Hypergeometric Functions and Loop Integrals - I

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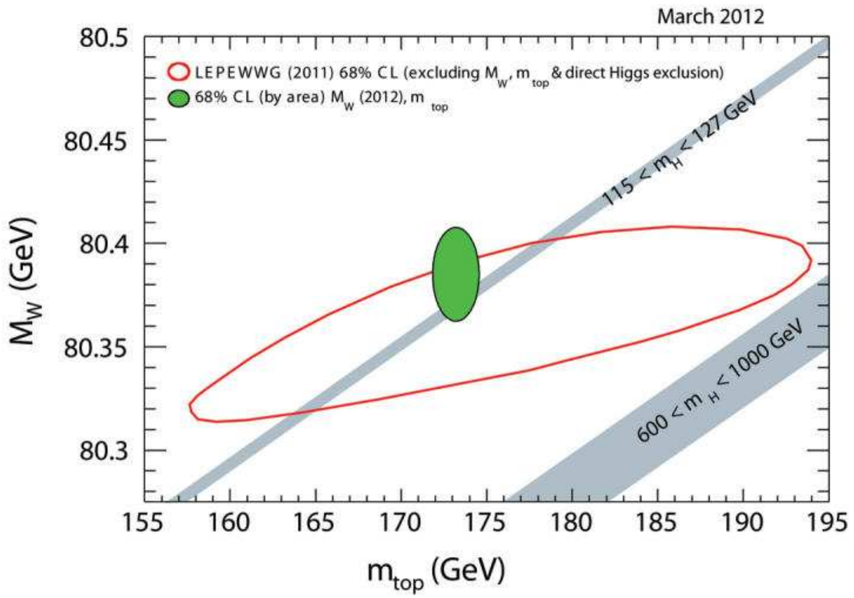
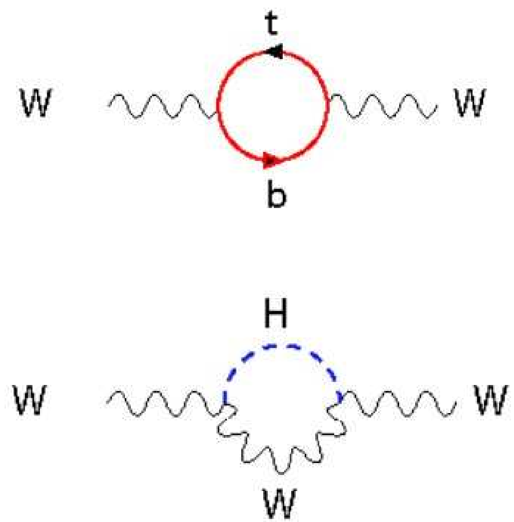
IPPP, Durham University

Integration, Summation and Special Functions in Quantum Field Theory,
9-13 July 2012, RISC

Why loop integrals?

- ✓ vital in making precise perturbative predictions in quantum field theory, in general, and in the Standard Model of particle physics, in particular.
- ✓ precise data enables information on new physics to be extracted indirectly (pre 4 July 2012)



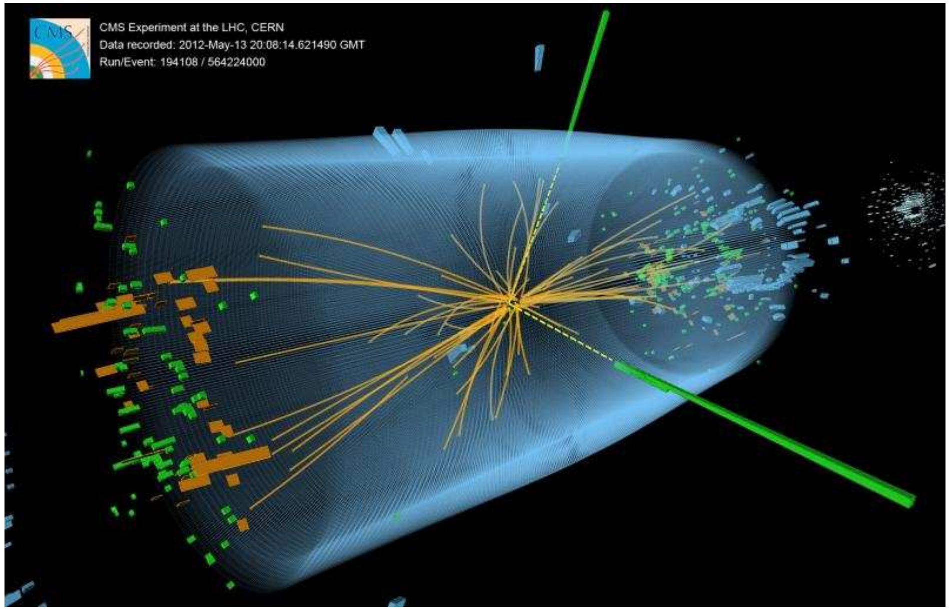


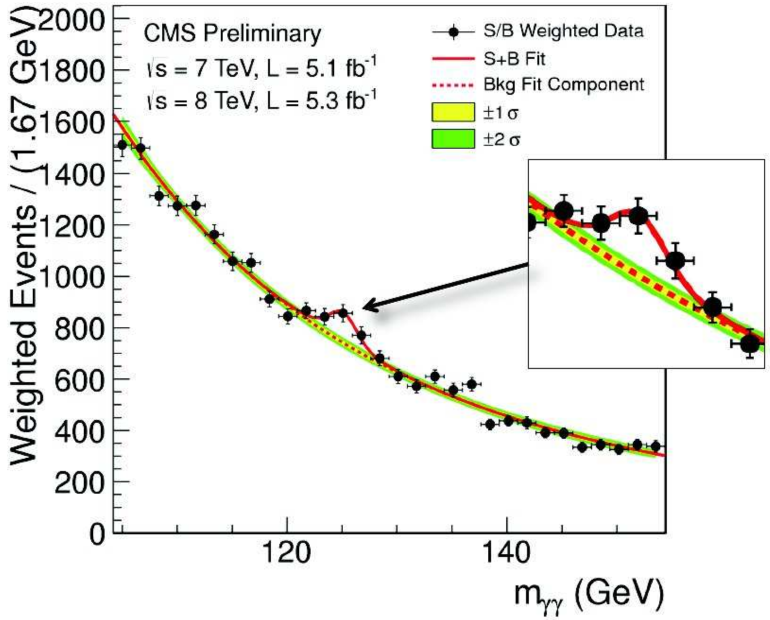
$\sim M_t^2$

$\sim \ln(m_H)$

Why loop integrals?

✓ equally important for interpretation of direct discovery - Latest update in the search for the Higgs boson, CERN, 4 July 2012

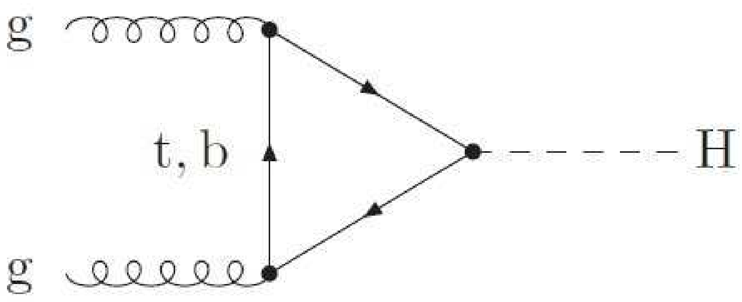




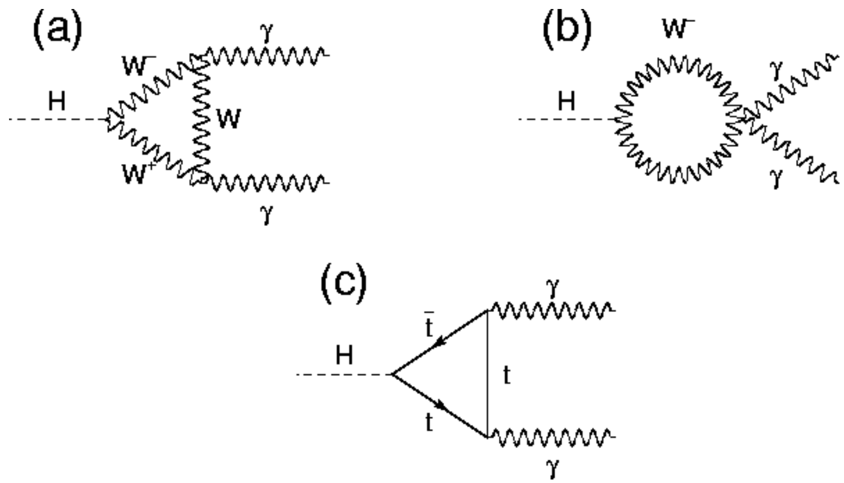
Why loop integrals?

✓ Interpreting size of bump as due to the production and decay of a Standard Model-like Higgs particle requires

✓ Higgs production cross section (σ)



✓ probability of decay into a particular final state (BR)



✓ The gluon fusion production cross section gets very large QCD corrections +70% at NLO and +20% at NNLO!

Why loop integrals?

✓ Loop integrals play an intrinsic part in

(a) the interpretation of experimental discoveries at the high energy frontier

(b) extracting precise information from precision experiments

(c) in making the case for the physics potential of future high energy facilities

Scalar Loop integrals

- ✓ complicated by the appearance of ultraviolet (UV) and infrared (IR) singularities, and it has become customary to use dimensional regularisation to extend the dimensionality of the loop integral away from 4-dimensions to $D = 4 - 2\epsilon$, to regulate the infrared and ultraviolet singularities.
- ✓ general scalar m-loop integral in momentum space with n propagators

$$I^D_{\nu_1\ldots\nu_n} = \int \frac{d^D\ell_1}{i\pi^{D/2}} \ldots \int \frac{d^D\ell_m}{i\pi^{D/2}} \frac{1}{A_1^{\nu_1}\ldots A_n^{\nu_n}}$$

- ✓ with the propagator $A_i =$ momentum^2 - mass^2 $+ i0$ e.g. $(\ell_1 + p_1)^2 + i0$

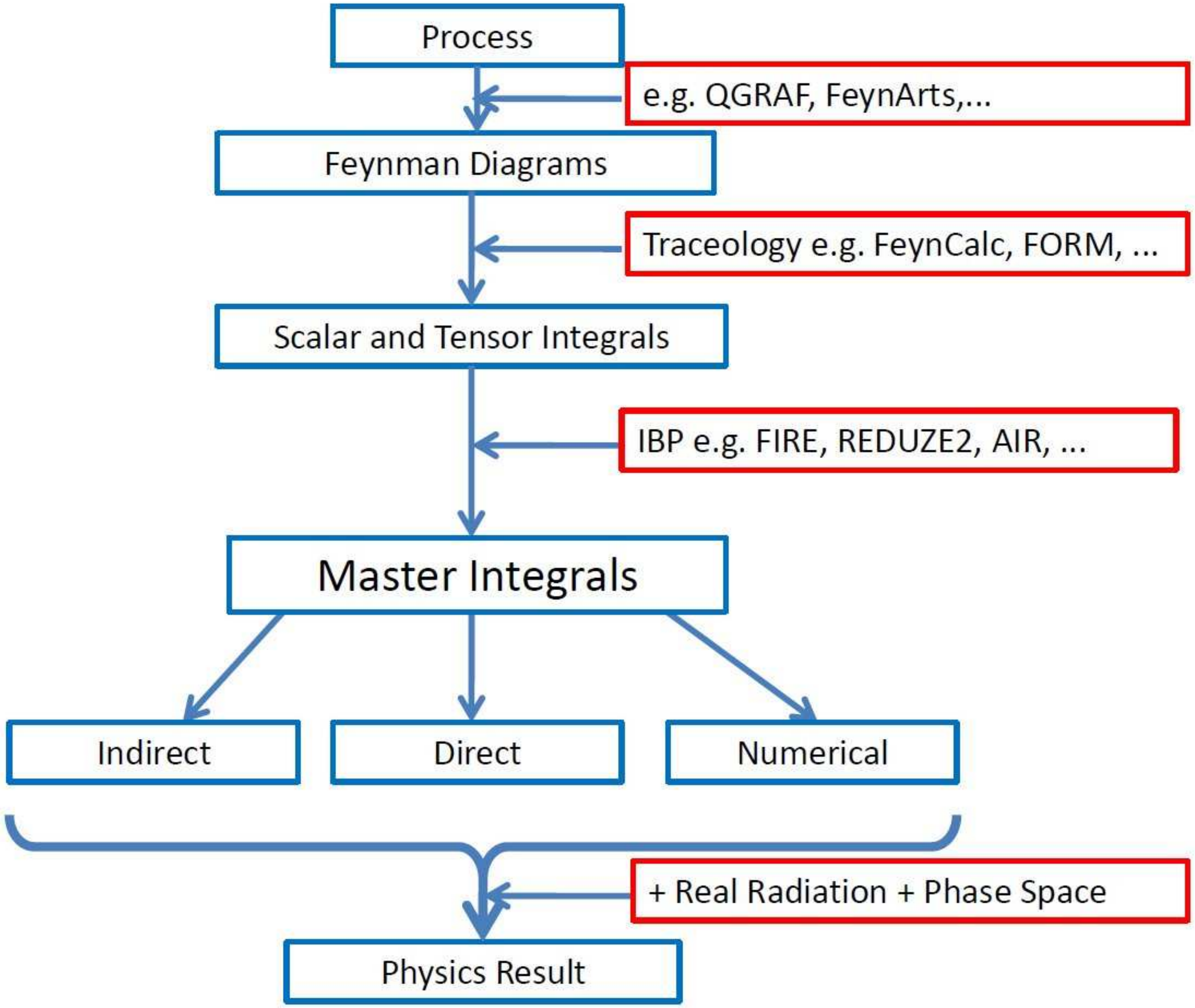
Tensor Loop integrals

- ✓ general tensor m-loop integral in momentum space with n propagators

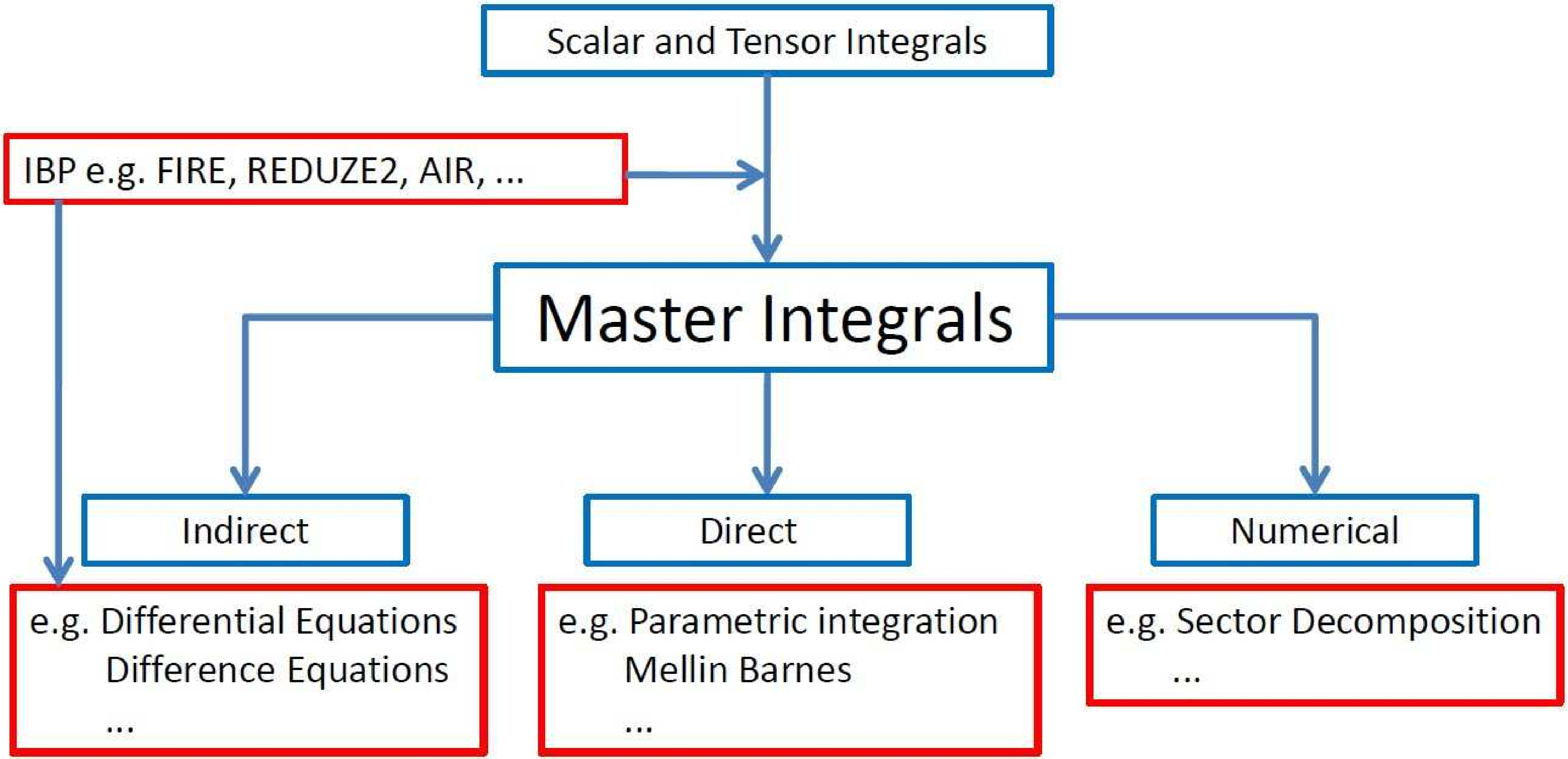
$$I^{D,\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_n} = \int \frac{d^D\ell_1}{i\pi^{D/2}} \ldots \int \frac{d^D\ell_m}{i\pi^{D/2}} \frac{\ell_1^{\mu_1}\ldots\ell_N^{\mu_n}}{A_1^{\nu_1}\ldots A_n^{\nu_n}}$$

- ✓ after extracting dependence on Lorentz indices, find scalar integrals (sometimes in shifted dimensions and with additional powers of propagators)
- ✓ Integration by Parts identities — Chetyrkin, Tkachov
- ✓ Dimensional shift equation — Tarasov

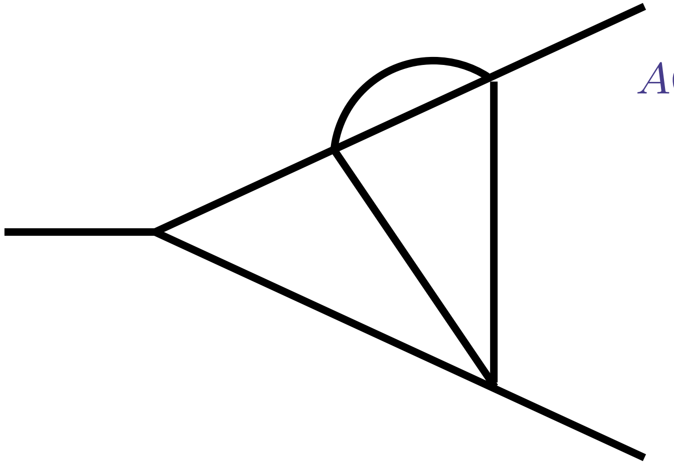
General work plan



Detailed work plan



Where do hypergeometric functions fit in?



$$A63 = (-Q^2)^{-3\epsilon} \frac{\Gamma(1-2\epsilon)\Gamma(1-3\epsilon)}{\Gamma(2-2\epsilon)\Gamma(2-3\epsilon)\Gamma(2-4\epsilon)} \times \left[\frac{\Gamma(1-3\epsilon)\Gamma(3\epsilon)\Gamma(2\epsilon)\Gamma(\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} + \frac{\Gamma(3\epsilon-1)\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} {}_3F_2(1, 1-\epsilon, 1-2\epsilon, 2-2\epsilon, 2-3\epsilon, 1) \right]$$

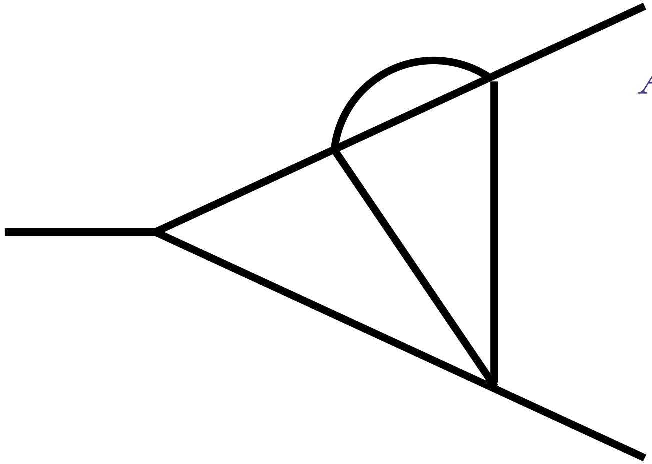
$P^2 = Q^2, p_1^2 = p_2^2 = 0$

Gehrmann, Heinrich, Huber, Studerus

- ✓ this is a one-scale integral (with unit propagators).
- ✓ All the scale dependence is in the prefactor. The hypergeometric function must therefore be evaluated at unity!
- ✓ evaluation of ${}_PF_{P-1}$ hypergeometric functions about integer or half-integer parameters can be done with the HypExp Mathematica package.

Maitre, Huber

Where do hypergeometric functions fit in?



$$A63 = \frac{1}{6\epsilon^3} + \frac{3}{2\epsilon^2} + \frac{1}{\epsilon}\left(\frac{55}{6} + \frac{\pi^2}{6}\right) + \left(\frac{3\pi^2}{2} + \frac{95}{2} - \frac{17\zeta_3}{3}\right) + \epsilon\left(\frac{1351}{6} + \frac{55\pi^2}{6} + \frac{\pi^4}{90} - 51\zeta_3\right) + \epsilon^3\left(\frac{2023}{2} + \frac{95\pi^2}{2} + \frac{\pi^4}{10} - \frac{935\zeta_3}{3} - \frac{10\pi^2\zeta_3}{3} - 65\zeta_5\right) + \mathcal{O}(\epsilon^3)$$

$P^2 = Q^2, p_1^2 = p_2^2 = 0$

Gehrmann, Heinrich, Huber, Studerus

- ✓ this is a one-scale integral (with unit propagators).
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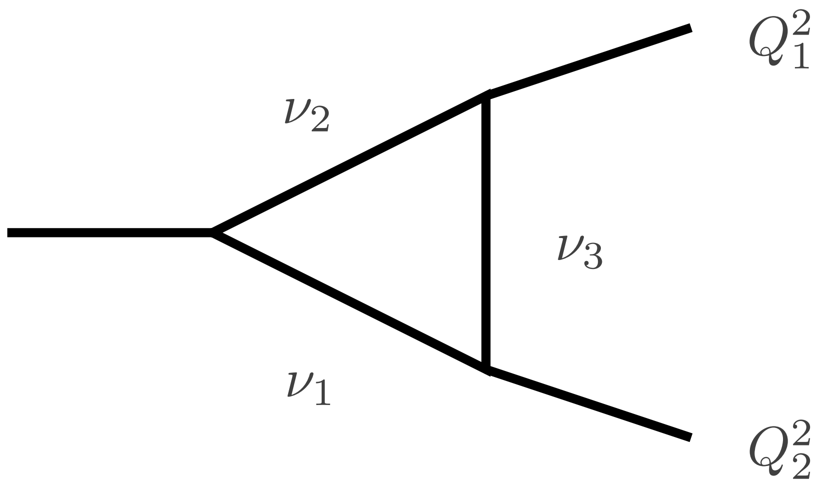
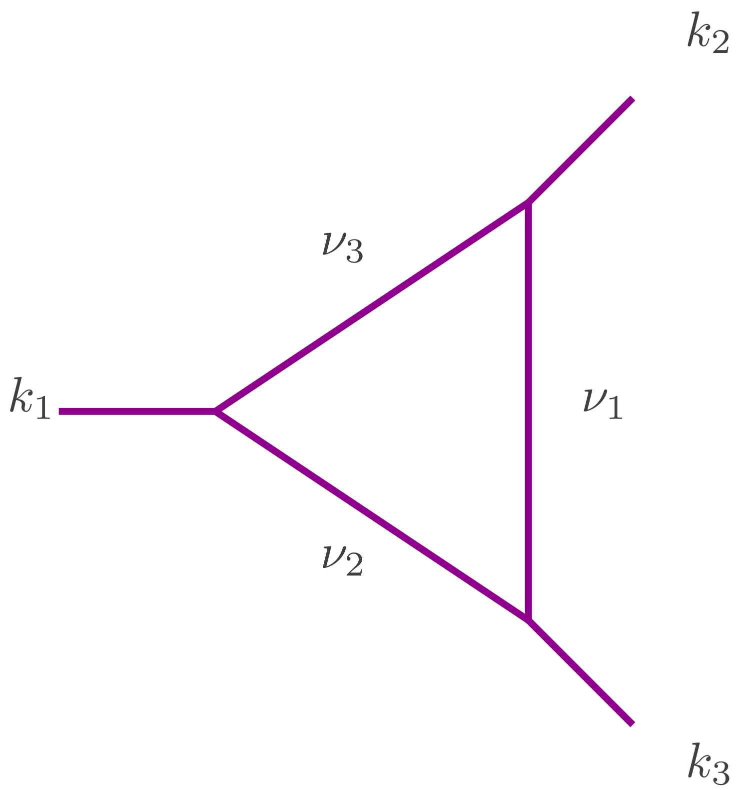
Where do hypergeometric functions fit in?

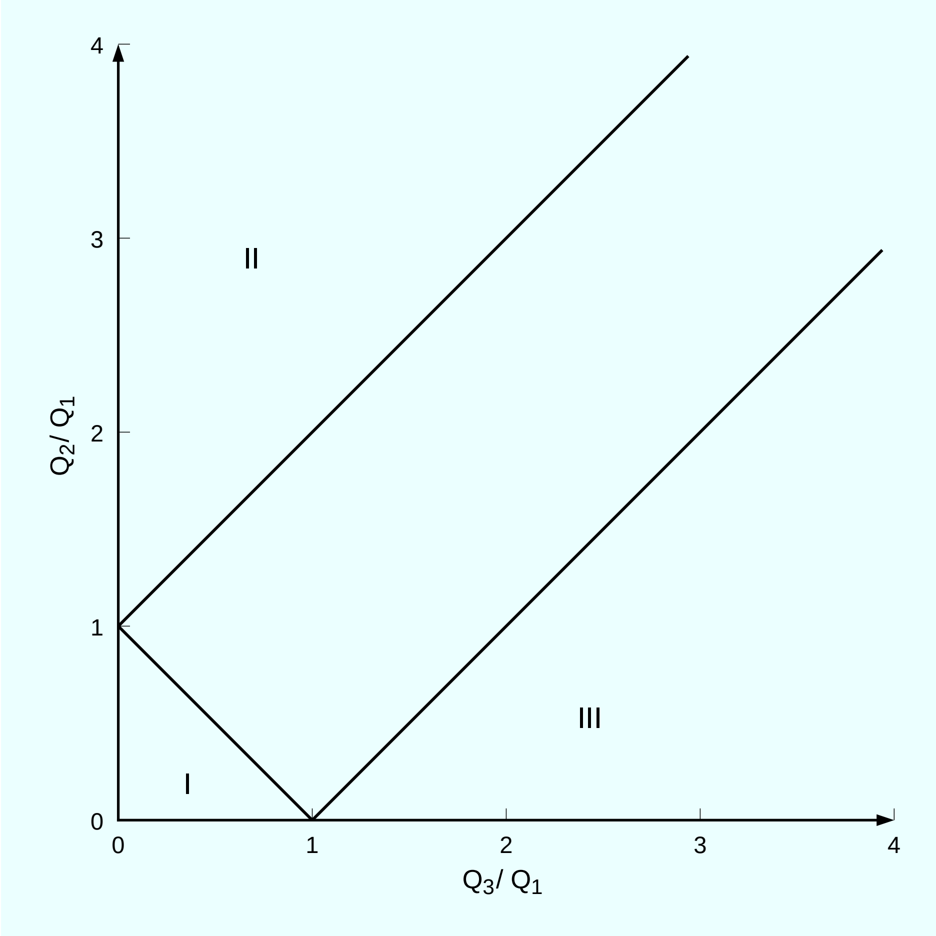


- ✓ massless one-loop triangle with two offshell legs, $Q_1^2 > Q_2^2$
- ✓ analytic continuation to $Q_1^2 < Q_2^2$

$$
\begin{aligned}
I^D_{\nu_1\nu_2\nu_3} \quad = \quad & (-1)^{\frac{D}{2}} \, (Q_1^2)^{\frac{D}{2}-\nu_1-\nu_2-\nu_3} \frac{\Gamma\left(\frac{D}{2}-\nu_1-\nu_2\right)\Gamma(\frac{D}{2}-\nu_1-\nu_3)\Gamma(\nu_1+\nu_2+\nu_3-\frac{D}{2})}{\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(D-\nu_1-\nu_2-\nu_3)} \\
& \times {}_2F_1\left(\nu_1, \nu_1+\nu_2+\nu_3-\frac{D}{2}, 1+\nu_1+\nu_3-\frac{D}{2}, \frac{Q_2^2}{Q_1^2}\right) \\
+ \quad & (-1)^{\frac{D}{2}} \, (Q_1^2)^{-\nu_2} \, (Q_2^2)^{\frac{D}{2}-\nu_1-\nu_3} \frac{\Gamma\left(\frac{D}{2}-\nu_1-\nu_2\right)\Gamma\left(\frac{D}{2}-\nu_3\right)\Gamma\left(\nu_1+\nu_3-\frac{D}{2}\right)}{\Gamma(\nu_1)\Gamma(\nu_3)\Gamma(D-\nu_1-\nu_2-\nu_3)} \\
& \times {}_2F_1\left(\nu_2, \frac{D}{2}-\nu_3, 1+\frac{D}{2}-\nu_1-\nu_3, \frac{Q_2^2}{Q_1^2}\right)
\end{aligned}
$$

Another example: the massless off-shell triangle





$$k_1^2 = Q_1^2,\ k_2^2 = Q_2^2,\ k_3^2 = Q_3^2$$

$$M_1 = M_2 = M_3 = 0$$

Mellin Barnes: Davydychev

Another example: the massless off-shell triangle

$$
\begin{aligned}
I^D_{\nu_1\nu_2\nu_3} = \; & (-1)^{\frac{D}{2}} \left(Q_1^2\right)^{\frac{D}{2}-\nu_{123}} \frac{\Gamma\left(\frac{D}{2}-\nu_{12}\right)\Gamma\left(\frac{D}{2}-\nu_{13}\right)\Gamma\left(\nu_{123}-\frac{D}{2}\right)}{\Gamma\left(\nu_2\right)\Gamma\left(\nu_3\right)\Gamma\left(D-\nu_{123}\right)} \\
& \times \; F_4\left(\nu_1, \nu_{123}-\frac{D}{2}, 1+\nu_{13}-\frac{D}{2}, 1+\nu_{12}-\frac{D}{2}, \frac{Q_2^2}{Q_1^2}, \frac{Q_3^2}{Q_1^2}\right) \\
+ \; & (-1)^{\frac{D}{2}} \left(Q_1^2\right)^{-\nu_2} \left(Q_2^2\right)^{\frac{D}{2}-\nu_{13}} \frac{\Gamma\left(\frac{D}{2}-\nu_{12}\right)\Gamma\left(\nu_{13}-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-\nu_3\right)}{\Gamma\left(\nu_1\right)\Gamma\left(\nu_3\right)\Gamma\left(D-\nu_{123}\right)} \\
& \times \; F_4\left(\nu_2, \frac{D}{2}-\nu_3, 1+\frac{D}{2}-\nu_{13}, 1+\nu_{12}-\frac{D}{2}, \frac{Q_2^2}{Q_1^2}, \frac{Q_3^2}{Q_1^2}\right) \\
+ \; & (-1)^{\frac{D}{2}} \left(Q_1^2\right)^{-\nu_3} \left(Q_3^2\right)^{\frac{D}{2}-\nu_{12}} \frac{\Gamma\left(\frac{D}{2}-\nu_{13}\right)\Gamma\left(\nu_{12}-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-\nu_2\right)}{\Gamma\left(\nu_1\right)\Gamma\left(\nu_2\right)\Gamma\left(D-\nu_{123}\right)} \\
& \times \; F_4\left(\nu_3, \frac{D}{2}-\nu_2, 1+\nu_{13}-\frac{D}{2}, 1+\frac{D}{2}-\nu_{12}, \frac{Q_2^2}{Q_1^2}, \frac{Q_3^2}{Q_1^2}\right) \\
+ \; & (-1)^{\frac{D}{2}} \left(Q_1^2\right)^{\nu_1-\frac{D}{2}} \left(Q_2^2\right)^{\frac{D}{2}-\nu_{13}} \left(Q_3^2\right)^{\frac{D}{2}-\nu_{12}} \frac{\Gamma\left(\nu_{12}-\frac{D}{2}\right)\Gamma\left(\nu_{13}-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-\nu_1\right)}{\Gamma\left(\nu_1\right)\Gamma\left(\nu_2\right)\Gamma\left(\nu_3\right)} \\
& \times \; F_4\left(D-\nu_{123}, \frac{D}{2}-\nu_1, 1+\frac{D}{2}-\nu_{13}, 1+\frac{D}{2}-\nu_{12}, \frac{Q_2^2}{Q_1^2}, \frac{Q_3^2}{Q_1^2}\right)
\end{aligned}
$$

Another example: the massless off-shell triangle

✓ F_4 is one of the Appell functions

$$F_4\left(\alpha,\beta,\gamma,\gamma',x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\alpha,m+n)(\beta,m+n)}{(\gamma,m)(\gamma',n)}\,\frac{x^m}{m!}\,\frac{y^n}{n!}$$

where (α,n) is the Pochhammer symbol

✓ converges when

$$\sqrt{x}+\sqrt{y}<1$$

i.e. precisely on the physical phase space

✓ analytic continuation gives solutions in other physical regions

✓ has a double integral representation

Another example: the massless off-shell triangle

✓ In fact, for unit propagators, $\nu_1 = \nu_2 = \nu_3 = 1$, the Appell F_4 functions collapse onto ${}_2F_1$ functions

$$F_4\left(\alpha, \beta, \beta, \beta, -\frac{x}{(1-x)(1-y)}, -\frac{y}{(1-x)(1-y)}\right)$$
$$= (1-x)^\alpha (1-y)^\alpha {}_2F_1\left(\alpha, 1+\alpha-\beta, \beta, xy\right),$$
$$F_4\left(\alpha, \beta, 1+\alpha-\beta, \beta, -\frac{x}{(1-x)(1-y)}, -\frac{y}{(1-x)(1-y)}\right)$$
$$= (1-y)^\alpha {}_2F_1\left(\alpha, \beta, 1+\alpha-\beta, -\frac{x(1-y)}{1-x}\right).$$

All one-loop integrals

- ✓ Using difference equations, and unit propagators, one can find expressions for arbitrary masses and kinematics

Fleischer, Jegerlehner, Tarasov

- ✓ two point functions: ${}_2F_1$
- ✓ three point functions: ${}_2F_1$ and F_1

$$F_1(\alpha, \beta_1, \beta_2, \gamma, x, y) = \sum \frac{(\alpha, m+n)(\beta_1, m)(\beta_2, n)}{(\gamma, m+n)m!n!} x^m y^n$$

- ✓ four point functions: ${}_2F_1$, F_1 and the Lauricella-Saran triple series

$$F_S(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_1, \gamma_1, x, y, z)$$
$$= \sum \frac{(\alpha_1, r)(\alpha_2, m+n)(\beta_1, r)(\beta_2, m)(\beta_3, n)}{(\gamma_1, r+m+n)r!m!n!} x^r y^m z^n$$

- ✓ In this case, the variables are complicated functions of the kinematic scales and internal masses

Connection to polylogarithms

- ✓ through expansion of integrand in integral representations e.g., F_1

$$F_1(\alpha, \beta_1, \beta_2, \gamma, x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 du u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta_1}(1-uy)^{-\beta_2}$$

- ✓ typically the parameters involve ϵ and can be expanded

$$(1-ux)^{-\epsilon} \to 1 - \epsilon \ln(1-ux) + \frac{\epsilon^2}{2} \ln^2(1-ux) + \mathcal{O}(\epsilon^3)$$

- ✓ Mathematica package HyperDIRE can reduce Appel $F_1, \ldots, F_4$ functions to a set of basis functions

Bytev, Kalmykov, Kniehl

Hypergeometric simplifications

- ✓ In general there are three types of identities that relate hypergeometric functions
  - ✓ Analytic continuations that connect functions in different regions of convergence
  - ✓ Reduction formulae which allow the functions to be expressed as simpler series for certain values of the parameters
  - ✓ Transformations which relate similar functions with different arguments
- ✓ For the functions of two variables, many of these relations are known in the literature
  - ✓ new relations found comparing different expressions of loop integrals

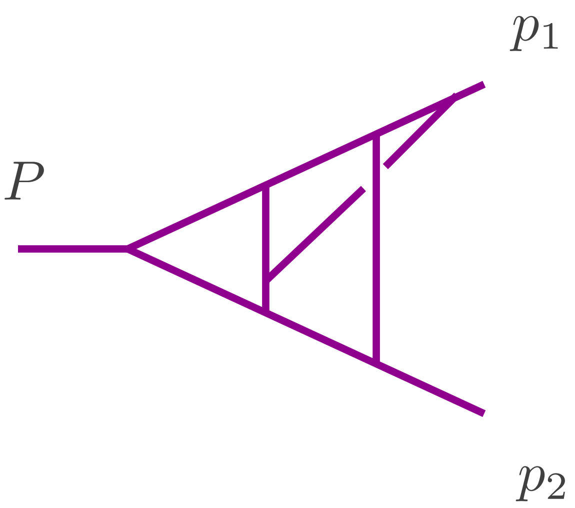
Kniehl, Tarasov

- ✗ To my knowledge, these relations and identities are less well known for functions of three or more variables - hope to learn more this week

Why hypergeometric sums?

- ✓ hypergeometric sums sometimes have integral representations themselves, in which an expansion in ϵ can be made, yielding expressions in logarithms, dilogarithms etc.
- ✓ when the series is convergent and well behaved in a particular region of phase space, it can be numerically evaluated. In fact, each hypergeometric sum immediately allows an asymptotic expansion of the integral in terms of ratios of momentum and mass scales.
- ✓ through analytic continuation formulae, the hypergeometric sums valid in one kinematic domain can be re-expressed in a different kinematic region.
- ✓ the convergence properties of the hypergeometric functions linked to the “physics" via the phase space boundaries
- ✓ still need to be able to turn the series into known numbers/functions?

Examples: three loop vertex



$$P^2 = Q^2, p_1^2 = p_2^2 = 0$$

$$
\begin{aligned}
A92 \quad = \quad & \frac{2}{9\epsilon^6} + \frac{5}{6\epsilon^5} + \frac{1}{\epsilon^4}\left(-\frac{20}{9} - \frac{7\pi^2}{27}\right) \\
& + \frac{1}{\epsilon^3}\left(+\frac{50}{9} - \frac{17\pi^2}{27} - \frac{91\zeta_3}{9}\right) \\
& + \frac{1}{\epsilon^2}\left(+\frac{4\pi^2}{3} - \frac{166\zeta_3}{9} - \frac{373\pi^4}{1080} - \frac{110}{9}\right) \\
& + \frac{1}{\epsilon}\left(+\frac{494\zeta_3}{9} + \frac{179\pi^2\zeta_3}{27} - 167\zeta_5 - \frac{16\pi^2}{9} - \frac{187\pi^4}{540} + \frac{170}{9}\right) \\
& + \left(+\frac{130}{9} - \frac{32\pi^2}{9} - \frac{1466\zeta_3}{9} + \frac{679\pi^4}{540} + \frac{682\pi^2\zeta_3}{27} - \frac{1390\zeta_5}{3}\right. \\
& \left. - \frac{59797\pi^6}{136080} + \frac{169\zeta_3^2}{9}\right) + \mathcal{O}(\epsilon)
\end{aligned}
$$

Heinrich, Huber, Kosower, Smirnov; Baikov, Chetrykin, Smirnov, Smirnov, Steinhauser; Lee, Smirnov, Smirnov

Loop integrals into hypergeometric sums

- ✓ The Negative Dimension (NDIM) method gives an efficient way of converting a loop integral into multiple hypergeometric sums.

  Halliday, Ricotta (87), Broadhurst (87) Suzuki, Schmidt (97), Anastasiou, NG, Oleari (99)

  - ✓ exploits analytic properties of integral
  - ✓ treats the case of arbitrary powers of propagators and arbitrary dimension
  - ✓ rapidly (and automatically) gives result as combinations of multiple hypergeometric sums
  - ✓ gives solutions valid in all regions of phase space
  - ✗ knowledge of convergence properties of hypergeometric sums needed to construct full solution in a particular region of phase space
  - ✗ still need to be able to turn multiple series into known numbers/functions

Schwinger parameterisation

- ✓ general scalar m-loop integral in momentum space with n propagators

$$I^D_{\nu_1\ldots\nu_n} = \int \frac{d^D\ell_1}{i\pi^{D/2}} \cdots \int \frac{d^D\ell_m}{i\pi^{D/2}} \frac{1}{A_1^{\nu_1}\ldots A_n^{\nu_n}}$$

- ✓ Schwinger parameterisation:

$$\frac{1}{A_i^{\nu_i}} = \frac{(-1)^{\nu_i}}{\Gamma(\nu_i)} \int_0^\infty dx_i x_i^{\nu_i-1} \; \exp(x_i A_i),$$

one (independent) parameter for each propagator, such that

$$I^D_{\nu_1\ldots\nu_n} \quad = \quad \left(\prod_{i=1}^n \frac{(-1)^{\nu_i}}{\Gamma(\nu_i)} \int_0^\infty dx_i x_i^{\nu_i-1}\right) \int \frac{d^D\ell_1}{i\pi^{D/2}} \cdots \int \frac{d^D\ell_m}{i\pi^{D/2}} \; \exp\left(\sum_{i=1}^n x_i A_i\right),$$

Schwinger parameterisation continued

- ✓ Completing the square and integrating out loop momenta

$$I^D_{\nu_1\ldots\nu_n} = \left(\prod_{i=1}^{n} \frac{(-1)^{\nu_i}}{\Gamma(\nu_i)} \int_0^\infty dx_i x_i^{\nu_i - 1}\right) \frac{1}{\mathcal{P}^{D/2}} \exp\left(\frac{\mathcal{Q}}{\mathcal{P}}\right) \exp\left(-\mathcal{M}\right).$$

- ✓ Here, $\mathcal{Q}$, $\mathcal{P}$ and $\mathcal{M}$ depend on the mass/momentum scales and the parameters, and can be simply extracted from loop integral
  - ✓ $\mathcal{M} = \sum x_i M_i^2$ where the sum runs over the propagators with mass M_i^2
  - ✓ $\mathcal{P}$ is the first Symanzik polynomial (sometimes called $\mathcal{U}$) composed of x_i
  - ✓ $\mathcal{Q}$ is the second Symanzik polynomial (sometimes called $\mathcal{F}$) composed of x_i and invariants $(p_i + p_j)^2$
- ✓ This is the starting point for the NDIM method