Special Functions in *Mathematica*

Part I: general overview

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Mathematica rules

- Capitalized function names, e.g. Plot, Solve, Integrate
- Square brackets around the arguments, i.e. Sin[x], not Sin (x)
- Curly brackets for lists and ranges
- SHIFT + ENTER to evaluate
- Take advantage of pallettes →

Examples: Stand-alone commands

In[1]:=
```
Fibonacci[36]
```

Out[1]= 14 930 352

In[2]:=
```
Sin[36 Degree]
```

Out[2]= $\sqrt{\frac{5}{8} - \frac{\sqrt{5}}{8}}$

In[3]:=
```
Sum[x^k / k^2, {k, 1, Infinity}]
```

Out[3]= PolyLog[2, x]

In[4]:=
```
Series[Sin[Tan[x]] - Tan[Sin[x]], {x, 0, 12}]
```

Out[4]= $-\frac{x^7}{30} - \frac{29\,x^9}{756} - \frac{1913\,x^{11}}{75\,600} + O[x]^{13}$

In[5]:=
```
DSolve[y''[x] - x y[x] == x , y, x]
```

Out[5]= {{y → Function[{x}, π AiryAiPrime[x] AiryBi[x] - π AiryAi[x] AiryBiPrime[x] + AiryAi[x] C[1] + AiryBi[x] C[2]]}}

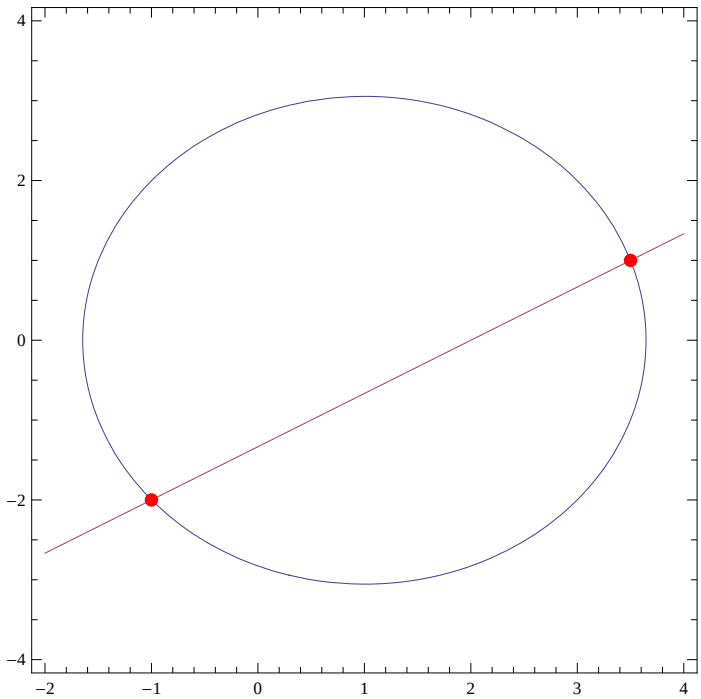
Examples: Command interaction

In[6]:=
```
sol = Solve[{4 (x - 1) ^2 + 3 y^2 == 28, 2 x - 3 y == 4}, {x, y}]
```

Out[6]= $\left\{\{x \to -1,\ y \to -2\},\ \left\{x \to \frac{7}{2},\ y \to 1\right\}\right\}$

In[7]:=
```
ContourPlot[{4 (x - 1) ^2 + 3 y^2 == 28, 2 x - 3 y == 4},
 {x, -2, 4}, {y, -4, 4},
 Epilog → {PointSize[Large], Red, Point[{x, y} /. sol]}]
```

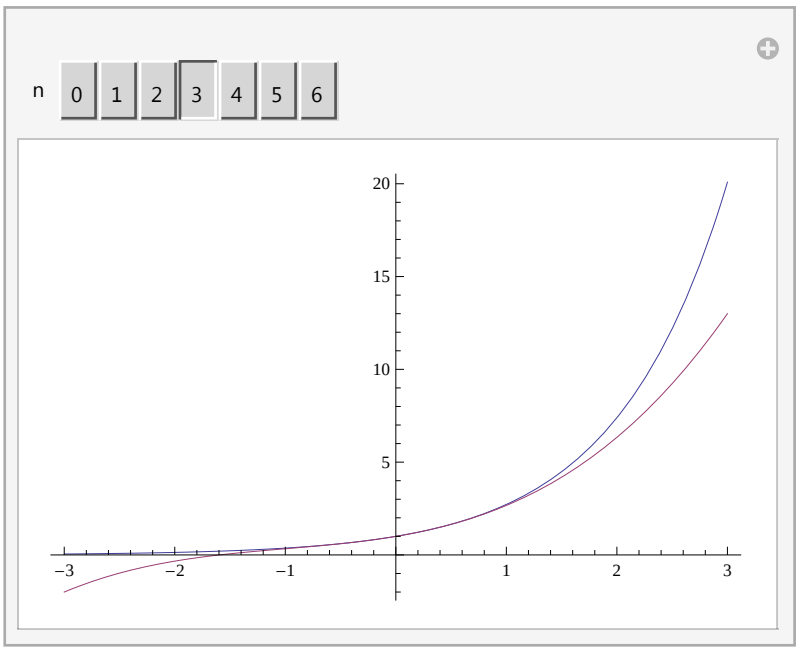
Out[7]=



Examples: Interactive exploration

In[8]:=
```
Manipulate[
 Plot[{Exp[x], Sum[x^k/k!, {k, 0, n}]}, {x, -3, 3}],
 {n, 0, 6, 1, SetterBar}]
```

Out[8]=



Special functions...

... are particular mathematical functions which have more or less established names and notations due to their importance in mathematical analysis, functional analysis, physics, or other applications.

There is no general formal definition, but the list of mathematical functions contains functions which are commonly accepted as special. In particular, *elementary functions are also considered as special functions*.

Comment The above is a quote from wikipedia [link]

… and why study them?

» to solve the problem they appeared in (physics, chemistry, statistics, engineering, etc.)

» because they provide a language to solve other problems with

The integral $\int \frac{\sin(x)}{x}\, dx$ is *not* expressible in elementary functions, see Liouville's theorem

But expand the language and the integral becomes doable:

```
Integrate[Sin[x] / x, x]
```

```
SinIntegral[x]
```

Sine integral provides a language to express answers to other problems as well:

$$\text{Sum}\left[\frac{(-1)\,\hat{}\,k}{2\,k+1}\,\frac{x^{2\,k+1}}{(2\,k+1)\,!},\ \{k,\ 0,\ \infty\}\right]$$

```
SinIntegral[x]
```

$$\text{Integrate}\left[\frac{\text{Exp}[-x\ t]}{1+t\,\hat{}\,2},\ \{t,\ 0,\ \text{Infinity}\},\ \text{Assumptions} \to x > 0\right]$$

$$\text{CosIntegral}[x]\ \text{Sin}[x] + \frac{1}{2}\,\text{Cos}[x]\ (\pi - 2\ \text{SinIntegral}[x])$$

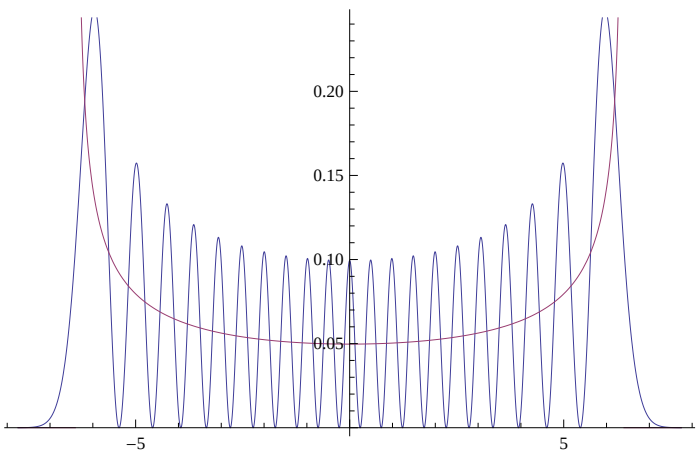
… and why study them?

» just for the fun of it

Example 1

Comparison of quantum and classical probability distributions for a harmonic oscillator:

```
Block[{n = 20},
 Plot[{e^(-x^2) HermiteH[n, x]^2/(2^n n! Sqrt[π]), Piecewise[
    {{1/(Sqrt[2 n + 1 - x^2] π), x^2 < 2 n + 1}}]}, {x, -Sqrt[3 n], Sqrt[3 n]}]]
```



Example 2

Given a word with character counts $\{n_1, \ldots, n_k\}$ the number of anagrams with no fixed letter:

```
DerangementsCount[nvec_List] := Integrate[Exp[-x]
    Product[(-1)^n LaguerreL[n, x], {n, nvec}], {x, 0, ∞}]
```

For "LHCPhenoNet":

```
LHCPhenoNet
1211 221  1
```

```
DerangementsCount[{1, 2, 1, 1, 2, 2, 1, 1}]
```

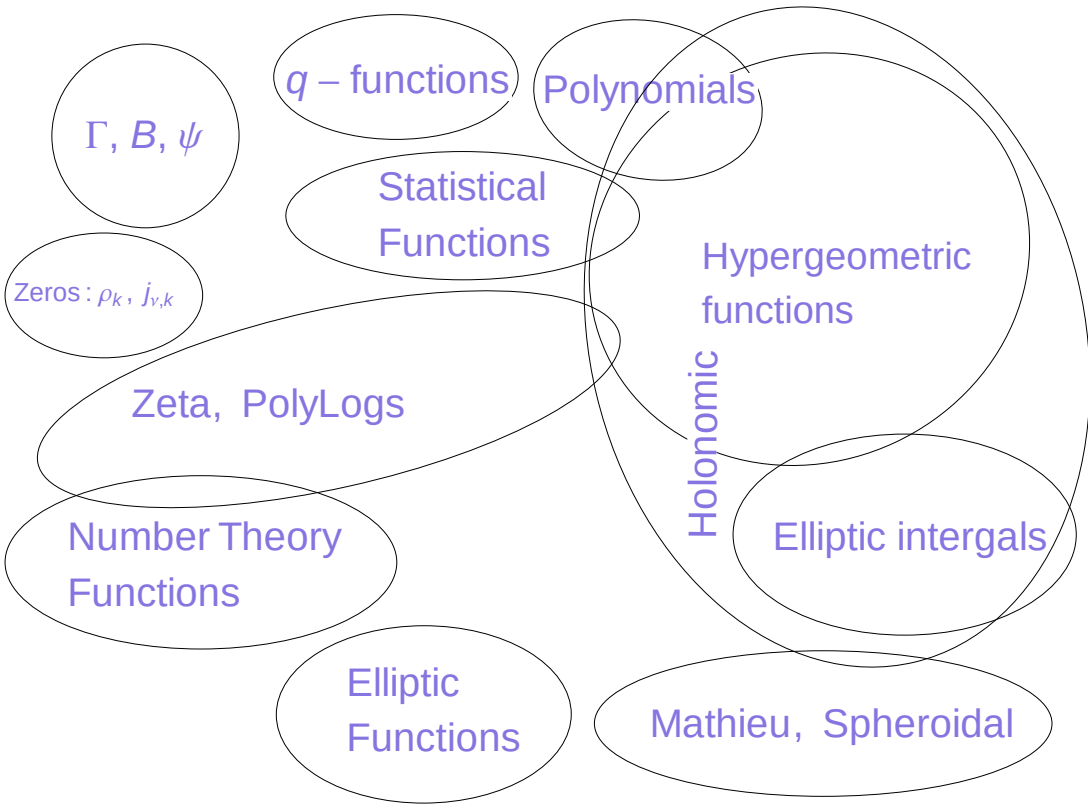
989 660

```
totalPermutationCount = Multinomial[1, 2, 1, 1, 2, 2, 1, 1]
```

4 989 600

Special functions in *Mathematica*

The chart of 180+ functions



▲ Landscape of special function groups, supported in *Mathematica* »

Hypergeometric function

Geometric series

$$\sum_{k=0}^{\infty} x^k$$

$$\frac{1}{1-x}$$

Generalized hypergeometric series

$$\sum_{k=0}^{\infty} c_k\, x^k$$

▲ here c_k is such that $c_{k+1} == c_k \frac{1}{k+1} \frac{(k+a_1)(k+a_2)\times\cdots\times(k+a_p)}{(k+b_1)(k+b_2)\times\cdots\times(k+b_q)}$.

The non-terminating series converges absolutely for $p < q+1$, or for $p = q+1$ and $|x| < 1$.

The series terminates if one $\{a_i\}$ is a non-positive integer.

A function defined by an *analytic continuation* of the above series is denoted as ${}_pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z)$ and in *Mathematica* as

```
HypergeometricPFQ[{a1, a2, …, ap}, {b1, b2, …, bq}, x]
```

Comment

Many elementary and special functions are hypergeometric functions.
Hypergeometric functions enjoy many nice properties warranting closer look in part II.

Special functions in *Mathematica*

extent of the support

- Numeric evaluation
- Exact evaluation
- Differentiation
- Series expansion
- Simplification rules, functional expansion
- Support in solvers (input & output)

Special functions in *Mathematica*

Exact evaluation

Sequences:

```
BernoulliB[60]
```

$-1\,215\,233\,140\,483\,755\,572\,040\,304\,994\,079\,820\,246\,041\,491 / 56\,786\,730$

```
Binomial[52, 6]
```

20 358 520

```
LegendreP[4, x]
```

$\frac{1}{8}\left(3 - 30\,x^2 + 35\,x^4\right)$

Evaluation at special points:

```
Erf[Infinity]
```

1

```
Zeta'[-1]
```

$\frac{1}{12} - \text{Log}[\text{Glaisher}]$

```
Hypergeometric2F1[n, n, 2 n + 1, 1]
```

$\frac{2^{2\,n}\,\text{Gamma}\left[\frac{1}{2} + n\right]}{\sqrt{\pi}\,\text{Gamma}[1 + n]}$

```
HermiteH[n, 0]
```

$\frac{2^n\,\sqrt{\pi}}{\text{Gamma}\left[\frac{1-n}{2}\right]}$

```
JacobiSN[EllipticK[m] / 2, m]
```

$\frac{1}{\sqrt{1 + \sqrt{1 - m}}}$

Evaluation to simpler functions:

```
Hypergeometric2F1[1, 1, 4, x]
```

$$-\frac{1}{2 x^3} 3 \left(2 x - 3 x^2 + 2 \operatorname{Log}[1 - x] - 4 x \operatorname{Log}[1 - x] + 2 x^2 \operatorname{Log}[1 - x]\right)$$

```
LegendreQ[3, 2, x]
```

$$\frac{\left(1 - x^2\right) \left(-8 + 25 x^2 - 15 x^4\right)}{\left(-1 + x^2\right)^2} + 15 x \left(1 - x^2\right) \left(-\frac{1}{2} \operatorname{Log}[1 - x] + \frac{1}{2} \operatorname{Log}[1 + x]\right)$$

Parity transformations, argument reduction make expression canonical:

```
Erf[-x]
```

$-\operatorname{Erf}[x]$

```
Sin[127 Degree]
```

$\operatorname{Cos}[37 °]$

```
JacobiSN[u + I EllipticK[1 - m] + 2 EllipticK[m], m]
```

$$-\frac{\operatorname{JacobiNS}[u, m]}{\sqrt{m}}$$

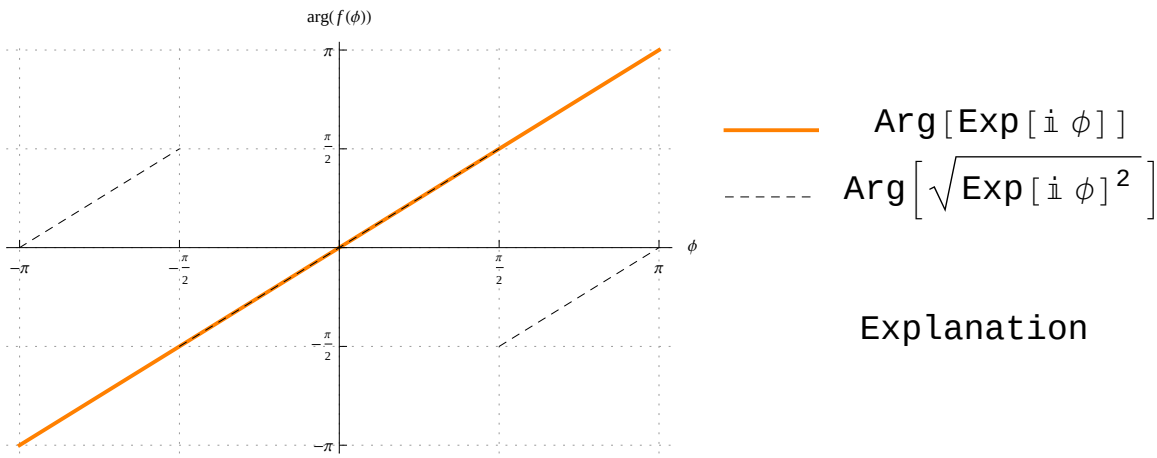
Special functions in *Mathematica*

Numeric evaluation: maximally extended domain

Special functions are functions *complex variable*.
Mathematical consistency —
choices of branch-cuts, and continuity at them, are consistent with functional relations between functions, and evaluation semantics.

Implications of argument reduction

$\sqrt{z^2} \neq z$, because in *Mathematica* argument is automatically reduced to $-\pi < \arg(z) \leq \pi$

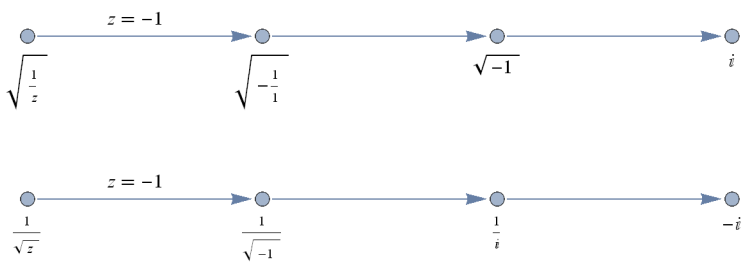


Explanation

Likewise $\sqrt{\frac{1}{z}} \neq \frac{1}{\sqrt{z}}$, $\log\left(\frac{1}{z}\right) \neq -\log(z)$, etc.

```
{{Sqrt[1/z], 1/Sqrt[z]}, {Log[1/z], -Log[z]}} /. z → -1
```

```
{{i, -i}, {i π, -i π}}
```



```
ExpIntegralE[1, x] // FunctionExpand
```

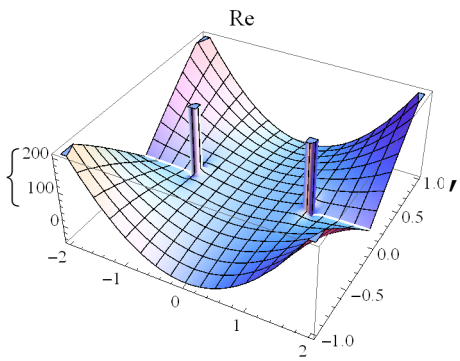
$$-\text{ExpIntegralEi}[-x] + \frac{1}{2}\left(-\text{Log}\left[-\frac{1}{x}\right] + \text{Log}[-x]\right) - \text{Log}[x]$$

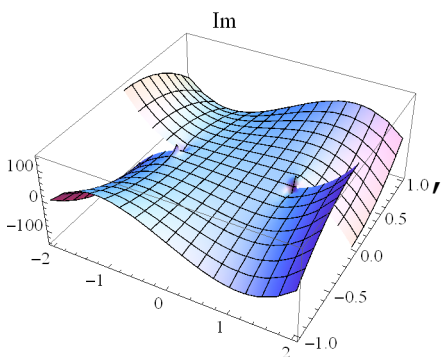
```
Refine[%, x > 0]
```

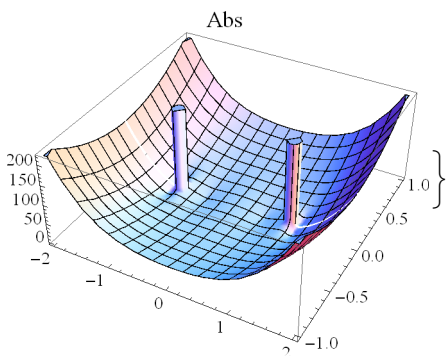
$$-\text{ExpIntegralEi}[-x]$$

Visualizing branch-cuts

Plot of Legendre function $Q_3^2(z)$ over a complex plane shows *singularities*, and *branch-cuts*:







$$\text{LegendreQ}[3, 2, z] == \frac{8-25 z^2+15 z^4}{-1+z^2} - 15 z \left(-1 + z^2\right) \text{ArcTanh}[z]$$

$Q_3^2(z)$ is continuous at branch line $z > 1$ from below and at $z < -1$ from above:

```
TableForm[
 Table[Limit[LegendreQ[3, 2, z], z → p, Direction → dir] //
   FullSimplify, {dir, {-I, I, 1}}, {p, {-2, 2}}],
 TableHeadings → {{"from above", "from below",
    "at the point"}, {z == -2, z == 2}}]
```

	z == -2	z == 2
from above	$\frac{148}{3} + 45\, i\, \pi - 45\, \mathrm{Log}[3]$	$\frac{148}{3} - 45\, i\, \pi - 45\, \mathrm{Log}[3$
from below	$\frac{148}{3} - 45\, i\, \pi - 45\, \mathrm{Log}[3]$	$\frac{148}{3} + 45\, i\, \pi - 45\, \mathrm{Log}[3$
at the point	$\frac{148}{3} + 45\, i\, \pi - 45\, \mathrm{Log}[3]$	$\frac{148}{3} + 45\, i\, \pi - 45\, \mathrm{Log}[3$

Special functions in *Mathematica*

Numeric evaluation: multiple algorithms

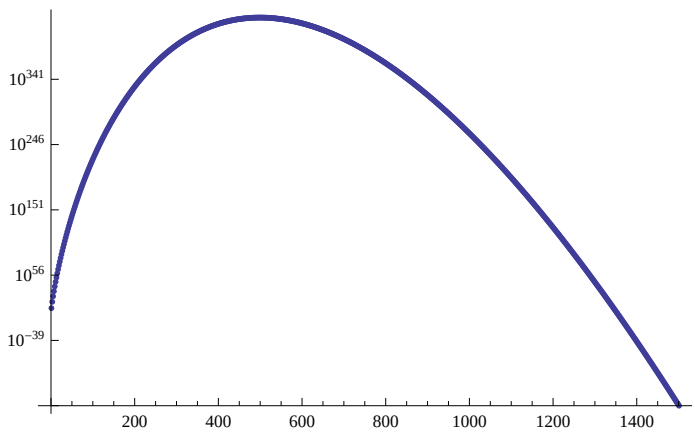
Use of asymptotic expansions

Direct summation of the defining series vs. asymptotic expansion:

```
BesselJ[3.3, 10.^3]
```

0.00691901

```
With[{ν = 3.3, x = 1000.},
 ListLogPlot[Table[(-1)^k (x/2)^(ν+2 k)/(k! Gamma[ν + 1 + k]), {k, 0, 1500}]]]
```




```
With[{ν = 3.3, x = 1000.},
  Max[Abs[Table[(-1)^k (x/2)^(ν+2 k)/(k! Gamma[ν + 1 + k]), {k, 0, 1500}]]]] //
 Log10
```

430.795020822201

```
With[{ν = 33/10, x = 1000``450},
 Sum[(-1)^k (x/2)^(ν+2 k)/(k! Gamma[ν + 1 + k]), {k, 0, 1500}]]
```

0.00691901025297857

```
Accuracy[%] + %%
```

448.402

Divide-and-conquer techniques

Index doubling:

```
AbsoluteTiming[ Fibonacci[10^7] // IntegerLength]
```

{0.1092002, 2 089 877}

```
ChebyshevT[10^7, 2] // IntegerLength // AbsoluteTiming
```

{0.2028004, 5 719 476}

```
v1 = ChebyshevT[10^4, 2]; // AbsoluteTiming
```

{0., Null}

```
poly = ChebyshevT[10^4, x]; // AbsoluteTiming
```

{0.7800014, Null}

```
v1 == poly /. x → 2
```

True

Use of divide-and-conquer summation at rationals (see [Haible, Papanikolaou], also [Karatsuba])

```
HermiteH[10^5, 2] // IntegerLength // AbsoluteTiming
```

{0.3276006, 243 338}

```
LegendreP[10^5, -3] // IntegerLength // AbsoluteTiming
```

{0.4836008, 76 553}

Truncated series for exponential: $\exp_n(1) = \sum_{k=0}^{n} \frac{(-n)_k}{(-n)_k} \frac{1}{k!}$

```
ra = HypergeometricPFQ[{-10 000}, {-10 000}, 1]; //
  AbsoluteTiming
```

{0.0312001, Null}

```
N[Exp[1], 35 700] - ra
```

$3.513382832584003155970194442193450 45 \times 10^{-35\,664}$

$\frac{1.0}{k!}$ /. k → 10 001

$3.513031564557862 \times 10^{-35\,664}$

Numeric vs. symbolic

Numeric evaluation is used:

```
HypergeometricPFQ[{1, 1, 1}, {3/2, 2}, 1`50]
```

2.46740110027233965470862274996903778382842485181 0

Symbolic evaluation is used first, then use N:

```
HypergeometricPFQ[{1, 1, 1}, {3/2, 2}, 1]
```

$$\frac{\pi^2}{4}$$

```
N[%, 50]
```

2.46740110027233965470862274996903778382842485181 02

Special functions in *Mathematica*

differentiation (see D, Derivative)

Expression for the derivative may be not unique, nicer one is chosen as a default:

```
D[BesselJ[ν, x], x]
```

$$\frac{1}{2}\left(\text{BesselJ}[-1+\nu, x]-\text{BesselJ}[1+\nu, x]\right)$$

```
% == ν BesselJ[ν, x]/x - BesselJ[1 + ν, x] ==
   BesselJ[-1 + ν, x] - ν BesselJ[ν, x]/x // FullSimplify
```

```
True
```

Special values of derivative:

```
D[BesselJ[ν, x], ν] /. ν → 0
```

$\text{BesselJ}^{(1,0)}[0, x]$

Comment Alternative input:

```
Derivative[1, 0][BesselJ][0, x]
```

$\text{BesselJ}^{(1,0)}[0, x]$

```
FunctionExpand[%]
```

Special functions in *Mathematica*

series expansion (see Series also Limit, Residue)

Expansion at generic points

Genererating function of Catalan numbers:

```
Series[2/(1 + Sqrt[1 - 4 x]), {x, 0, 12}]
```

$$1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + 132x^6 + 429x^7 + 1430x^8 + 4862x^9 + 16\,796x^{10} + 58\,786x^{11} + 208\,012x^{12} + O[x]^{13}$$

Expansion of Ai(x) at a generic point:

```
Series[AiryAi[x], {x, a, 4}]
```

$$\text{AiryAi}[a] + \text{AiryAiPrime}[a]\,(x-a) + \frac{1}{2}\,a\,\text{AiryAi}[a]\,(x-a)^2 + \frac{1}{6}\,(\text{AiryAi}[a] + a\,\text{AiryAiPrime}[a])\,(x-a)^3 + \frac{1}{24}\left(a^2\,\text{AiryAi}[a] + 2\,\text{AiryAiPrime}[a]\right)(x-a)^4 + O[x-a]^5$$

Expansion at poles

Explicit point:

```
Series[Cot[π x], {x, 1, 4}]
```

$$\frac{1}{\pi\,(x-1)} - \frac{1}{3}\,\pi\,(x-1) - \frac{1}{45}\,\pi^3\,(x-1)^3 + O[x-1]^5$$

Expansion of Euler Γ-function at symbolic negative integer point:

```
Series[Gamma[s], {s, -k, 1},
 Assumptions → k ∈ Integers && k ≥ 0]
```

$$\frac{(-1)^{-k}}{k!\,(s+k)} + \frac{(-1)^{-k}\,\text{PolyGamma}[0,\,1+k]}{k!} + \frac{1}{6\,k!}\,(-1)^{-k}\left(\pi^2 + 3\,\text{PolyGamma}[0,\,1+k]^2 - 3\,\text{PolyGamma}[1,\,1+k]\right)(s+k) + O[s+k]^2$$

Expansion at regular singular points

Wronskian of independent solutions of Bessel's ODE:

```
Series[Det[(BesselJ[ν, x]                BesselY[ν, x]
            D[BesselJ[ν, x], x]  D[BesselY[ν, x], x])],
  {x, 0, 2}]
```

$$x^{2\nu}\left(\left(2^{-2-2\nu}\,\mathrm{Cos}[\pi(1+\nu)]\,\mathrm{Gamma}[-1-\nu]\,x\right)/(\pi\,\mathrm{Gamma}[1+\nu])+O[x]^3\right)+x^{2\nu}\left(-\frac{2^{-2-2\nu}\,\mathrm{Cos}[\pi\nu]\,\mathrm{Gamma}[-\nu]\,x}{\pi\,\mathrm{Gamma}[2+\nu]}+O[x]^3\right)+\left(\frac{2}{\pi x}+\left(\frac{1}{4\pi(-1+\nu)\nu}+\frac{1}{4\pi\nu(1+\nu)}-\frac{\mathrm{Gamma}[-1+\nu]}{4\pi\,\mathrm{Gamma}[1+\nu]}-\frac{\mathrm{Gamma}[\nu]}{4\pi\,\mathrm{Gamma}[2+\nu]}\right)x+O[x]^2\right)+x^{2\nu}\left(\frac{2^{-2\nu}\,\mathrm{Cos}[\pi\nu]\,\mathrm{Gamma}[-\nu]}{\pi\,\mathrm{Gamma}[\nu]\,x}-\left(2^{-2-2\nu}(1+2\nu)\,\mathrm{Cos}[\pi\nu]\,\mathrm{Gamma}[-\nu]\,x\right)/(\pi\nu(1+\nu)\,\mathrm{Gamma}[\nu])+O[x]^2\right)+x^{2\nu}\left(-\left(2^{-2\nu}\,\mathrm{Cos}[\pi(-1+\nu)]\,\mathrm{Gamma}[1-\nu]\right)/(\pi\,\mathrm{Gamma}[1+\nu]\,x)+\left(2^{-2-2\nu}(1+2\nu)\,\mathrm{Cos}[\pi(-1+\nu)]\,\mathrm{Gamma}[1-\nu]\,x\right)/(\pi\nu(1+\nu)\,\mathrm{Gamma}[1+\nu])+O[x]^2\right)$$

```
FullSimplify[%]
```

$$x^{2\nu}\,O[x]^2+\left(\frac{2}{\pi x}+O[x]^2\right)$$

Compare:

```
Wronskian[{BesselJ[ν, x], BesselY[ν, x]}, x]
```

$$\frac{2}{\pi x}$$

Expansion at branch point

Point $z = 1$ is a *regular* singular point and a branch-point of Gauss's hypergeometric function:

```
Series[Hypergeometric2F1[a, b, a + b + 1, z], {z, 1, 1}]
```

$$\text{Floor}\left[-\frac{\text{Arg}[-1+z]}{2\pi}\right]\left(-\frac{2\,i\,\pi\,\text{Gamma}[1+a+b]\,(z-1)}{\text{Gamma}[a]\,\text{Gamma}[b]}+O[z-1]^2\right)+\left(\frac{\text{Gamma}[1+a+b]}{\text{Gamma}[1+a]\,\text{Gamma}[1+b]}-\frac{1}{\text{Gamma}[a]\,\text{Gamma}[b]}\text{Gamma}[1+a+b]\,(-1+2\,\text{EulerGamma}+i\,\pi+\text{Log}[-1+z]+\text{PolyGamma}[0,1+a]+\text{PolyGamma}[0,1+b])\,(z-1)+O[z-1]^2\right)$$

Branch-cut corrections are *artificial*, but needed:

```
Series[Hypergeometric2F1[a, b, a + b + 1, 1 - w], {w, 0, 1}]
```

$$\frac{\text{Gamma}[1+a+b]}{\text{Gamma}[1+a]\,\text{Gamma}[1+b]}+\frac{1}{\text{Gamma}[a]\,\text{Gamma}[b]}\text{Gamma}[1+a+b]\,(-1+2\,\text{EulerGamma}+\text{Log}[w]+\text{PolyGamma}[0,1+a]+\text{PolyGamma}[0,1+b])\,w+O[w]^2$$

Expansion at irregular singular points

```
Series[BesselI[ν, x], {x, Infinity, 3}] // Expand
```

$$e^{x}\left(\frac{\sqrt{\frac{1}{x}}}{\sqrt{2\pi}}-\frac{\left(-1+4\nu^2\right)\left(\frac{1}{x}\right)^{3/2}}{8\sqrt{2\pi}}+\frac{\left(9-40\nu^2+16\nu^4\right)\left(\frac{1}{x}\right)^{5/2}}{128\sqrt{2\pi}}+O\left[\frac{1}{x}\right]^{7/2}\right)+$$

$$e^{-x}\left(\frac{i\,e^{i\pi\nu}\sqrt{\frac{1}{x}}}{\sqrt{2\pi}}+\frac{i\,e^{i\pi\nu}\left(-1+4\nu^2\right)\left(\frac{1}{x}\right)^{3/2}}{8\sqrt{2\pi}}+\left(i\,e^{i\pi\nu}\left(9-40\nu^2+16\nu^4\right)\left(\frac{1}{x}\right)^{5/2}\right)\Big/\left(128\sqrt{2\pi}\right)+O\left[\frac{1}{x}\right]^{7/2}\right)$$

```
Series[Gamma[k + α]/Gamma[k + β], {k, ∞, 1}]
```

$$\left(\frac{1}{k}\right)^{-\alpha+\beta}\left(1+\frac{-\alpha+\alpha^2+\beta-\beta^2}{2k}+O\left[\frac{1}{k}\right]^{2}\right)$$

For further reference: Stirling's formula

Generalized Stirling's formula:

$$\log\Gamma(x+a)=\left(x+a-\frac{1}{2}\right)\log(x)-x+\frac{1}{2}\log(2\pi)+\sum_{m=1}^{p}\frac{(-1)^{m-1}}{m(m+1)}\frac{B_{m+1}(a)}{x^m}+o\left(x^{-p}\right)$$

```
With[{p = 10},
  Series[LogGamma[x + a] == (x + a - 1/2) Log[x] - x + 1/2 Log[2 π] +
     Sum[(-1)^(m-1)/(m (m + 1)) BernoulliB[m + 1, a] 1/x^m,
      {m, 1, p}], {x, ∞, p}]] // Simplify
```

True

Special functions in *Mathematica*

functional expansion: to simpler, less general (see FunctionExpand)

Expand Meijer's G-function

```
MeijerG[{{0, 0}, {}}, {{1, 1}, {}}, x]
```

```
MeijerG[{{0, 0}, {}}, {{1, 1}, {}}, x]
```

```
FunctionExpand[%]
```

$$\frac{x\,(2-2\,x+(1+x)\,\mathrm{Log}[x])}{(-1+x)^3}$$

Expansion hypergeometric function in terms on simpler special functions:

```
HypergeometricPFQ[{3/2}, {2, 3}, x]
```

$$\mathrm{HypergeometricPFQ}\left[\left\{\frac{3}{2}\right\}, \{2, 3\}, x\right]$$

```
FunctionExpand[%]
```

$$\frac{4\,\mathrm{BesselI}\left[1, \sqrt{x}\right]^2}{x}$$

Expansion of parabolic cylinder function in terms of elementary functions:

```
ParabolicCylinderD[3, x]
```

```
ParabolicCylinderD[3, x]
```

```
FunctionExpand[%]
```

$$\frac{e^{-\frac{x^2}{4}}\left(-6\sqrt{2}\,x+2\sqrt{2}\,x^3\right)}{2\sqrt{2}}$$

Special values:

```
Assuming[n ∈ Integers && n ≥ 1,
 FunctionExpand[Zeta[2 n]] // Refine]
```

$$-\frac{(-1)^n\,2^{-1+2\,n}\,\pi^{2\,n}\,\mathrm{BernoulliB}[2\,n]}{\mathrm{Gamma}[1+2\,n]}$$

Use FunctionExpand to expand holonomic DifferenceRoot/DifferentialRoot in special function:

```
FunctionExpand[DifferentialRoot[Function[{y, x},
    {y''[x] == x y[x], y[0] == c1, y'[0] == c2}]][z]]
```

$$\left(\frac{\text{c2 AiryAi}[z]}{3^{2/3}} - \frac{\text{c2 AiryBi}[z]}{3 \times 3^{1/6}}\right) \text{Gamma}\left[-\frac{2}{3}\right] + \frac{1}{2} 3^{2/3} \text{c1 AiryAi}[z] \text{ Gamma}\left[\frac{2}{3}\right] + \frac{1}{2} 3^{1/6} \text{c1 AiryBi}[z] \text{ Gamma}\left[\frac{2}{3}\right]$$

Special functions in *Mathematica*

simplification (see FullSimplify)

Simplification *blindly* applies applicable transformations to subexpressions, keeping only those that reduce complexity.

Identities with related functions

```
Sin[π ν] BesselY[ν, x] + BesselJ[-ν, x] // FullSimplify
```

```
BesselJ[ν, x] Cos[π ν]
```

```
ExpIntegralEi[Log[x]] // FullSimplify
```

```
LogIntegral[x]
```

```
Gamma[a] Gamma[1 - b]
   Hypergeometric1F1[a, b, z] / Gamma[1 + a - b] +
  z^(1-b) Gamma[b - 1] Hypergeometric1F1[1 + a - b, 2 - b, z] //
 FullSimplify
```

```
Gamma[a] HypergeometricU[a, b, z]
```

Recurrence equations

Differentiation rule for the fraction of Bessel functions:

```
D[BesselJ[ν + 1, x] / BesselJ[ν, x], x]
```

$$-((\text{BesselJ}[-1+\nu, x] - \text{BesselJ}[1+\nu, x])\,\text{BesselJ}[1+\nu, x]) / \left(2\,\text{BesselJ}[\nu, x]^2\right) + \frac{\text{BesselJ}[\nu, x] - \text{BesselJ}[2+\nu, x]}{2\,\text{BesselJ}[\nu, x]}$$

```
Simplify`SimplifyRecurrence[%]
```

$$1 + \frac{(-1-2\nu)\,\text{BesselJ}[1+\nu, x]}{x\,\text{BesselJ}[\nu, x]} + \frac{\text{BesselJ}[1+\nu, x]^2}{\text{BesselJ}[\nu, x]^2}$$

Thus we established:

```
g'[x] == 1 - (2 ν + 1)/x g[x] + g[x]^2 /.
  g → Function[x, BesselJ[ν + 1, x]/BesselJ[ν, x]] // FullSimplify
```

```
True
```

Verify solutions of differential equations:

```
x (1 - x) y''[x] + (γ - (α + β + 1) x) y'[x] - α β y[x] /.
  y -> Function[{x}, c1 Hypergeometric2F1[α, β, γ, x] +
     x^(1-γ) c2 Hypergeometric2F1[1 + α - γ,
       1 + β - γ, 2 - γ, x]] // FullSimplify
```

```
0
```

Word of caution

FullSimplify may reject transformations that increase complexity of intermediate expressions:

```
PolyLog[2, z] + PolyLog[2, 1/z] //
 FullSimplify[#, 0 < z < 1] &
```

$$\text{PolyLog}\left[2, \frac{1}{z}\right] + \text{PolyLog}[2, z]$$

Applying FunctionExpand forces those transformations to be applied:

```
PolyLog[2, z] + PolyLog[2, 1/z] // FunctionExpand //
 FullSimplify[#, 0 < z < 1] &
```

Special functions in *Mathematica*

support in solvers (Integrate, Sum, DSolve, etc.)

Used as output

Integrals:

```
Integrate[Exp[-z t] t^(1/2) (1 + t)^(-1/2),
 {t, 0, ∞}, Assumptions → z > 0]
```

$$\frac{1}{2}\sqrt{\pi}\ \text{HypergeometricU}\left[\frac{3}{2}, 2, z\right]$$

```
FunctionExpand[%]
```

$$-\frac{1}{2} e^{z/2}\ \text{BesselK}\left[0, \frac{z}{2}\right] + \frac{1}{2} e^{z/2}\ \text{BesselK}\left[1, \frac{z}{2}\right]$$

```
Integrate[t^(a-1) (1 - t)^(c-a-1) (1 - t z)^(-b),
 {t, 0, 1}, Assumptions → z < 1 && c > a > 0]
```

```
Gamma[a] Gamma[-a + c]
 Hypergeometric2F1Regularized[a, b, c, z]
```

```
FunctionExpand[%]
```

```
(Gamma[a] Gamma[-a + c] Hypergeometric2F1[a, b, c, z]) /
 Gamma[c]
```

Sums:

```
Sum[Binomial[n, k]^2 z^k, {k, 0, n}]
```

$$(1 - z)^n\ \text{LegendreP}\left[n, \frac{1 + z}{1 - z}\right]$$

```
Sum[Binomial[n, k]^4 z^k, {k, 0, n}]
```

```
HypergeometricPFQ[{-n, -n, -n, -n}, {1, 1, 1}, z]
```

Differential equations:

```
DSolve[y''[x] + 3 x y'[x] - 4 x^2 y[x] == 0, y, x]
```

$$\left\{\left\{y \to \text{Function}\left[\{x\}, e^{-2x^2} C[1]\, \text{HermiteH}\left[-\frac{4}{5}, \sqrt{\frac{5}{2}}\, x\right] + e^{-2x^2} C[2]\, \text{Hypergeometric1F1}\left[\frac{2}{5}, \frac{1}{2}, \frac{5x^2}{2}\right]\right]\right\}\right\}$$

Recurrence equations:

```
RSolve[(2 n + 1) f[n] == (n + 1) f[n - 1] ∧ f[1] == 1, f, n]
```

$$\left\{\left\{f \to \text{Function}\left[\{n\}, \frac{3 \times 2^{-2-n}\, \text{Pochhammer}[1, 1+n]}{\text{Pochhammer}\left[\frac{1}{2}, 1+n\right]}\right]\right\}\right\}$$

Used as input

In integrals:

```
Integrate[ArcTan[x] BesselK[0, z x],
 {x, 0, ∞}, Assumptions → z > 0]
```

$$\frac{1}{8}\, \text{MeijerG}\left[\left\{\{0\}, \left\{\frac{1}{2}\right\}\right\}, \left\{\left\{-\frac{1}{2}, 0, 0, 0\right\}, \{\}\right\}, \frac{z^2}{4}\right]$$

```
N[% /. z → 2]
```

0.211963

```
NIntegrate[ArcTan[x] BesselK[0, 2 x], {x, 0, ∞}]
```

0.211963

Sums:

```
Sum[CatalanNumber[k] CatalanNumber[n - 1 - k], {k, 0, n - 1}]
```

CatalanNumber[n]

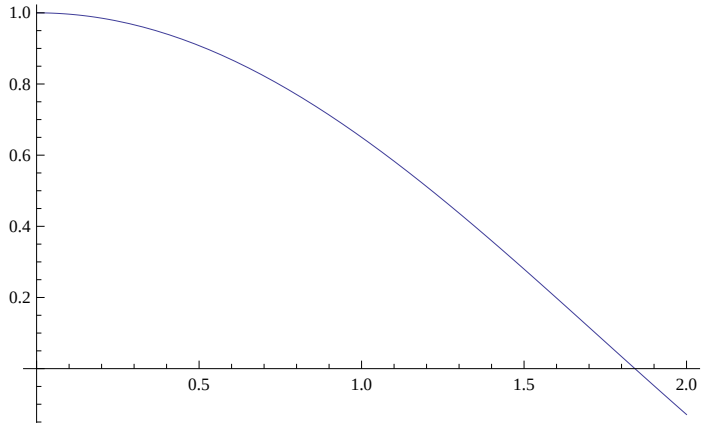
Solving equations:

```
Reduce[BesselJ[0, x] == BesselJ[2, x] && 0 < x < 2, x]
```

Reduce::nint: Warning: Reduce used numeric integration to show that the solution set found is complete. ≫

x == Root[{BesselJ[0, #1] - BesselJ[2, #1] &, 1.841183781340659302643629513 64}]

```
Plot[BesselJ[0, x] - BesselJ[2, x], {x, 0, 2}]
```



Solving differential equations:

```
DSolve[y''[x] + y[x] == SinIntegral[x], y, x]
```

$$\left\{\left\{y \to \text{Function}\left[\{x\}, C[1]\,\text{Cos}[x] + C[2]\,\text{Sin}[x] + \frac{1}{2}\left(\text{CosIntegral}[2\,x]\,\text{Sin}[x] - \text{Log}[x]\,\text{Sin}[x] + 2\,\text{Cos}[x]^2\,\text{SinIntegral}[x] + 2\,\text{Sin}[x]^2\,\text{SinIntegral}[x] - \text{Cos}[x]\,\text{SinIntegral}[2\,x]\right)\right]\right\}\right\}$$

Neat example

Holonomic representation of a hypergeometric function:

```
DifferenceRootReduce[
 Hypergeometric2F1[n + 1/2, n + 1/2, 2 n + 1, z], n]
```

$$\text{DifferenceRoot}\left[\text{Function}\left[\{\dot{y}, \dot{n}\}, \left\{64\,(1+\dot{n})\,(2+\dot{n})\,\dot{y}[\dot{n}] + 32\,(1+\dot{n})\,(2+\dot{n})\,(-2+z)\,\dot{y}[1+\dot{n}] + (3+2\,\dot{n})^2\,z^2\,\dot{y}[2+\dot{n}] == 0,\ \dot{y}[0] == \frac{2\,\text{EllipticK}[z]}{\pi},\ \dot{y}[1] == \text{Hypergeometric2F1}\left[\frac{3}{2}, \frac{3}{2}, 3, z\right]\right\}\right]\right][n]$$

Notice that for $z == 2$ the middle term of the recurrence relation vanishes.
Use FunctionExpand to try to solve the obtained recurrence equation:

```
FunctionExpand[% /. z → 2]
```

$$\left(e^{\frac{i n \pi}{2}}\,\text{EllipticK}[2]\,\text{Gamma}\left[\frac{3}{4}\right]^2\text{Gamma}[1+n]\right)\Big/\left(\pi\,\text{Gamma}\left[\frac{3}{4}+\frac{n}{2}\right]^2\right)+\left((-1)^n e^{\frac{i n \pi}{2}}\,\text{EllipticK}[2]\,\text{Gamma}\left[\frac{3}{4}\right]^2\text{Gamma}[1+n]\right)\Big/\left(\pi\,\text{Gamma}\left[\frac{3}{4}+\frac{n}{2}\right]^2\right)+\left(4\,i\,e^{\frac{i n \pi}{2}}\,\text{EllipticE}[2]\,\text{Gamma}\left[\frac{5}{4}\right]^2\text{Gamma}[1+n]\right)\Big/\left(\pi\,\text{Gamma}\left[\frac{3}{4}+\frac{n}{2}\right]^2\right)-\left(4\,i\,(-1)^n e^{\frac{i n \pi}{2}}\,\text{EllipticE}[2]\,\text{Gamma}\left[\frac{5}{4}\right]^2\text{Gamma}[1+n]\right)\Big/\left(\pi\,\text{Gamma}\left[\frac{3}{4}+\frac{n}{2}\right]^2\right)$$

```
Hypergeometric2F1[n + 1 / 2, n + 1 / 2, 2 n + 1, 2] ==
 FullSimplify[%, n ∈ Integers && n ≥ 0]
```

$$\text{Hypergeometric2F1}\left[\frac{1}{2}+n,\ \frac{1}{2}+n,\ 1+2n,\ 2\right] == \left(2\,(-1)^n\,n!\,\left(\text{Cos}\left[\frac{n\pi}{2}\right]\text{EllipticK}[2]\,\text{Gamma}\left[\frac{3}{4}\right]^2+4\,\text{EllipticE}[2]\,\text{Gamma}\left[\frac{5}{4}\right]^2\text{Sin}\left[\frac{n\pi}{2}\right]\right)\right)\Big/\left(\pi\,\text{Gamma}\left[\frac{3}{4}+\frac{n}{2}\right]^2\right)$$

Verify:

```
Table[%, {n, 0, 6}] // FullSimplify
```

{True, True, True, True, True, True, True}

Defining your own function

Define a function as symbolic expression in terms of known functions

Wave function of harmonic oscillator

```
ψ[n_, x_] := e^(-x^2/2) HermiteH[n, x] / Sqrt[2^n π^(1/2) n!]
```

Comment Advantage: solvers will see the input in terms of objects they know how to deal with.

```
Integrate[ψ[4, x] x^2 ψ[4, x], {x, -∞, ∞}]
```

$$\frac{9}{2}$$

Define atomic special function

Let $f(x) = \frac{x\cos(x)-\sin(x)}{x^2}$.

```
Clear[f];
f[0] := 0
f[(Pi k_.)] /; IntegerQ[k] := (-1)^k / (Pi k)
```

Evaluation for reals and complexes, using the form with no division by zero problem:

```
f[x_Real] := -x / 3 HypergeometricPFQ[{}, {5/2}, -x^2/4]
f[z_Complex] := -z / 3 HypergeometricPFQ[{}, {5/2}, -z^2/4]
f /: Derivative[1][f] :=
 Function[z, -(2 z Cos[z] + (-2 + z^2) Sin[z]) / z^3]
```

Comment Disadvantage of such an approach is that solvers see this function as a black box:

```
Integrate[f[x], x]
```

$$\int f[x]\, dx$$

```
Integrate[(x Cos[x] - Sin[x]) / x^2, x]
```

$$\frac{\text{Sin}[x]}{x}$$

Thank you

Initialization