Multiple hypergeometric series
Appell series and beyond

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Outline

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Recall the Pochhammer symbol notation for the shifted factorial:

$$(a)_n := \begin{cases} a(a+1)\dots(a+n-1) & \text{if } n = 1, 2, \dots, \\ 1 & \text{if } n = 0. \end{cases}$$

The (generalized) hypergeometric series is defined by

$${}_rF_s\left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; x\right) = \sum_{n\geq 0} \frac{(a_1)_n\,(a_2)_n \dots (a_r)_n}{n!\,(b_1)_n \dots (b_s)_n} x^n.$$

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Goal: We would like to generalize the Gauß hypergeometric function

$${}_2F_1\left(\begin{matrix} a, \; b \\ c \end{matrix}; x\right) = \sum_{n \geq 0} \frac{(a)_n (b)_n}{n! \, (c)_n} x^n$$

to a double series depending on two variables.

In the following, we follow, to great extent, the classical expositions from
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We consider the product

$$ {}_2F_1\left(\begin{matrix} a, & b \\ & c \end{matrix}; x\right) {}_2F_1\left(\begin{matrix} a', & b' \\ & c' \end{matrix}; y\right) = \sum_{m\geq 0}\sum_{n\geq 0} \frac{(a)_m\,(a')_n\,(b)_m\,(b')_n}{m!\,n!\,(c)_m\,(c')_n} x^m y^n. $$

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Now replace one, two or three of the products $(a)_m\,(a')_n$, $(b)_m\,(b')_n$, $(c)_m\,(c')_n$ by the corresponding expressions

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There are five possibilities, one of which gives the series

$$ \sum_{m\ge 0}\sum_{n\ge 0} \frac{(a)_{m+n}\,(b)_{m+m}}{m!\,n!\,(c)_{m+n}} x^m y^n = {}_2F_1\left(\begin{matrix} a, & b \\ & c \end{matrix}; x+y\right). $$

Four remaining possibilities (Paul Appell [1855–1930], 1880; and P. Appell & Marie-Joseph Kampé de Fériet [1893–1982], 1926):

$$F_1(a;b,b';c;x,y) := \sum_{m\geq 0}\sum_{n\geq 0} \frac{(a)_{m+n}\,(b)_m\,(b')_n}{m!\,n!\,(c)_{m+n}} x^m y^n, \qquad |x|,|y|<1.$$

$$F_2(a;b,b';c,c';x,y) := \sum_{m\geq 0}\sum_{n\geq 0} \frac{(a)_{m+n}\,(b)_m\,(b')_n}{m!\,n!\,(c)_m\,(c')_n} x^m y^n, \qquad |x|+|y|<1.$$

$$F_3(a,a';b,b';c;x,y) := \sum_{m\geq 0}\sum_{n\geq 0} \frac{(a)_m\,(a')_n\,(b)_m\,(b')_n}{m!\,n!\,(c)_{m+n}} x^m y^n, \qquad |x|,|y|<1.$$

$$F_4(a;b;c,c';x,y) := \sum_{m\geq 0}\sum_{n\geq 0} \frac{(a)_{m+n}\,(b)_{m+n}}{m!\,n!\,(c)_m\,(c')_n} x^m y^n, \qquad |x|^{\frac{1}{2}}+|y|^{\frac{1}{2}}<1.$$

Simple observations:

$$F_1(a;b,b';c;x,y) = \sum_{m\geq 0} \frac{(a)_m\,(b)_m}{m!\,(c)_m} x^m\, {}_2F_1\left(\begin{matrix} a+m,\ b' \\ c+m \end{matrix};y\right).$$

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$$F_1(a;b,b';c;x,0) = F_2(a;b,b';c,c';x,0) = F_3(a,a';b,b';c;x,0) \\ = F_4(a;b;c,c';x,0) = {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix};x\right).$$

Simple observations:

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$$F_1(a; b, b'; c; x, 0) = F_2(a; b, b'; c, c'; x, 0) = F_3(a, a'; b, b'; c; x, 0)$$
$$= F_4(a; b; c, c'; x, 0) = {}_2F_1\left(\begin{matrix} a, \; b \\ c \end{matrix}; x\right).$$

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$$= F_3(a, a'; b, 0; c; x, y) = {}_2F_1\left(\begin{matrix} a, \; b \\ c \end{matrix}; x\right).$$

Contiguous relations

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All contiguous relations for the F_1 can be derived from these four:

$$\begin{aligned}(a-b-b')\,F_1(a;b,b';c;x,y)-a\,F_1(a+1;b,b';c;x,y)\\ +b\,F_1(a;b+1,b';c;x,y)+b'\,F_1(a;b,b'+1;c;x,y)=0,\end{aligned}$$

$$\begin{aligned}c\,F_1(a;b,b';c;x,y)-(c-a)\,F_1(a;b,b';c+1;x,y)\\ -a\,F_1(a+1;b,b';c+1;x,y)=0,\end{aligned}$$

$$\begin{aligned}c\,F_1(a;b,b';c;x,y)+c(x-1)\,F_1(a;b+1,b';c;x,y)\\ -(c-a)x\,F_1(a;b+1,b';c+1;x,y)=0,\end{aligned}$$

$$\begin{aligned}c\,F_1(a;b,b';c;x,y)+c(y-1)\,F_1(a;b,b'+1;c;x,y)\\ -(c-a)y\,F_1(a;b,b'+1;c+1;x,y)=0.\end{aligned}$$

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$$(a-b-b')\,F_1(a;b,b';c;x,y) - a\,F_1(a+1;b,b';c;x,y) + b\,F_1(a;b+1,b';c;x,y) + b'\,F_1(a;b,b'+1;c;x,y) = 0,$$

$$c\,F_1(a;b,b';c;x,y) - (c-a)\,F_1(a;b,b';c+1;x,y) - a\,F_1(a+1;b,b';c+1;x,y) = 0,$$

$$c\,F_1(a;b,b';c;x,y) + c(x-1)\,F_1(a;b+1,b';c;x,y) - (c-a)x\,F_1(a;b+1,b';c+1;x,y) = 0,$$

$$c\,F_1(a;b,b';c;x,y) + c(y-1)\,F_1(a;b,b'+1;c;x,y) - (c-a)y\,F_1(a;b,b'+1;c+1;x,y) = 0.$$

Similar sets of relations exist for the other Appell functions. See
R.G. Buschman, *Contiguous relations for Appell functions*,
J. Indian Math. Soc. 29 (1987), 165–171.

Partial differential equations

Partial differential equations

Let

$$z = F_1(a; b, b'; c; x, y) = \sum_{m\geq 0} \sum_{n\geq 0} A_{m,n} x^m y^n.$$

Then

$$A_{m+1,n} = \frac{(a+m+n)(b+m)}{(1+m)(c+m+n)} A_{m,n},$$

and

$$A_{m,n+1} = \frac{(a+m+n)(b'+n)}{(1+n)(c+m+n)} A_{m,n}.$$

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Denoting

$$\theta = x\frac{\partial}{\partial x} \qquad \text{and} \qquad \phi = y\frac{\partial}{\partial y},$$

we see that F_1 satisfies the partial differential equations

$$\Big[(\theta+\phi+a)(\theta+b) - \frac{1}{x}\theta(\theta+\phi+c-1)\Big]z = 0,$$
$$\Big[(\theta+\phi+a)(\phi+b') - \frac{1}{x}\phi(\theta+\phi+c-1)\Big]z = 0.$$

Now let

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial z}{\partial x}\frac{\partial z}{\partial y}, \quad s = \frac{\partial z^2}{\partial x^2}, \quad t = \frac{\partial z^2}{\partial y^2}.$$

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Then $z = F_1$ satisfies the partial differential equations

$$x(1-x)r + y(1-x)s + [c-(a+b+1)x]p - byq - abz = 0,$$
$$y(1-y)t + x(1-y)s + [c-(a+b'+1)y]q - b'xp - ab'z = 0.$$

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Then $z = F_1$ satisfies the partial differential equations

$$x(1-x)r + y(1-x)s + [c - (a+b+1)x]p - byq - abz = 0,$$
$$y(1-y)t + x(1-y)s + [c - (a+b'+1)y]q - b'xp - ab'z = 0.$$

Similarly, $z = F_2$ satisfies the partial differential equations

$$x(1-x)r - xys + [c - (a+b+1)x]p - byq - abz = 0,$$
$$y(1-y)t - xys + [c' - (a+b'+1)y]q - b'xp - ab'z = 0.$$

Similarly, $z = F_3$ satisfies the partial differential equations

$$\begin{aligned} x(1-x)r + ys + [c-(a+b+1)x]p - abz &= 0, \\ y(1-y)t + xs + [c-(a'+b'+1)y]q - a'b'z &= 0. \end{aligned}$$

Similarly, $z = F_3$ satisfies the partial differential equations

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$$y(1-y)t + xs + [c - (a'+b'+1)y]q - a'b'z = 0.$$

and $z = F_4$ satisfies the partial differential equations

$$x(1-x)r - y^2t - 2xys + cp - (a+b+1)(xp+yq) - abz = 0,$$
$$y(1-y)t - x^2r - 2xys + c'q - (a+b+1)(xp+yq) - abz = 0.$$

Integral representations

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Consider the integral

$$I = \iint u^{b-1} v^{b'-1} (1-u-v)^{c-b-b'-1} (1-ux-vy)^{-a} \, \mathrm{d}u \, \mathrm{d}v,$$

taken over the triangular region $u \geq 0, \ v \geq 0, \ u+v \leq 1$.
(We also assume suitable conditions of the parameters a, b, b', c such that the integral converges.)

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(We also assume suitable conditions of the parameters a, b, b', c such that the integral converges.)

Now, provided $|vy/(1-ux)| < 1$, we have, by binomial expansion,

$$\begin{aligned}(1-ux-vy)^{-a} &= (1-ux)^{-a} \sum_{m \geq 0} \frac{(a)_m}{(1)_m} \left(\frac{vy}{1-ux} \right)^m \\ &= \sum_{m \geq 0} \frac{(a)_m}{(1)_m} v^m y^m (1-ux)^{-a-m} \\ &= \sum_{m \geq 0} \frac{(a)_m}{(1)_m} v^m y^m \sum_{n \geq 0} \frac{(a+m)_n}{(1)_n} u^n x^n.\end{aligned}$$

Thus,

$$I = \sum_{m\geq 0}\sum_{n\geq 0}\frac{(a)_{m+n}}{(1)_m(1)_n}x^n y^m \iint u^{b-1+n}v^{b'-1+m}(1-u-v)^{c-b-b'-1}\,\mathrm{d}u\,\mathrm{d}v$$

$$= \sum_{m\geq 0}\sum_{n\geq 0}\frac{(a)_{m+n}}{(1)_m(1)_n}x^n y^m\,\Gamma\begin{bmatrix} b+n, b'+m, c-b-b' \\ c+m+n \end{bmatrix},$$

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which yields

$$I = \Gamma\begin{bmatrix} b, b', c-b-b' \\ c \end{bmatrix} F_1\big(a; b, b'; c; x, y\big).$$

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(While I is a double integral, a single integral for F_1 even exists. We will turn to that later.)

Simlarly,

$$\int_0^1\int_0^1 u^{b-1}v^{b'-1}(1-u)^{c-b'-1}(1-v)^{c'-b'-1}(1-ux-vy)^{-a}\,\mathrm{d}u\,\mathrm{d}v$$
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and

$$\iint u^{b-1}v^{b'-1}(1-u-v)^{c-b-b'-1}(1-ux)^{-a}(1-vy)^{-a'}\,\mathrm{d}u\,\mathrm{d}v$$

$$=\Gamma\begin{bmatrix} b, b', c-b-b' \\ c' \end{bmatrix} F_3(a, a'; b, b'; c'; x, y),$$

the last integral taken over the triangular region $u \geq 0$, $v \geq 0$, $u+v \leq 1$.

The double integral for F_4 is more complicated:

$$\int_0^1\int_0^1 u^{a-1}v^{b-1}(1-u)^{c-a-1}(1-v)^{c'-b-1}(1-ux)^{-b}(1-vy)^{-a} \times\left(1-\frac{uvxy}{(1-ux)(1-vy)}\right)^{c+c'-a-b-1}\mathrm{d}u\,\mathrm{d}v$$

$$=\Gamma\left[\begin{matrix}a, b, c-a, c'-b\\ c,\ c'\end{matrix}\right] F_4\big(a; b; c, c'; x(1-y), y(1-x)\big).$$

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$$\begin{aligned} I' &= \sum_{m\geq 0}\sum_{n\geq 0}\int_0^1 u^{a-1}(1-u)^{c-a-1}\frac{(b)_m}{(1)_m}u^m x^m \frac{(b')_n}{(1)_n}u^n y^n\,\mathrm{d}u \\ &= \sum_{m\geq 0}\sum_{n\geq 0}\frac{(b)_m(b')_n}{(1)_m(1)_n}x^m y^n \int_0^1 u^{a+m+n-1}(1-u)^{c-a-1}\,\mathrm{d}u \\ &= \sum_{m\geq 0}\sum_{n\geq 0}\frac{(b)_m(b')_n}{(1)_m(1)_n}x^m y^n\,\Gamma\left[\begin{matrix} a+m+n, c-a \\ c+m+n \end{matrix}\right], \end{aligned}$$

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hence

$$I' = \Gamma\begin{bmatrix} a, c-a \\ c \end{bmatrix} F_1\big(a; b, b'; c; x, y\big).$$

It follows that the incomplete elliptic integrals F and E and the complete elliptic integral Π can all be expressed in terms of special cases of the F_1:

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$$F(\phi, k) = \int_0^{\phi} \frac{\mathrm{d}\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$
$$= \sin\phi \, F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \sin^2\phi, k^2 \sin^2\phi\right), \qquad |\Re\phi| < \frac{\pi}{2},$$

$$E(\phi, k) = \int_0^{\phi} \sqrt{1 - k^2 \sin^2 \theta} \, \mathrm{d}\theta$$
$$= \sin\phi \, F_1\left(\frac{1}{2}; \frac{1}{2}, -\frac{1}{2}; \frac{3}{2}; \sin^2\phi, k^2 \sin^2\phi\right), \qquad |\Re\phi| < \frac{\pi}{2},$$

$$\Pi(n, k) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{(1 - n \sin^2\theta)\sqrt{1 - k^2 \sin^2\theta}} = \frac{\pi}{2} F_1\left(\frac{1}{2}; 1, \frac{1}{2}; 1; n, k^2\right).$$

Transformations

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In the single integral for the F_1 series,

$$F_1\left(a;b,b';c;x,y\right) = \Gamma\left[\begin{matrix} c \\ a, c-a \end{matrix}\right] \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-ux)^{-b}(1-uy)^{-b'}\,\mathrm{d}u,$$

one may use the substitution of variables

$$u = 1 - v$$

to prove

$$F_1\left(a;b,b';c;x,y\right) = (1-x)^{-b}(1-y)^{-b'} F_1\left(c-a;b,b';c;\frac{x}{x-1},\frac{y}{y-1}\right).$$

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For $b' = 0$ this reduces to the Pfaff–Kummer transformation for the ${}_2F_1$:

$${}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}; x\right) = (1-x)^{-b}\,{}_2F_1\left(\begin{matrix} c-a,\ b \\ c \end{matrix}; \frac{x}{x-1}\right).$$

Similarly, the substitution of variables

$$u = \frac{v}{1 - x + vx}$$

can be used to prove

$$F_1(a; b, b'; c; x, y) = (1-x)^{-a} F_1\left(a; -b - b' + c, b'; c; \frac{x}{x-1}, \frac{y-x}{1-x}\right).$$

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For $b' = 0$ this reduces again to the Pfaff–Kummer transformation for the ${}_2F_1$ series.

On the other hand, if $c = b + b'$, then

$$\begin{aligned} F_1\big(a; b, b'; b + b'; x, y\big) &= (1-x)^{-a}\, {}_2F_1\left(\begin{matrix} a,\ b' \\ b + b' \end{matrix}; \frac{y-x}{1-x}\right) \\ &= (1-y)^{-a}\, {}_2F_1\left(\begin{matrix} a,\ b \\ b + b' \end{matrix}; \frac{x-y}{1-y}\right). \end{aligned}$$

Similarly,

$$F_1(a;b,b';c;x,y) = (1-y)^{-a} F_1\left(a;b,c-b-b';c;\frac{x-y}{1-y},\frac{y}{y-1}\right),$$

$$F_1(a;b,b';c;x,y)$$
$$= (1-x)^{c-a-b}(1-y)^{-b'} F_1\left(c-a;c-b-b',b';c;x,\frac{x-y}{1-y}\right),$$

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Similarly,

$$F_2(a; b, b'; c, c'; x, y) = (1-x)^{-a} F_2\left(a; c-b, b'; c, c'; \frac{x}{x-1}, \frac{y}{1-x}\right),$$

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Also quadratic transformations are known for Appell functions. See B.C. Carlson, *Quadratic transformations of Appell functions*, SIAM J. Math. Anal. 7 (1976), 291–304.

Reduction formulae

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By Euler's transformation this is

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$F_1[c = b + b']$:

$$F_1(a; b, b'; b+b'; x, y) = (1-y)^{-a}\, {}_2F_1\left(\begin{matrix} a,\ b \\ b+b' \end{matrix}; \frac{x-y}{1-y}\right).$$

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$F_2[c = b]$:

$$F_2\left(a; b, b'; b, c'; x, y\right) = (1-x)^{-a}\,{}_2F_1\left(\begin{matrix} a,\ b' \\ c' \end{matrix}; \frac{y}{1-x}\right).$$

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Since

$$F_1(a;b,b';c;x,y)=\sum_{m\geq 0}\frac{(a)_m\,(b)_m}{(1)_m\,(c)_m}x^m\,{}_2F_1\left(\begin{matrix}a+m,\ b'\\ c+m\end{matrix};y\right)$$

and

$${}_2F_1\left(\begin{matrix}a,\ b\\ c\end{matrix};1\right)=\Gamma\left[\begin{matrix}c,\ c-a-b\\ c-a,\ c-b\end{matrix}\right],\qquad \Re(c-a-b)>0,$$

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we have

$$F_1(a; b, b'; c; x, 1) = \Gamma\left[\begin{matrix} c, \ c-a-b' \\ c-a, \ c-b' \end{matrix}\right] {}_2F_1\left(\begin{matrix} a, \ b \\ c-b' \end{matrix}; x\right),$$

for $\Re(c-a-b') > 0$.

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we have

$$\begin{aligned}F_1(a;b,b';c;x,y)&=(1-y)^{-b'}\sum_{m\geq 0}\frac{(a)_m\,(b)_m}{(1)_m\,(c)_m}x^m\,{}_2F_1\left(\begin{matrix}c-a,\ b'\\ c+m\end{matrix};\frac{y}{y-1}\right)\\&=(1-y)^{-b'}F_3\left(a,c-a;b,b';c;x,\frac{y}{y-1}\right).\end{aligned}$$

Hence, any F_1 function can be expressed in terms of an F_3 function. The converse is only true when $c=a+a'$.

Since the F_1 function reduces to an ordinary ${}_2F_1$ function when $c = b + b'$, we have

$$F_3\left(a, c-a; b, c-b; c; x, \frac{y}{y-1}\right) = (1-x)^{-a}(1-y)^{c-b}\, {}_2F_1\left(\begin{matrix} a,\ c-b \\ c \end{matrix}; \frac{y-x}{1-x}\right).$$

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Similarly, any F_2 function reduces to an F_1 function when $c' = a$:

$$F_2(a; b, b'; c, a; x, y) = (1-y)^{-b'} F_1\left(b; a-b', b'; c; x, \frac{x}{1-y}\right).$$

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If further $c = a$, then

$$F_2(a; b, b'; a, a; x, y) = (1-x)^{-b}(1-y)^{-b'}\,{}_2F_1\left(\begin{matrix} b,\ b' \\ a \end{matrix}; \frac{xy}{(1-x)(1-y)}\right).$$

J.L Burchnall & T.W. Chaundy, 1940, 1941:

$$F_4\big(a;b;c,c';x(1-y),y(1-x)\big) = \sum_{m\geq 0} \frac{(a)_m\,(b)_m\,(1+a+b-c-c')_m}{m!\,(c)_m\,(c')_m} x^m y^m \times {}_2F_1\left(\begin{matrix} a+m,\ b+m \\ c+m \end{matrix};x\right) {}_2F_1\left(\begin{matrix} a+m,\ b+m \\ c'+m \end{matrix};y\right).$$

This expansion has applications to classical orthogonal polynomials.
It can also be used to deduce the double integral representation for F_4.

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It can also be used to deduce the double integral representation for F_4.

The $c' = 1+a+b-c$ special case gives the product formula

$$F_4\big(a;b;c,1+a+b-c;x(1-y),y(1-x)\big) = {}_2F_1\left(\begin{matrix}a,\ b\\ c\end{matrix};x\right) {}_2F_1\left(\begin{matrix}a,\ b\\ c'\end{matrix};y\right).$$

On the other hand, the $c' = b$ special case gives the reduction formula

$$F_4\big(a; b; c, b; x(1-y), y(1-x)\big)$$
$$= (1-x)^{-a}(1-y)^{-a}\, F_1\left(a; 1+a-c, c-b; c; \frac{xy}{(1-x)(1-y)}, \frac{x}{x-1}\right),$$

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while putting in addition $c = a$ gives the attractive summation formula

$$F_4\big(a; b; a, b; x(1-y), y(1-x)\big) = (1-x)^{1-b}(1-y)^{1-a}(1-x-y)^{-1}.$$

Written out in explicit terms, this is

$$\sum_{m\geq 0}\sum_{n\geq 0} \frac{(a)_{m+n}\,(b)_{m+n}}{m!\,n!\,(a)_m\,(b)_n} x^m(1-y)^m y^n(1-x)^n$$
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$$=(1-x)^{1-b}(1-y)^{1-a}(1-x-y)^{-1}.$$

For $y = 0$ this reduces to Newton's binomial expansion formula

$${}_1F_0\left(\begin{matrix}b\\ -\end{matrix};x\right)=(1-x)^{-b}.$$

Extensions

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In 1937, M.-J. Kampé de Fériet introduced the following bivariate extension of the generalized hypergeometric series:

$$F^{p:q}_{r:s}\left(\begin{matrix} a_1,\dots,a_p : b_1, b_1'; \dots; b_q, b_q'; \\ c_1,\dots,c_r : d_1, d_1'; \dots; d_s, d_s'; \end{matrix} x, y\right) = \sum_{m\geq 0}\sum_{n\geq 0} \frac{(a_1)_{m+n}\dots(a_p)_{m+n}}{(c_1)_{m+n}\dots(c_r)_{m+n}} \frac{(b_1)_m(b_1')_n\dots(b_q)_m(b_q')_n}{(d_1)_m(d_1')_n\dots(d_s)_m(d_s')_n} \frac{x^m y^n}{m!n!}.$$

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Numerous identities exist for special instances of such series.

P.W. Karlsson, 1994:

$$F^{0:3}_{1:1}\left(\begin{matrix} - : a, d-a; b, d-b; c, -c; \\ d : e, d+e-a-b-c; \end{matrix} 1, 1\right) = \Gamma\left[\begin{matrix} e, \; e+d-a-b-c \\ e-c, \; e+d-a-b \end{matrix}\right],$$

where $\Re(e) > 0$ and $\Re(d+e-a-b-c) > 0$.

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$$F^{0:3}_{1:1}\left(\begin{matrix} - : a, d-a; b, d-b; c, e-c-1; \\ d : e, d+e-a-b-c; \end{matrix} 1, 1\right)$$
$$= \Gamma\left[\begin{matrix} 1-a,\ 1-b,\ e,\ e-d,\ d+e-a-b-c \\ 1-d,\ e-a,\ e-b,\ e-c,\ 1+d-a-b \end{matrix}\right],$$

where $\Re(d+e-a-b-c) > 0$, and $d-a$ or $d-b$ is a negative integer.

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$$F_B^{(n)}(a_1, \dots, a_n; b_1, \dots, b_n; c; x_1, \dots, x_n) = \sum_{m_1 \geq 0} \cdots \sum_{m_n \geq 0} \frac{(a_1)_{m_1} \dots (a_n)_{m_n} (b_1)_{m_1} \dots (b_n)_{m_n}}{(c)_{m_1+\cdots+m_n} (1)_{m_1} \dots (1)_{m_n}} x_1^{m_1} \dots x_n^{m_n},$$

where $|x_1|, \dots, |x_n| < 1$.

$$F_C^{(n)}(a;b;c_1,\dots,c_n;x_1,\dots,x_n) = \sum_{m_1\geq 0}\cdots\sum_{m_n\geq 0}\frac{(a)_{m_1+\cdots+m_n}(b)_{m_1+\cdots+m_n}}{(c_1)_{m_1}\dots(c_n)_{m_n}(1)_{m_1}\dots(1)_{m_n}}x_1^{m_1}\dots x_n^{m_n},$$

where $|x_1|^{\frac{1}{2}}+\cdots+|x_n|^{\frac{1}{2}}<1$.

$$F_C^{(n)}(a; b; c_1, \ldots, c_n; x_1, \ldots, x_n)$$
$$= \sum_{m_1 \geq 0} \cdots \sum_{m_n \geq 0} \frac{(a)_{m_1+\cdots+m_n} (b)_{m_1+\cdots+m_n}}{(c_1)_{m_1} \ldots (c_n)_{m_n} (1)_{m_1} \ldots (1)_{m_n}} x_1^{m_1} \ldots x_n^{m_n},$$

where $|x_1|^{\frac{1}{2}} + \cdots + |x_n|^{\frac{1}{2}} < 1$.

$$F_D^{(n)}(a; b_1, \ldots, b_n; c; x_1, \ldots, x_n)$$
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where $|x_1|, \dots, |x_n| < 1$.

We have

$$F_A^{(2)} = F_2, \qquad F_B^{(2)} = F_3, \qquad F_C^{(2)} = F_4, \qquad F_D^{(2)} = F_1.$$

Integral representation of $F_D^{(n)}$

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where $\Re c > \Re a > 0$.

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where $\Re c > \Re a > 0$.

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Further important extension (not considered here):
Multivariate hypergeometric functions in the sense of I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinky, late 1980's.

An application: Mizan Rahman's evaluation of an integral

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and

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In our paper, we claimed that these integrals would be difficult to prove with standard methods.

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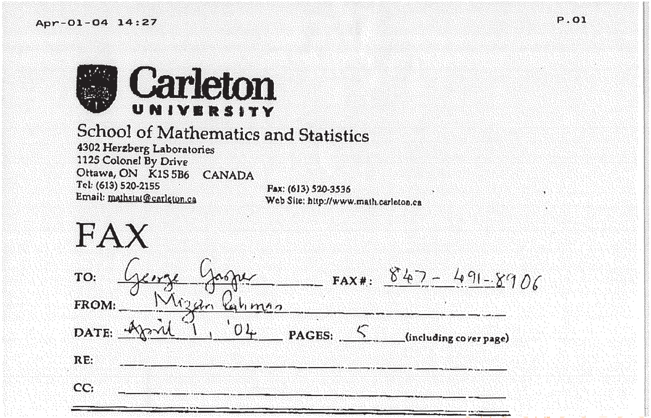

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FROM: Mizan Rahman

DATE: April 1, '04 PAGES: 5 (including cover page)

RE:

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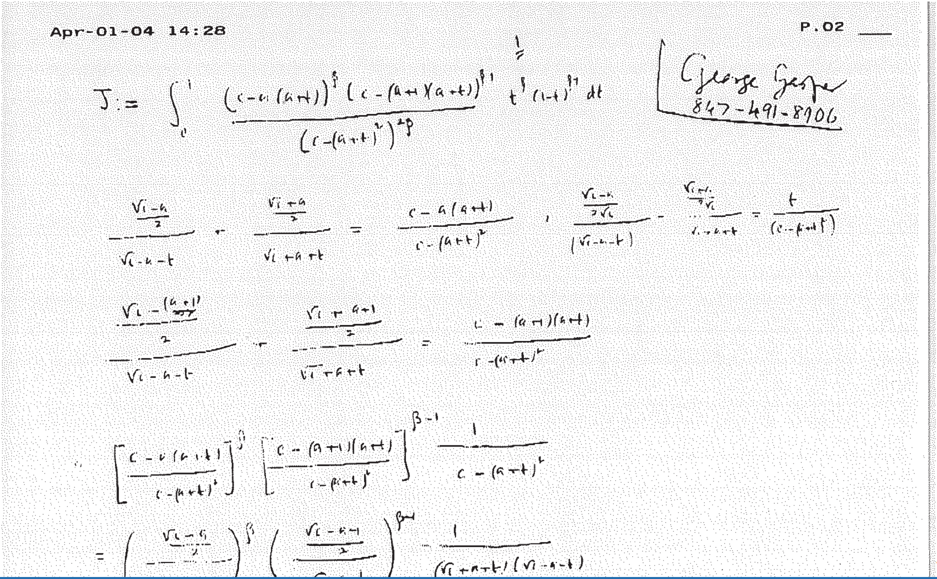
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George Gasper
847-491-8906

$$J := \int_0^1 \frac{(c-a(a+t))^{\beta}\,(c-(a+1)(a+t))^{\beta-1}}{(c-(a+t)^2)^{2\beta}}\, t^{\beta}(1-t)^{\beta-1}\,dt$$

$$\frac{\frac{\sqrt{c}-a}{2}}{\sqrt{c}-a-t} + \frac{\frac{\sqrt{c}+a}{2}}{\sqrt{c}+a+t} = \frac{c-a(a+t)}{c-(a+t)^2}, \quad \frac{\frac{\sqrt{c}-a}{2\sqrt{c}}}{(\sqrt{c}-a-t)} - \frac{\frac{\sqrt{c}+a}{2\sqrt{c}}}{\sqrt{c}+a+t} = \frac{t}{(c-(a+t)^2)}$$

$$\frac{\frac{\sqrt{c}-(a+1)}{2}}{\sqrt{c}-a-t} + \frac{\frac{\sqrt{c}+a+1}{2}}{\sqrt{c}+a+t} = \frac{c-(a+1)(a+t)}{c-(a+t)^2}$$

$$\left[\frac{c-a(a+t)}{c-(a+t)^2}\right]^{\beta} \left[\frac{c-(a+1)(a+t)}{c-(a+t)^2}\right]^{\beta-1} \frac{1}{c-(a+t)^2}$$

$$= \left(\frac{\frac{\sqrt{c}-a}{2}}{\cdots}\right)^{\beta} \left(\frac{\frac{\sqrt{c}-a-1}{2}}{\cdots}\right)^{\beta-1} \frac{1}{(\sqrt{c}+a+t)(\sqrt{c}-a-t)}$$

$$\left[\frac{\quad}{(1-(a+t))^2}\right]\left[\frac{\quad}{(1-(a+t))^2}\right]\quad c-(a+t)$$

$$= \left(\frac{\frac{\sqrt{i}-a}{2}}{\sqrt{i}-a-t}\right)^{\beta}\left(\frac{\frac{\sqrt{i}-a-1}{2}}{\sqrt{i}-a-t}\right)^{\beta-1}\frac{1}{(\sqrt{i}+a+t)(\sqrt{i}-a-t)}$$

$$\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-\beta)_m(1-\beta)_n}{m!\,n!}\left(\frac{\sqrt{i}+a}{\sqrt{i}-a}\right)^m\left(\frac{\sqrt{i}-a-t}{\sqrt{i}+a+t}\right)^{m+n}(-1)^{m+n}\left(\frac{\sqrt{i}+a+1}{\sqrt{i}-a-1}\right)^n$$

$$= \left(\frac{\sqrt{i}-a}{2}\right)^{\beta}\left(\frac{\sqrt{i}-a-1}{2}\right)^{\beta-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-\beta)_m(1-\beta)_n}{m!\,n!}(-1)^{m+n}\left(\frac{\sqrt{i}+a}{\sqrt{i}-a}\right)^m\left(\frac{\sqrt{i}+a+1}{\sqrt{i}-a-1}\right)^n$$

$$\times(\sqrt{i}-a-t)^{m+n-2\beta}(\sqrt{i}+a+t)^{-m-n-1}$$

$$= 2^{1-2\beta}\frac{(\sqrt{i}-a-1)^{\beta-1}}{(\sqrt{i}+a)(\sqrt{i}-a)^{\beta}}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-\beta)_m(1-\beta)_n}{m!\,n!}(-1)^{m+n}\left(\frac{(\sqrt{i}-a)(\sqrt{i}+a+1)}{(\sqrt{i}+a)(\sqrt{i}-a-1)}\right)^n$$

$$\times\left(1-\frac{t}{\sqrt{i}-a}\right)^{m+n-2\beta}\left(1+\frac{t}{\sqrt{i}+a}\right)^{-m-n-1}.$$

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We have

$$\frac{\frac{\sqrt{c}-a}{2}}{\sqrt{c}-a-t}+\frac{\frac{\sqrt{c}+a}{2}}{\sqrt{c}+a+t}=\frac{c-a(a+t)}{c-(a+t)^2}$$

and

$$\frac{\frac{\sqrt{c}-a}{2\sqrt{c}}}{\sqrt{c}-a-t}-\frac{\frac{\sqrt{c}+a}{2\sqrt{c}}}{\sqrt{c}+a+t}=\frac{t}{c-(a+t)^2}$$

So

$$\left\{\frac{t\,(c-a(a+t))}{(c-(a+t)^2)^2}\right\}=\frac{1}{4\sqrt{c}}\left\{\left(\frac{\sqrt{c}-a}{\sqrt{c}-a-t}\right)^2-\left(\frac{\sqrt{c}+a}{\sqrt{c}+a+t}\right)^2\right\}.$$

Also

$$\frac{\frac{\sqrt{c}+a+1}{2}}{\sqrt{c}+a+t}+\frac{\frac{\sqrt{c}-a-1}{2}}{\sqrt{c}-a-t}=\frac{c-(a+1)(a+t)}{c-(a+t)^2}$$

$$\frac{\;[\text{illegible}]\;}{\sqrt{c}+a+t} + \frac{\frac{\sqrt{c}-a-1}{2}}{\sqrt{c}-a-t} = \frac{c-(a+1)(a+t)}{c-(a+t)^2}$$

and
$$\frac{\frac{\sqrt{c}+a+1}{2\sqrt{c}}}{\sqrt{c}+a+t} - \frac{\frac{\sqrt{c}-a-1}{2\sqrt{c}}}{\sqrt{c}-a-t} = \frac{1-t}{c-(a+t)^2}$$

So
$$\frac{(1-t)\,(c-(a+1)(a+t))}{\left(c-(a+t)^2\right)^2} = \frac{1}{4\sqrt{c}}\left\{\left(\frac{\sqrt{c}+a+1}{\sqrt{c}+a+t}\right)^2 - \left(\frac{\sqrt{c}-a-1}{\sqrt{c}-a-t}\right)^2\right\}.$$

Hence
$$\frac{\left(t\,(c-a(a+t))\right)^{\beta}}{\left(c-(a+t)^2\right)^{2\beta}}\left((1-t)(c-(a+1)(a+t))\right)^{\beta-1}$$

$$= \frac{1}{(4\sqrt{c})^{\beta}}\left(\frac{(\sqrt{c}-a)^2}{(\sqrt{c}-a-t)^2}\right)^{\beta}\left(1-\left(\frac{(\sqrt{c}-a-t)\,(\sqrt{c}+a)}{(\sqrt{c}+a+t)\,(\sqrt{c}-a)}\right)^2\right)^{\beta}$$

$$\times\frac{1}{(4\sqrt{c})^{\beta-1}}\left(c-(a+t)^2\right)^{2\beta-2}\left(\frac{\sqrt{c}+a+1}{\sqrt{c}+a+t}\right)^{2\beta-2}\left(1-\left(\frac{(\sqrt{c}-a-1)\,(\sqrt{c}+a+t)}{(\sqrt{c}+a+1)\,(\sqrt{c}-a-t)}\right)^2\right)^{\beta-1}$$

$$= \frac{1}{(4\sqrt{c})^{2\beta-1}}\,\frac{(\sqrt{c}-a)^{2\beta}\,(\sqrt{c}+a+1)^{2\beta-2}}{(\sqrt{c}-a-t)^2}\left\{1-\left(\frac{(\sqrt{c}+a)\,(\sqrt{c}-a-t)}{(\sqrt{c}-a)\,(\sqrt{c}+a+t)}\right)^2\right\}^{\beta}\left\{1-\left(\frac{(\sqrt{c}-a-1)(\sqrt{c}+a+t)}{(\sqrt{c}+a+1)(\sqrt{c}-a-t)}\right)^2\right\}^{\beta-1}.$$

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$$= \frac{1}{(4\sqrt{i})^{2\beta+1}} (\sqrt{i}-a)^{2\beta} (\sqrt{i}+a+1)^{2\beta-2} \sum_m \sum_n \frac{(-\beta)_m (1-\beta)_n}{m!\, n!}$$

$$\left(\frac{\sqrt{i}+a}{\sqrt{i}-a}\right)^{2m} \left(\frac{\sqrt{i}-a-1}{\sqrt{i}+a+1}\right)^{2n} (\sqrt{i}-a-t)^{2m-2n-2} (\sqrt{i}+a+t)^{2n-2m}$$

$$= \frac{(\sqrt{i}-a)^{2\beta-2} (\sqrt{i}+a+1)^{2\beta-2}}{(4\sqrt{i})^{2\beta+1}} \sum_m \sum_n \frac{(-\beta)_m (1-\beta)_n}{m!\, n!} \left(\frac{\sqrt{i}+a}{\sqrt{i}-a}\right)^{2m} \left(\frac{\sqrt{i}-a-1}{\sqrt{i}+a+1}\right)^{2n}$$

$$\left(1-\frac{t}{\sqrt{i}-a}\right)^{2m-2n-2} \left(1+\frac{t}{\sqrt{i}+a}\right)^{2n-2m},$$

where it is assumed that $a>0$ and $\sqrt{i} > a+1$.

$$\therefore J := \int_0^1 t^{\beta} (1-t)^{\beta-1} \,(i-a(a+t))^{\beta}\, (i-(a+1)(a+t))^{\beta-1}\, dt$$

where it is assumed that $a>0$ and $\sqrt{c}>a+1$.

$$\therefore\ J:=\int_0^1\frac{t^{\beta}(1-t)^{\beta-1}\,\big(c-a(a+t)\big)^{\beta}\,\big(c-(a+1)(a+t)\big)^{\beta-1}}{\big(c-(a+t)^2\big)^{2\beta}}\,dt$$

$$=\frac{\big((\sqrt{c}-a)(\sqrt{c}+a+1)\big)^{2\beta-2}}{(4\sqrt{c})^{2\beta-1}}\sum_m\sum_n\frac{(-\beta)_m(1-\beta)_n}{m!\,n!}\left(\frac{(\sqrt{c}+a)(\sqrt{c}-a-1)}{(\sqrt{c}-a)(\sqrt{c}+a+1)}\right)^{2n}$$

$$\cdot\,F_1\left(1;\,2n+2-2m,\,2m-2n;\,2;\,\frac{1}{\sqrt{c}-a},\,\frac{-1}{\sqrt{c}+a}\right).$$

By Bailey [ ]

$$F_1=\frac{\sqrt{c}-a}{\sqrt{c}+a+1}\,{}_2F_1\left[\begin{matrix}1,\ 2n-2m+2\\ 2\end{matrix};\frac{2\sqrt{c}}{(\sqrt{c}-a)(\sqrt{c}+a+1)}\right]$$

$$=\frac{(c-a^2)}{2\sqrt{c}\,(2n-2m+1)}\left\{\left(\frac{(\sqrt{c}+a)(\sqrt{c}-a-1)}{(\sqrt{c}-a)(\sqrt{c}+a+1)}\right)^{2m-2n-1}-1\right\}.$$

$$\therefore J = \frac{(r-a^2)\left((\sqrt{i}-a)(\sqrt{i}+a+1)\right)^{2\beta-2}}{(4\sqrt{i})^{2\beta}} \left\{ \sum_m \sum_n \frac{(-\beta)_m (1-\beta)_n}{m!\, n!\, (\frac{1}{2}-m+n)} \right.$$

$$\times \left[\left(\frac{(\sqrt{i}+a)(\sqrt{i}-a-1)}{(\sqrt{i}-a)(\sqrt{i}+a+1)} \right)^{2m-1} - \left(\frac{(\sqrt{i}+a)(\sqrt{i}-a-1)}{(\sqrt{i}-a)(\sqrt{i}+a+1)} \right)^{2n} \right]$$

$$= \frac{(r-a^2)\left((\sqrt{i}-a)(\sqrt{i}+a+1)\right)^{2\beta-2}}{(4\sqrt{i})^{2\beta}} \left\{ \frac{(\sqrt{i}-a)(\sqrt{i}+a+1)}{(\sqrt{i}+a)(\sqrt{i}-a-1)} \sum_{m=0}^{\infty} \frac{(-\beta)_m}{m!} \left(\frac{(\sqrt{i}+a)(\sqrt{i}-a-1)}{(\sqrt{i}-a)(\sqrt{i}+a+1)} \right)^{2m} \right.$$

$$\times \frac{1}{\frac{1}{2}-m} \sum_{n=0}^{\infty} \frac{(1-\beta)_n \, (\frac{1}{2}-m)_n}{n!\, (\frac{3}{2}-m)_n}$$

$$\left. - \sum_{n=0}^{\infty} \frac{(1-\beta)_n}{n!} \left(\frac{(\sqrt{i}+a)(\sqrt{i}-a-1)}{(\sqrt{i}-a)(\sqrt{i}+a+1)} \right)^{2n} \cdot \frac{1}{n+\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(-\beta)_m \, (-n-\frac{1}{2})_m}{m!\, (\frac{1}{2}-n)_m} \right\}$$

$$(r-a^2)\left((\sqrt{i}-a)(\sqrt{i}+a+1)\right)^{2\beta-2} \left\{ \Gamma(\tfrac{1}{2})\,\Gamma(\beta)\,(\sqrt{i}-a)(\sqrt{i}+a+1)\; F\left[-\beta;\ \tfrac{1}{2}-\beta;\ (\sqrt{i}+a)(\ldots) \right. \right.$$

$$-\sum_{n=0}^{\infty} \frac{(1-\beta)_n}{n!} \left(\frac{(\sqrt{c}+a)(\sqrt{c}-a-1)}{(\sqrt{c}-a)(\sqrt{c}+a+1)}\right) \cdot \frac{1}{n+1/2} \sum_{m=0}^{\infty} \frac{(-\beta)_m \, (-n-1/2)_m}{m! \, (\frac{1}{2}-n)_m} \Bigg\}$$

$$= \frac{(c-a^2)\left((\sqrt{c}-a)(\sqrt{c}+a+1)\right)^{2\beta-2}}{(4\sqrt{c})^{2\beta}} \Bigg\{ \frac{\Gamma(\frac{1}{2})\Gamma(\beta)}{\Gamma(\beta+\frac{1}{2})} \frac{(\sqrt{c}-a)(\sqrt{c}+a+1)}{(\sqrt{c}+a)(\sqrt{c}-a-1)} F\left[\begin{matrix}-\beta, \frac{1}{2}-\beta \\ \frac{1}{2}\end{matrix}; \left(\frac{(\sqrt{c}+a)(\sqrt{c}-a-1)}{(\sqrt{c}-a)(\sqrt{c}+a+1)}\right)^2\right]$$

$$- \frac{2\,\Gamma(\frac{1}{2})\Gamma(\beta+1)}{\Gamma(\beta+\frac{1}{2})} F\left[\begin{matrix}1-\beta, \frac{1}{2}-\beta \\ \frac{1}{2}\end{matrix}; \left(\frac{(\sqrt{c}+a)(\sqrt{c}-a-1)}{(\sqrt{c}-a)(\sqrt{c}+a+1)}\right)^2\right] \Bigg\}.$$

$$= \frac{(c-a^2)\left((\sqrt{c}-a)(\sqrt{c}+a+1)\right)^{2\beta-2}}{(4\sqrt{c})^{2\beta}} \Bigg\{ \frac{\Gamma(\frac{1}{2})\Gamma(\beta)}{\Gamma(\beta+\frac{1}{2})} \frac{(\sqrt{c}-a)(\sqrt{c}+a+1)}{2(\sqrt{c}+a)(\sqrt{c}-a-1)} \left[\left(\frac{2(c-a(a+1))}{(\sqrt{c}-a)(\sqrt{c}+a+1)}\right)^{2\beta} + \left(\frac{2\sqrt{c}}{(\sqrt{c}-a)(\sqrt{c}+a+1)}\right)^{2\beta} \right]$$

$$- \frac{\Gamma(\frac{1}{2})\Gamma(\beta+1)}{\Gamma(\beta+\frac{1}{2})} \frac{(\sqrt{c}-a)(\sqrt{c}+a+1)}{(\sqrt{c}+a)(\sqrt{c}-a-1)} \frac{1}{2\beta} \left[\left(\frac{2(c-a(a+1))}{(\sqrt{c}-a)(\sqrt{c}+a+1)}\right)^{2\beta} - \left(\frac{2\sqrt{c}}{(\sqrt{c}-a)(\sqrt{c}+a+1)}\right)^{2\beta} \right] \Bigg\}$$

$$= \frac{(c-a^2)\left((\sqrt{c}-a)(\sqrt{c}+a+1)\right)^{2\beta-2}}{(4\sqrt{c})^{2\beta}} \frac{(\sqrt{c}-a)(\sqrt{c}+a+1)}{(\sqrt{c}+a)(\sqrt{c}-a-1)} \frac{(2\sqrt{c})^{2\beta}}{\left((\sqrt{c}-a)(\sqrt{c}+a+1)\right)^{2\beta}} \frac{\Gamma(\frac{1}{2})\Gamma(\beta)}{\Gamma(\beta+\frac{1}{2})}$$

$$= 2^{-2\beta} \frac{\Gamma(\frac{1}{2})\Gamma(\beta)}{\Gamma(\beta+\frac{1}{2})} \frac{(c-a^2)(\sqrt{c}-a)(\sqrt{c}+a+1)}{(\sqrt{c}+a)(\sqrt{c}-a-1)(\sqrt{c}-a)^2(\sqrt{c}+a+1)^2}$$

$$= 2^{-2\beta} \frac{\Gamma(\frac{1}{2})\Gamma(\beta)}{\Gamma(\beta+\frac{1}{2})} \frac{1}{c-(a+1)^2}.$$

More sums

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We denote the q-shifted factorial by

$$(a;q)_0 = 1, \qquad (a;q)_k = (1-a)(1-aq)\dots(1-aq^{k-1}),$$
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The continuous *q*-ultraspherical polynomials are given by

$$C_n(x;\beta|q) = \sum_{k=0}^{n} \frac{(\beta;q)_k\,(\beta;q)_{n-k}}{(q;q)_k\,(q;q)_{n-k}} e^{i(n-2k)\theta}, \qquad x = \cos\theta.$$

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They satisfy (for $|q|,|\beta|<1$) the orthogonality relation

$$\frac{1}{2\pi}\int_{-1}^{1} C_m(x;\beta|q)\,C_n(x;\beta|q)\,\frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{(\beta e^{2i\theta},\beta e^{-2i\theta};q)_\infty}\frac{\mathrm{d}x}{\sqrt{1-x^2}}$$
$$= \frac{(\beta,\beta q;q)_\infty}{(q,\beta^2;q)_\infty}\frac{(\beta^2;q)_n}{(q;q)_n}\frac{(1-\beta)}{(1-\beta q^n)}\,\delta_{m,n}.$$

In 1895, L.J. Rogers derived the following linearization formula for the continuous q-ultraspherical polynomials:

$$C_m(x;\beta|q)\,C_n(x;\beta|q) = \sum_{k=0}^{\min(m,n)} \frac{(q;q)_{m+n-2k}(\beta;q)_{m-k}(\beta;q)_{n-k}(\beta;q)_k(\beta^2;q)_{m+n-k}}{(\beta^2;q)_{m+n-2k}(q;q)_{m-k}(q;q)_{n-k}(q;q)_k(\beta q;q)_{m+n-k}} \times \frac{(1-\beta q^{m+n-2k})}{(1-\beta)}\,C_{m+n-2k}(x;\beta|q).$$

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$$
\begin{aligned}
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&\qquad\qquad\times\frac{(1-\beta q^{m+n-2k})}{(1-\beta)}\,C_{m+n-2k}(x;\beta|q).
\end{aligned}
$$

For simplicity, write this as

$$
C_m(x;\beta|q)\,C_n(x;\beta|q)=\sum_k f_{m,n}^k\,C_k(x;\beta|q),
$$

with explicity determined structure coefficients $f_{m,n}^k=f_{m,n}^k(\beta,q)$.
By definition, $f_{m,n}^k=f_{n,m}^k$.

Linearization of the triple product $C_l(x;\beta|q)\,C_m(x;\beta|q)\,C_n(x;\beta|q)$ in two different ways gives

$$\sum_j \sum_k f^j_{l,k}\, f^k_{m,n}\, C_j(x;\beta|q) = \sum_j \sum_k f^j_{n,k}\, f^k_{m,l}\, C_j(x;\beta|q),$$

as we must have symmetry between n and l.

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More generally, the r-fold sum

$$\sum_{k_1,\dots,k_r} f^j_{m_0,k_1}\, f^{k_1}_{m_1,k_2}\, f^{k_2}_{m_2,k_3} \dots f^{k_{r-1}}_{m_{r-1},k_r}\, f^{k_r}_{m_r,m_{r+1}}$$

is symmetric in $\{m_0, m_1, \dots, m_{r+1}\}$, resulting in transformation formulae for multivariate basic hypergeometric series.

The $r = 1$ case, after analytic continuation can be written as the following transformation formula for a very-well-poised ${}_{14}\phi_{13}$ series (R. Langer, MJS & S.O. Warnaar, 2009):

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{(1-aq^{2k})}{(1-a)} \frac{(aq/b;q)_{2k}}{(ab;q)_{2k}} \\
&\qquad \times \frac{(a,b,c,d,ab/c,ab/d,abq^n,q^{-n};q)_k}{(q,aq/b,aq/c,aq/d,cq/b,dq/b,q^{1-n}/b,aq^{n+1};q)_k} \left(\frac{q}{b}\right)^{2k} \\
&= \frac{(aq,\hat{a}q/c,\hat{a}q/d,aq/cd;q)_n}{(\hat{a}q,aq/c,aq/d,\hat{a}q/cd;q)_n} \sum_{k=0}^{n} \frac{(1-\hat{a}q^{2k})}{(1-\hat{a})} \frac{(\hat{a}q/b;q)_{2k}}{(\hat{a}b;q)_{2k}} \\
&\qquad \times \frac{(\hat{a},b,c,d,\hat{a}b/c,\hat{a}b/d,\hat{a}bq^n,q^{-n};q)_k}{(q,\hat{a}q/b,\hat{a}q/c,\hat{a}q/d,cq/b,dq/b,q^{1-n}/b,\hat{a}q^{n+1};q)_k} \left(\frac{q}{b}\right)^{2k},
\end{aligned}$$

where $\hat{a} = q^{-n}cd/ab$.

By inverse relations, one obtains the following double sum identity:

$$\sum_{l,k\geq 0} \frac{(1-abq^{2l+2k})}{(1-ab)} \frac{(aq^m, q^{-m}; q)_{l+k}}{(bq^{1-m}, abq^{m+1}; q)_{l+k}} \frac{(b, ab/c, ab/d, aq/cd; q)_l}{(q, aq/c, aq/d, ab/cd; q)_l} q^l$$

$$\times \frac{(1-cdq^{k-l}/ab)}{(1-cdq^{-l}/ab)} \frac{(cdq^{1-l}/ab^2; q)_k}{(cdq^{-l}/a; q)_k} \frac{(b, c, d, cd/a; q)_k}{(q, cq/b, dq/b, cdq/ab; q)_k} q^k$$

$$= \frac{(aq/b; q)_{2m}}{(ab; q)_{2m}} \frac{(abq, b, c, d, ab/c, ab/d; q)_m}{(1/b, aq/b, aq/c, aq/d, cq/b, dq/b; q)_m} \left(\frac{q}{b^2}\right)^m.$$

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These identities can be extended to the elliptic setting. For the latter, we have

$$\sum_{l,k\geq 0} \frac{\theta(abq^{2l+2k};p)}{\theta(ab;p)} \frac{(aq^m, q^{-m};q,p)_{l+k}}{(bq^{1-m}, abq^{m+1};q,p)_{l+k}} \frac{(b, ab/c, ab/d, aq/cd;q,p)_l}{(q, aq/c, aq/d, ab/cd;q,p)_l} q^l$$
$$\times \frac{\theta(cdq^{k-l}/ab;p)}{\theta(cdq^{-l}/ab;p)} \frac{(cdq^{1-l}/ab^2;q,p)_k}{(cdq^{-l}/a;q,p)_k} \frac{(b,c,d,cd/a;q,p)_k}{(q, cq/b, dq/b, cdq/ab;q,p)_k} q^k$$
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$$\theta(x;p) := (x, p/x; p)_\infty = \prod_{j=0}^{\infty}(1 - p^j x)(1 - p^{j+1}/x).$$

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Theta shifted factorials:

$$(a;q,p)_k := \theta(a;p)\theta(aq;p)\cdots\theta(aq^{k-1};p) \quad \text{for} \quad k = 0,1,2,\dots.$$

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Compact notations:

$$\theta(x_1, \ldots, x_m; p) := \theta(x_1;p)\cdots\theta(x_m;p),$$

$$(a_1, \ldots, a_m; q; p)_k := (a_1;q,p)_k \cdots (a_m;q,p)_k.$$

Inversion formula:

$$\theta(1/x;p) = -\frac{1}{x}\,\theta(x;p).$$

Quasi-periodicity:

$$\theta(px;p) = -\frac{1}{x}\,\theta(x;p).$$

Riemann relation:

$$\theta(xy, x/y, uv, u/v; p) - \theta(xv, x/v, uy, u/y; p) = \frac{u}{y}\,\theta(yv, y/v, xu, x/u; p).$$

Elliptic hypergeometric series:

$$\sum_{k\geq 0} c_k,$$

where $c_0 = 1$ and $g(k) = c_{k+1}/c_k$ is an elliptic (doubly periodic, meromorphic) function of k with k considered as a complex variable.

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Without loss of generality,

$$g(x) = \frac{\theta(a_0 q^x, a_1 q^x, \ldots, a_s q^x)}{\theta(q^{1+x}, b_1 q^x, \ldots, b_s q^x)}\, z,$$

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If we write $q = e^{2\pi i\sigma}$, $p = e^{2\pi i\tau}$, with complex σ, τ, then $g(x)$ is periodic in x with periods σ^{-1} and $\tau\sigma^{-1}$.

General solution:

$$
{}_{s+1}E_s\left[\begin{matrix} a_0, a_1, \ldots, a_s \\ b_1, b_2, \ldots, b_s \end{matrix}; q, p; z\right] := \sum_{k=0}^{\infty} \frac{(a_0, a_1, \ldots, a_s; q, p)_k}{(q, b_1 \ldots, b_s; q, p)_k} z^k,
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where $a_0 a_1 \cdots a_s = q b_1 b_2 \cdots b_s$.

For convergence, one usually requires $a_s = q^{-n}$ (n being a nonnegative integer), so that the sum is finite.

Elliptic hypergeometric series first appeared as elliptic solutions of the Yang–Baxter equation in work by Date, Jimbo, Kuniba, Miwa and Okado in 1987, and ten years later by I. B. Frenkel and V. Turaev.

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Frenkel and Turaev's ${}_{10}V_9$ summation:

$$\sum_{k=0}^{n} \frac{\theta(aq^{2k};p)}{\theta(a;p)} \frac{(a,b,c,d,e,q^{-n};q,p)_k}{(q,aq/b,aq/c,aq/d,aq/e,aq^{n+1};q,p)_k} q^k = \frac{(aq,aq/bc,aq/bd,aq/cd;q,p)_n}{(aq/b,aq/c,aq/d,aq/bcd;q,p)_n},$$

where $a^2q^{n+1} = bcde$.