Generalization of Risch's Algorithm to Special Functions

Clemens G. Raab (RISC)

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Overview

1. Introduction to symbolic integration
2. Relevant classes of functions and Risch's algorithm
3. Basics of differential fields
4. A generalization of Risch's algorithm
   1. Introduction
   2. Inside the algorithm
5. Application to definite integrals depending on parameters

Introduction to symbolic integration

Symbolic integration

Computer algebra

1. Model the functions by algebraic structures
2. Computations in the algebraic framework
3. Interpret result in terms of functions

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Different approaches and structures

- Differential algebra: differential fields

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- Differential algebra: differential fields
- Holonomic systems: Ore algebras

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2. Computations in the algebraic framework
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Different approaches and structures

- Differential algebra: differential fields
- Holonomic systems: Ore algebras
- Rule-based: expressions, tables of transformation rules
- ...

Indefinite integration

Antiderivatives

$$\int f(x)\,dx = g(x)$$

Indefinite integration

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Examples

$$\begin{aligned}
\int \frac{\mathrm{Li}_3(x) - x\mathrm{Li}_2(x)}{(1-x)^2}\,dx &= \frac{x}{1-x}\left(\mathrm{Li}_3(x) - \mathrm{Li}_2(x)\right) + \frac{\ln(1-x)^2}{2} \\
\int \mathrm{Ai}'(x)^2\,dx &= \tfrac{1}{3}\left(x\mathrm{Ai}'(x)^2 + 2\mathrm{Ai}(x)\mathrm{Ai}'(x) - x^2\mathrm{Ai}(x)^2\right) \\
\int \frac{1}{xJ_n(x)Y_n(x)}\,dx &= \frac{\pi}{2}\ln\left(\frac{Y_n(x)}{J_n(x)}\right)
\end{aligned}$$

Definite integration

Integrals depending on parameters

$$\int_a^b f(x,y)\,dx = g(y)$$

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Examples

$$\int_0^\infty \frac{zx}{e^x - z}\,dx = \mathrm{Li}_2(z)$$

$$\int_0^\infty e^{-sx}\gamma(a,x)\,dx = \frac{\Gamma(a)}{s(s+1)^a}$$

$$\int_0^1 e^{-2n\pi ix}\ln(\sin(\tfrac{\pi}{2}x))\,dx = -\frac{1}{4n} + \frac{i}{n\pi}\sum_{k=1}^n \frac{1}{2k-1}$$

Example: Gamma function

$$\Gamma(z) := \int_0^\infty \underbrace{x^{z-1}e^{-x}}_{=:f(z,x)} dx \qquad \text{for } z > 0$$

We compute

$$zf(z,x) - f(z+1,x) = \frac{d}{dx}x^z e^{-x}$$

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After integrating from 0 to ∞ we obtain

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In other words, we proved

$$z\Gamma(z) - \Gamma(z+1) = 0$$

Linear Relations

Integrals depending on one parameter

- $c_0(y)f(x,y) + \cdots + c_m(y)\frac{\partial^m f}{\partial y^m}(x,y) = \frac{d}{dx}g(x,y)$

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- $c_0(n)f(x,n)+\dots+c_m(n)f(x,n+m)=\frac{d}{dx}g(x,n)$ yields a recurrence for

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Parametric integration

Compute linear relation of integrals

Given $f\ \ (x)$, find $g(x)$ s.t.

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Given $f_0(x), \dots, f_m(x)$, find $g(x)$ and $c_0, \dots, c_m$ const. w.r.t. x s.t.

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Transfer this to a relation of corresponding integrals

$$c_0 \int_a^b f_0(x)dx + \cdots + c_m \int_a^b f_m(x)dx = g(b) - g(a)$$

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Certificate

$g(x)$ is a certificate for the relation

$$c_0 \int_a^b f_0(x)dx + \cdots + c_m \int_a^b f_m(x)dx = r$$

It is easy to verify

$$c_0 f_0(x) + \cdots + c_m f_m(x) = g'(x) \quad \text{and} \quad r = g(b) - g(a)$$

Relevant classes of functions and Risch's algorithm

Elementary functions

Characterization

Functions constructed from rational functions by

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Examples

algebraic functions, logarithms, c^x, x^c, trigonometric/hyperbolic functions and their inverses, ...

$$\frac{\ln(x+3)^2 - 4x}{\exp(\exp(x) - \frac{1}{x})\sqrt{\cos(2x)}} \qquad \frac{\arctan(\tanh(\frac{x}{2}))}{x^{x\ln(x)}\tan(x)}$$

Elementary integrals of elementary functions

Problem

- Given an elementary function $f(x)$
- Decide whether there is an elementary function $g(x)$ with $g'(x) = f(x)$ and compute such a $g(x)$ if it exists

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Algorithm

Risch 1969, Bronstein 1990

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Examples

$$\int \frac{1}{x^2 - 2}\,dx = \frac{\sqrt{2}}{4}\log\left(\frac{x-\sqrt{2}}{x+\sqrt{2}}\right)$$

$$\int \exp(x^2)\,dx \text{ is not elementary}$$

Example

$$\int \frac{x^4 + 2x^3 - x^2 + 3}{(x+1)(x+2)^2}\, dx = ?$$

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Split numerator $a(x) = x^4 + 2x^3 - x^2 + 3$ of integrand

$$a(x) = b(x) \cdot (-(x+1)) + c(x) \cdot (x+2).$$

By EEA we compute

$$b(x) = -1 \qquad c(x) = x^3 - x + 1.$$

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So

$$\begin{aligned} \int \frac{x^4 + 2x^3 - x^2 + 3}{(x+1)(x+2)^2}\, dx &= \frac{b(x)}{x+2} + \int \frac{c(x) - (x+1)b'(x)}{(x+1)(x+2)}\, dx \\ &= -\frac{1}{x+2} + \int \frac{x^3 - x + 1}{(x+1)(x+2)}\, dx. \end{aligned}$$

Example (cont.)

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For determining the residues we compute the Gröbner basis of

$$\{a(x) - zb'(x), b(x)\}$$

w.r.t. $z < x$ with numerator $a(x) = x^3 - x + 1$ and denominator $b(x) = (x+1)(x+2)$:

$$\{(z-1)(z-5), x + \tfrac{1}{4}z + \tfrac{3}{4}\}$$

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So with the residues $z = 1$ and $z = 5$

$$\int \frac{x^3 - x + 1}{(x+1)(x+2)}\,dx = 1\ln(x + \tfrac{1}{4} + \tfrac{3}{4}) + 5\ln(x + \tfrac{5}{4} + \tfrac{3}{4}) + \int x - 3\,dx$$

Example (cont.)

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Ansatz:

$$x - 3 = \frac{d}{dx}(a_2 x^2 + a_1 x)$$

Comparing coefficients leads to

$$a_2 = \tfrac{1}{2} \qquad a_1 = -3.$$

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Altogether, we obtained

$$\int \frac{x^4 + 2x^3 - x^2 + 3}{(x+1)(x+2)^2}\, dx = -\frac{1}{x+2} + \ln(x+1) + 5\ln(x+2) + \tfrac{1}{2}x^2 - 3x$$

Liouvillian functions

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Examples

elementary functions, exponential integrals, polylogarithms, error functions, Fresnel integrals, incomplete gamma function, . . .

$$\mathrm{Ei}(2\ln(x)) \qquad \mathrm{Li}_2(e^x) \qquad e^{-x^2}\left(\tfrac{\pi}{2}\mathrm{erfi}(x) - \tfrac{1}{2}\mathrm{Ei}(x^2)\right)$$

$$\int_{-\infty}^{x} \cos(\tfrac{\pi}{2}u^2)(C(u) + \tfrac{1}{2})(S(u) - \tfrac{1}{2})\, du$$

Generalization

Characterization

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- basic arithmetic operations $+, -, *, /$
- taking solutions of algebraic equations $y(x)^m + a_{m-1}(x)y(x)^{m-1} + \cdots + a_0(x) = 0$
- taking solutions of 2-dimensioinal differential systems

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}' = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} + \begin{pmatrix} b_1(x) \\ b_2(x) \end{pmatrix}$$

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Examples

Liouvillian functions, orthogonal polynomials, associated Legendre functions, complete elliptic integrals, Airy/Scorer functions, Bessel/Struve/Anger/Weber/Lommel/Kelvin functions, Whittaker functions, hypergeometric functions, Heun functions, Mathieu functions, . . .

Basics of differential fields

Differential algebra

Differential field

(F, D) such that for any $f, g \in F$

$$D(f+g) = Df + Dg \quad \text{and} \quad D(fg) = (Df)g + f(Dg)$$

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Constant field: $\operatorname{Const}(F) := \{c \in F \mid Dc = 0\}$

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Examples

$$(\mathbb{Q}(x), \tfrac{d}{dx}) \qquad (\mathbb{Q}(e^x), \tfrac{d}{dx}) \qquad (\mathbb{R}(n, x, x^n, \ln(x)), \tfrac{d}{dx})$$

$$(\mathbb{C}(n, x, J_n(x), J_{n+1}(x), Y_n(x), Y_{n+1}(x)), \tfrac{d}{dx})$$

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NB

$f(x), g(x) \in F \quad \Rightarrow \quad f(x) + g(x), f(x)g(x), \frac{f(x)}{g(x)}, f'(x) \in F$,
but $f(x)^{g(x)}$, $f(g(x))$, and $\int f(x)\,dx$ in general are not in F

Differential field extensions

Adjoin new elements

To a differential field (F, D) we can adjoin new elements $t_1, \dots, t_n$ to get a field $F(t_1, \dots, t_n)$.

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To a differential field (F, D) we can adjoin new elements $t_1, \dots, t_n$ to get a field $F(t_1, \dots, t_n)$.

The result is a differential field extension of (F, D) if

- $Dt_i \in F(t_1, \dots, t_n)$ and
- D can be extended consistently to $F(t_1, \dots, t_n)$.

Monomial extensions

Definition

t is a monomial over (F, D) if

- t is transcendental over F and
- Dt is a polynomial in t with coefficients from F

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- $\frac{d}{dx}\ln(x) = \frac{1}{x}$
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Towers of monomial extensions

We consider differential fields $(C(t_1, \ldots, t_n), D)$ such that each t_i is a monomial over $(C(t_1, \ldots, t_{i-1}), D)$.

Elementary extensions

Elementary extension

Any (E, D) generated from (F, D) by adjoining

- algebraics: $y(x)^m + a_{m-1}(x)y(x)^{m-1} + \cdots + a_0(x) = 0$
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NB

The definition is relative to F. An elementary extension E contains non-elementary functions if F does.

A generalization of Risch's algorithm

Introduction

Parametric elementary integration

Problem

- Given (F, D) and $f_0, \dots, f_m \in F$

Parametric elementary integration

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- Given (F, D) and $f_0, \ldots, f_m \in F$
- Find all $c_0, \ldots, c_m \in \mathrm{Const}(F)$ s.t.

$$c_0 f_0 + \cdots + c_m f_m$$

has an elementary integral over (F, D)

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We say that $f \in F$ has an elementary integral over (F, D)

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Definition

We say that $f \in F$ has an elementary integral over (F, D) if there exists an elementary extension (E, D) of (F, D) and $g \in E$ s.t.

$$Dg = f$$

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We say that $f \in F$ has an elementary integral over (F, D) if there exists an elementary extension (E, D) of (F, D) and $g \in E$ s.t.

$$Dg = f$$

NB

The definition is relative to F. The integral g need not be an elementary function.

Admissible integrands

Admissible differential fields

We call a tower of monomial extensions $(F, D) = (C(t_1, \dots, t_n), D)$ admissible, if $\mathrm{Const}(F) = C$ and

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We call a tower of monomial extensions $(F, D) = (C(t_1, \ldots, t_n), D)$ admissible, if $\mathrm{Const}(F) = C$ and for each t_i and $F_{i-1} := C(t_1, \ldots, t_{i-1})$ either

1. t_i is a Liouvillian monomial over F_{i-1}, i.e., either
   1. $Dt_i \in F_{i-1}$ (primitive), or
   2. $\frac{Dt_i}{t_i} \in F_{i-1}$ (hyperexponential); or

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   1. $Dt_i \in F_{i-1}$ (primitive), or
   2. $\frac{Dt_i}{t_i} \in F_{i-1}$ (hyperexponential); or
2. there is a $q \in F_{i-1}[t_i]$ with $\deg(q) \geq 2$ such that
   1. $Dt_i = q(t_i)$ and
   2. $Dy = q(y)$ does not have a solution $y \in \overline{F_{i-1}}$.

Admissible integrands

Admissible differential fields

We call a tower of monomial extensions $(F, D) = (C(t_1, \dots, t_n), D)$ admissible, if $\mathrm{Const}(F) = C$ and for each t_i and $F_{i-1} := C(t_1, \dots, t_{i-1})$ either

1. t_i is a Liouvillian monomial over F_{i-1}, i.e., either
   1. $Dt_i \in F_{i-1}$ (primitive), or
   2. $\frac{Dt_i}{t_i} \in F_{i-1}$ (hyperexponential); or
2. there is a $q \in F_{i-1}[t_i]$ with $\deg(q) \geq 2$ such that
   1. $Dt_i = q(t_i)$ and
   2. $Dy = q(y)$ does not have a solution $y \in \overline{F_{i-1}}$.

NB

In a tower of monomial extensions all generators t_i are algebraically independent over C.

History

Risch 1969, Mack 1976

complete algorithm for regular elementary (F, D)

Singer et al. 1985

complete algorithm for regular Liouvillian (F, D)

Bronstein 1990, 1997

partial results for (F, D) a tower of monomial extensions

CGR 2012

complete algorithm for (F, D) a tower of monomial extensions subject to some technical conditions

A generalization of Risch's algorithm

Inside the algorithm

Decision procedure

Recursive reduction algorithm

Exploit tower structure: focus on topmost generator only

1. integrands from $K(t_n) = C(t_1, \ldots, t_n)$

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2. compute parts of the integral involving t_n

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Exploit tower structure: focus on topmost generator only

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2. compute parts of the integral involving t_n
3. subtract its derivative $\Rightarrow$ remaining integrands are from $K = C(t_1, \ldots, t_{n-1})$

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At each level

1. Hermite Reduction for reducing denominator

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At each level

1. Hermite Reduction for reducing denominator
2. Residue Criterion for computing elementary extensions
3. Treat reduced integrands by solving auxiliary problems in K
4. remaining integrands are from K, reduce elementary integration over $K(t_n)$ to elementary integration over K

Structural observations: orders of poles

Rational integrand

$$\int \frac{2x^3+3x-3}{(x+1)^3(x+2)^2}\,dx = -\frac{2x^2+x+1}{(x+1)^2(x+2)}$$

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Elementary integrand

$$\int \frac{x\ln(x)-1}{x(\ln(x)+1)^2}\,dx = \frac{x+1}{\ln(x)+1}$$

$$\int \frac{xe^x+1}{(e^x-x-2)^2}\,dx = -\frac{x+1}{e^x-x-2}$$

Hermite reduction

Principle

- Consider squarefree factorization of denominator
- Use exponents of factors instead of orders of poles

Repeat the basic step

1. Splitting of the integrand

$$\int \frac{a}{u \cdot v^m} = \int \frac{b \cdot (1-m) Dv}{v^m} + \int \frac{c}{u \cdot v^{m-1}}$$

2. Integration by parts

$$\int b \cdot \frac{(1-m) Dv}{v^m} = \frac{b}{v^{m-1}} - \int \frac{Db}{v^{m-1}}$$

Exceptional cases

Special factors

$$\int \frac{(6x+1)e^x - 4x}{(e^x)^2(e^x-1)^2}\,dx = -\frac{2x+1}{(e^x)^2(e^x-1)}$$

$$\int \frac{20x\tan(x)^3+1}{\tan(x)^2(\tan(x)^2+1)^2} = -\frac{5x\tan(x)+1}{\tan(x)(\tan(x)^2+1)^2}$$

Structural observations: new functions

Rational integrand

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$$\int \frac{\operatorname{li}(x)}{x^2}\,dx = \ln(\ln(x)) - \frac{\operatorname{li}(x)}{x}$$

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Complete elliptic integrals

$$\int \frac{xE(x)^2}{(1-x^2)(E(x)-K(x))^2}\,dx = \frac{E(x)}{E(x)-K(x)} - \ln(x)$$

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Liouville's theorem

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$$\int f = u_0 + \sum_{i=1}^{j} c_i \log(u_i)$$

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Algorithms to compute the c_i and u_i

- Lazard-Rioboo-Rothstein-Trager (based on subresultants)
- Czichowski (based on Gröbner bases)

Examples

Rational integrand

$$\int \frac{2x^2 + 6x + 1}{(x^2 + 1)(3x^2 + 6x + 2)}\, dx = \arctan(x) + \frac{\operatorname{arctanh}(\sqrt{3}(x + 1))}{\sqrt{3}}$$

Examples

Rational integrand

$$\int \frac{2x^2+6x+1}{(x^2+1)(3x^2+6x+2)}\,dx = \sum_{\alpha^4+\frac{1}{6}\alpha^2-\frac{1}{48}=0} \alpha \ln\left(x+3\alpha^2+2\alpha+\tfrac{3}{4}\right)$$

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Bessel functions

$$\int \frac{1}{xJ_n(x)Y_n(x)}\,dx = \frac{\pi}{2}\ln\left(\frac{Y_n(x)}{J_n(x)}\right)$$

Structural observations: polynomial degree

Polynomials in x

$$\int 6x^2 - 6x + 1\,dx = 2x^3 - 3x^2 + x$$

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Polynomials in $\ln(x)$

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Polynomials in e^x

$$\int x(e^x)^2 + \tfrac{x^2+1}{(x+1)^2}e^x\,dx = \tfrac{2x-1}{4}(e^x)^2 + \tfrac{x-1}{x+1}e^x$$

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Polynomials in $\tan(x)$

$$\int \frac{x}{x+1}\tan(x)^2 + \frac{1}{(x+1)^2}\tan(x) + \frac{x^2-2}{x+1}\,dx = \frac{x}{x+1}\tan(x) + \frac{x^2-4x}{2}$$

Algorithm

Basic principle

Given: monomial t over (K, D) with $d := \deg(Dt)$
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$$Dg = f$$
3. Solve for coefficients $g_1, \ldots, g_{n+1-d} \in K$:
   - for $d \geq 2$ this is easy
   - for $d \leq 1$ this means solving differential equations in K

Recursive call

Question

When does $f \in K$ have an elementary integral over $(K(t_n), D)$?
How to determine this by computing elementary integrals over (K, D) only?

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Answer

- refined versions of Liouville's theorem
- highly depends on t_n
- may introduce new integrands, e.g., determine if there exists a $c \in \mathrm{Const}(K)$ s.t.

$$f - c{\cdot}Dt \in k \qquad \text{or} \qquad f - c{\cdot}\frac{Dt}{t} \in k$$

has an elementary integral over (K, D).

Sample computation

Using the field $F = \mathbb{Q}(x, \ln(x), \mathrm{li}(x))$ we compute

$$\int \frac{(x+1)^2}{x \ln(x)} + \mathrm{li}(x)\, dx \quad =$$

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$$\begin{aligned}\int \frac{(x+1)^2}{x\ln(x)} + \mathrm{li}(x)\,dx &= x\,\mathrm{li}(x) + \int \frac{2x+1}{x\ln(x)}\,dx \\ &= (x+c)\mathrm{li}(x) + \int \frac{2x+1}{x\ln(x)} - c\frac{1}{\ln(x)}\,dx\end{aligned}$$

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&= (x+2)\mathrm{li}(x) + \ln(\ln(x))
\end{aligned}$$

Application to definite integrals depending on parameters

Recall

Compute linear relation of integrals

Given $f_0(x), \ldots, f_m(x)$, find $g(x)$ and $c_0, \ldots, c_m$ const. w.r.t. x s.t.

$$c_0 f_0(x) + \cdots + c_m f_m(x) = g'(x)$$

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Transfer this to a relation of corresponding integrals

$$c_0 \int_a^b f_0(x)dx + \cdots + c_m \int_a^b f_m(x)dx = g(b) - g(a)$$

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Choose the f_i

- For obtaining an ODE compute

$$c_0(y)f(x,y) + \cdots + c_m(y)\frac{\partial^m f}{\partial y^m}(x,y) = \frac{d}{dx}g(x,y)$$

- For obtaining a recurrence compute

$$c_0(n)f(x,n) + \cdots + c_m(n)f(x,n+m) = \frac{d}{dx}g(x,n)$$

Example

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Our algorithm finds

$$f(n+1,x) - \tfrac{n}{n+1} f(n,x) = \\ \tfrac{d}{dx} \tfrac{e^{-2(n+1)\pi ix}}{2(n+1)\pi i} \left(\tfrac{1}{4(n+1)} + \tfrac{e^{\pi ix}}{2n+1} + \tfrac{e^{2\pi ix}}{4n} + (e^{2\pi ix}-1)\ln(\sin(\tfrac{\pi}{2}x)) \right)$$

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Solution:

$$I(n) = -\frac{1}{4n} + \frac{i}{n\pi} \sum_{k=1}^{n} \frac{1}{2k-1}$$

Example: connection coefficients

$$c_{m,n} = \int_{-1}^{1} C_m^{\mu}(x) C_n^{\nu}(x)(1-x^2)^{\nu-\frac{1}{2}}\,dx \qquad \text{for } m, m \in \mathbb{N},\ \mu, \nu > -\frac{1}{2}$$

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Solution:

$$c_{m,n} = \begin{cases} B(\frac{1}{2},\nu+\frac{1}{2})\frac{(\mu)_k(\mu-\nu)_{k-n}(2\nu)_n}{n!(k-n)!(\nu+1)_k} & \text{if } m+n=2k \\ 0 & \text{if } m+n=2k+1 \end{cases}$$