(2\pi i z))]^{24}$$

which is holomorphic in the upper half plane **H** with $\Im z > 0$, where it transforms as

$$\Delta\left(\frac{az+b}{cz+d}\right) = (cz+d)^{12}\Delta(z)$$

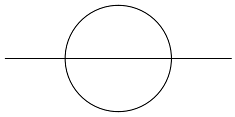
under the modular group

$$\mathrm{SL}(2,\mathbf{Z}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}, a,b,c,d \in \mathbf{Z}, ad-bc=1\right\}$$

with generators

$$\Delta(z+1) = \Delta(z), \quad \Delta(-1/z) = z^{12}\Delta(z).$$

Sunrise, at last:



At N loops, in 2 spacetime dimensions, the on-shell scalar sunrise integral is

$$\overline{S}_{N+2} = 2^N \int_0^\infty I_0(t)[K_0(t)]^{N+1} t \, \mathrm{d}t$$

where the subscript denotes the number of Bessel functions. The irregular Bessel function $K_0(t)$ results from internal edges with a common mass; its regular cousin $I_0(t)$ comes from an external line, with the same mass. We may also consider

$$\overline{T}_{N+3} = 2^N \int_0^\infty I_0^2(t)[K_0(t)]^{N+1} t \, \mathrm{d}t$$

$$\overline{U}_{N+4} = 2^N \int_0^\infty I_0^3(t)[K_0(t)]^{N+1} t \, \mathrm{d}t$$

$$\overline{W}_{N+5} = 2^N \int_0^\infty I_0^4(t)[K_0(t)]^{N+1} t \, \mathrm{d}t$$

with more than one external line on-shell.

To obtain $\overline{S}_{N+2}$ as an integral over Schwinger parameters, α_1 to α_N, let $\mathbf{A}$ be the diagonal $N \times N$ matrix with entries $A_{i,j} = \alpha_i \delta_{i,j}$, let $\mathbf{u}$ be the column vector of length N with unit entries, $u_i = 1$, and transpose $\tilde{\mathbf{u}}$. Then, with $\mathbf{M} = \mathbf{A} + \mathbf{u}\tilde{\mathbf{u}}$, we obtain

$$\overline{S}_{N+2} = \int_{\alpha_i > 0} \frac{\mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_N}{\mathrm{Det}(\mathbf{M})(\mathrm{Tr}(\mathbf{A}) + \tilde{\mathbf{u}}\mathbf{M}^{-1}\mathbf{u})}$$

and these evaluations up to 3 loops

$$\overline{S}_3 = \int_0^\infty \frac{\mathrm{d}a}{a^2 + a + 1} = \frac{2\pi}{3\sqrt{3}}$$

$$
\begin{aligned}
\overline{S}_4 &= \int_0^\infty \int_0^\infty \frac{\mathrm{d}a\,\mathrm{d}b}{(a+b)(a+1)(b+1)} = \frac{\pi^2}{4} \\
\overline{S}_5 &= \int_0^\infty \int_0^\infty \int_0^\infty \frac{\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c}{(abc+ab+bc+ca)(a+b+c)+(ab+bc+ca)} \\
&= \frac{\pi^3}{2}\left(1-\frac{1}{\sqrt{5}}\right)\left(1+2\sum_{n>0}\exp\left(-\sqrt{15}\pi n^2\right)\right)^4 \\
&= \frac{1}{30\sqrt{5}}\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right)
\end{aligned}
$$

with a 3-loop result given in terms the square of a complete elliptic integral at the 15th singular value, which yields Γ values via the Chowla-Selberg theorem.

A weight-3 modular form at 3 loops

At 3 loops, the denominator of Schwinger's integrand implicates the weight-3 modular form

$$f_3(q) = \sum_{n>0} A_{3,n} q^n = \eta(q)\eta(q^3)\eta(q^5)\eta(q^{15}) \sum_{m,n\in\mathbf{Z}} q^{m^2+mn+4n^2}$$

Its Dirichlet L function

$$L_3(s) = \sum_{n>0} \frac{A_{3,n}}{n^s}$$

obeys the functional equation

$$\Lambda_3(s) = \Lambda_3(3-s), \text{ with } \Lambda_3(s) = \Gamma(s)L_3(s)/(2\pi/\sqrt{15})^s$$

with a fast evaluation of

$$L_3(2) = \sum_{n>0} \frac{A_{3,n}}{n^2} \left(1 + \frac{4\pi n}{\sqrt{15}}\right) \exp\left(-\frac{2\pi n}{\sqrt{15}}\right)$$

inside the critical strip. Then, to high precision, we find that

$$3L_3(2) = \overline{T}_5 = \frac{\sqrt{15}}{4\pi}\,\overline{S}_5$$

A weight-4 modular form at 4 loops

Here the modular form is simpler:

$$f_4(q) = \sum_{n>0} A_{4,n} q^n = \left[\eta(q)\eta(q^2)\eta(q^3)\eta(q^6)\right]^2$$

Its Dirichlet L function

$$L_4(s) = \sum_{n>0} \frac{A_{4,n}}{n^s}$$

obeys the functional equation

$$\Lambda_4(s) = \Lambda_4(4-s), \text{ with } \Lambda_4(s) = \Gamma(s)L_4(s)/(2\pi/\sqrt{6})^s$$

The Mellin transform

$$\Lambda_4(s) = \sum_{n>0} A_{4,n} \int_0^\infty \frac{\mathrm{d}x}{x} x^s \exp\left(-\frac{2\pi n x}{\sqrt{6}}\right)$$

may be analytically continued to give

$$\Lambda_4(s) = \sum_{n>0} A_{4,n} \int_1^\infty \frac{\mathrm{d}x}{x} \left(x^s + x^{4-s}\right) \exp\left(-\frac{2\pi n x}{\sqrt{6}}\right)$$

by virtue of the inversion symmetry

$$M_4(\lambda) \equiv \lambda^2 \sum_{n>0} A_{4,n} \exp\left(-\frac{2\pi n \lambda}{\sqrt{6}}\right) = M_4(1/\lambda)$$

that gives the reflection symmetry $\Lambda_4(s) = \Lambda_4(4-s)$.

Hence we obtain fast numerical evaluations of

$$L_4(2) = \sum_{n>0} \frac{A_{4,n}}{n^2} \left(2 + \frac{4\pi n}{\sqrt{6}}\right) \exp\left(-\frac{2\pi n}{\sqrt{6}}\right)$$

$$L_4(3) = \sum_{n>0} \frac{A_{4,n}}{n^3} \left(1 + \frac{2\pi n}{\sqrt{6}} + \frac{2\pi^2 n^2}{3}\right) \exp\left(-\frac{2\pi n}{\sqrt{6}}\right)$$

inside the critical strip and discover (at a few digits accuracy) and check to thousands of digits that

$$\overline{S}_6 = 48\zeta(2)L_4(2)$$
$$\overline{T}_6 = 12L_4(3)$$
$$\overline{U}_6 = 6L_4(2)$$

with a multiple $48\zeta(2) = 8\pi^2$ for the 4-loop sunrise diagram, to surprise Bloch and Brown.

No Bessel-related weight-5 modular form?

In related studies of Kloosterman sums no corresponding modular form was found at weight 5. Instead the combinatorics implicate integers whose residues modulo a prime p derive from the norms of complex eigenvalues of a weight-3 Hecke newform. I found no way to turn this integer data into an L function that might yield moments of 7 Bessel functions.

A modular form of weight 6

For moments of 8 Bessel functions, there is a relevant modular form:

$$f_6(q) = \sum_{n>0} A_{6,n} q^n = g(q)g(q^2), \text{ with } g(q) = \Big[\eta(q)\eta(q^3)\Big]^2 \sum_{m,n\in\mathbf{Z}} q^{m^2+mn+n^2}$$

Proceeding along the lines of the 6-Bessel problem I accelerated the convergence of

$$L_6(3) = \sum_{n>0} \frac{A_{6,n}}{n^3}\left(2 + \frac{4\pi n}{\sqrt{6}} + \frac{2\pi^2 n^2}{3}\right)\exp\left(-\frac{2\pi n}{\sqrt{6}}\right)$$

$$L_6(4) = \sum_{n>0} \frac{A_{6,n}}{n^4}\left(1 + \frac{2\pi n}{\sqrt{6}} + \frac{4\pi^2 n^2}{9} + \frac{4\pi^3 n^3}{9\sqrt{6}}\right)\exp\left(-\frac{2\pi n}{\sqrt{6}}\right)$$

$$L_6(5) = \sum_{n>0} \frac{A_{6,n}}{n^5}\left(1 + \frac{2\pi n}{\sqrt{6}} + \frac{\pi^2 n^2}{3} + \frac{2\pi^3 n^3}{9\sqrt{6}} + \frac{\pi^4 n^4}{27}\right)\exp\left(-\frac{2\pi n}{\sqrt{6}}\right)$$

and obtained the following integer multiples

$$\overline{T}_8 = 216 L_6(5), \quad \overline{U}_8 = 36 L_6(4), \quad \overline{W}_8 = L_6(3)$$

and the surprising relation

$$7L_6(5) = 4\zeta(2)L_6(3)$$

which was also checked at high precision.

Comments and puzzles

1. The algebraic geometry of integrands, written in terms of Schwinger parameters, gives clues about analytical structure.

2. In particular, it has motivated numerically successful guesses for massive Feynman integrals involving 5, 6 and 8 Bessel functions.

3. One of the many integrals needed for the 4-loop contributions to the magnetic moment of the electron has been evaluated

$$\overline{S}_6 = 8\pi^2 \sum_{n>0} \frac{A_{4,n}}{n^2}\left(2 + \frac{4\pi n}{\sqrt{6}}\right)\exp\left(-\frac{2\pi n}{\sqrt{6}}\right)$$

with coefficients readily obtained from the weight-4 modular form

$$\left[\eta(q)\eta(q^2)\eta(q^3)\eta(q^6)\right]^2 = \sum_{n>0} A_{4,n}q^n$$

4. What sort of number might the K3 in ϕ^4 produce at 8 loops?