Likelihood Ratio Tests for Non-nested Models: the Case of the SM4

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“Prejudice meets Reality” workshop
Bonn, August 2012

Outline

1. Introduction
2. Status of the SM4
3. Likelihood Ratio Tests
4. Numerical Computation of *p*-values
5. Conclusions

SM4 Matter Content

$$\text{quarks:} \quad \begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} t \\ b \end{pmatrix}, \begin{pmatrix} t' \\ b' \end{pmatrix}$$

$$\text{leptons:} \quad \begin{pmatrix} e \\ \nu_e \end{pmatrix}, \begin{pmatrix} \mu \\ \nu_\mu \end{pmatrix}, \begin{pmatrix} \tau \\ \nu_\tau \end{pmatrix}, \begin{pmatrix} \ell' \\ \nu' \end{pmatrix}$$

SM4 Parameters

masses: m_u , m_c , m_t , $m_{t'}$

m_d , m_s , m_b , $m_{b'}$

m_e , m_μ , m_τ , $m_{\ell'}$

m_{ν_e} , m_{ν_μ} , m_{ν_τ} , $m_{\nu'}$

CKM mixing angles: θ_{12} , θ_{13} , θ_{23} , θ_{14} , θ_{24} , θ_{34}

CKM phases: δ_{13} , δ_{14} , δ_{24}

PMNS matrix: the same again

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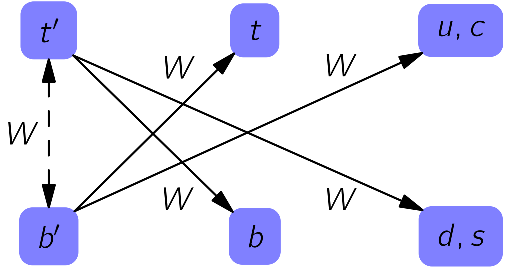
- PDG reviews claim since 1998 that a fourth generation is ruled out by electroweak precision observables (EWPOs).

  But: this statement is only true for degenerate fermion masses. (Since 2002 they say this explicitly.)

Direct mass limits

- $t' \to bW$: $m_{t'} > 557$ GeV [arXiv:1203.5410]
- $b' \to tW$: $m_{b'} > 611$ GeV [arXiv:1204.1088]
- inclusive: $m_{t'}, m_{b'} \gtrsim 650$ GeV [CMS-PAS-EXO-11-098]

But: these limits depend on the decay mode.



Electroweak Precision Observables

- Chiral fermions have couplings proportional to their mass

  $\Rightarrow$ they do not decouple from the theory when they are heavy.

- Electroweak precision observables (EWPOs) receive non-decoupling contributions from 4th generation fermions

$$\Delta S = \frac{1}{6\pi}\left(4 - \ln\frac{m_{t'}^2}{m_{b'}^2} + \ln\frac{m_{\nu'}^2}{m_{l'}^2}\right)$$

Higgs signal strengths

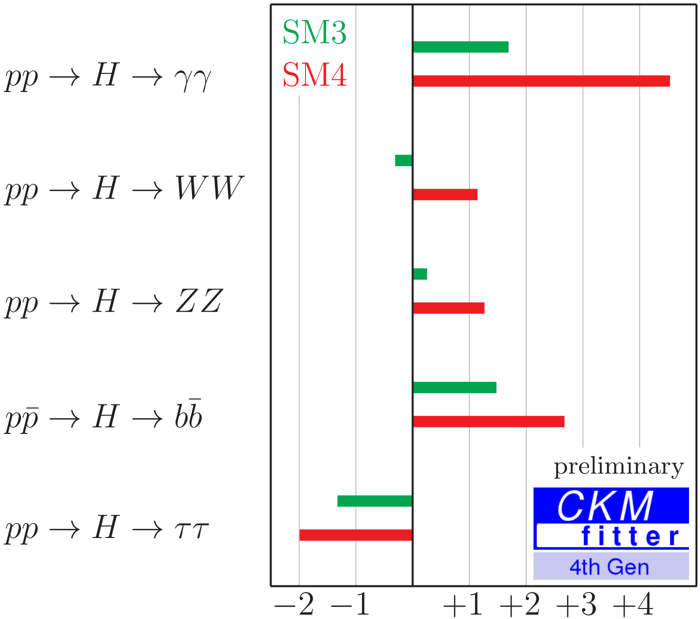
- Higgs production via gluon fusion is enhanced by a factor of 9 due to extra heavy quarks in the loop.
- $\mathrm{Br}(H \to \gamma\gamma)$ is reduced due to destructive interference with gauge boson loops.
- Higer order corrections are relevant for all search channels due to the large yukawa couplings. (Perturbativity?) [Denner, Dittmaier, Mück, Passarino, Spira, Sturm, Uccirati, Weber; arXiv:1111.6395]
- For $m_{\nu'} < m_H/2$ the Higgs can decay invisibly into $\nu'\bar{\nu}'$. This simultaneously reduces all other signal strengths. [Belotsky et al. (2003); Rozanov, Vysotsky (2010); Keung, Schwaller (2011); Cetin et al. (2011)]

Global analysis

We performed a global fit of Higgs signal strengths and EWPOs within the SM4.
[Eberhardt, Herbert, Lacker, Lenz, Menzel, Nierste, M.W.; arXiv:1207.0438]

- EWPOs in the SM4 were calculated with ZFitter and the method from [Gonzalez, Rohrwild, M.W.; arXiv:1105.3434]. (Using S, T and U is inconsistent!)
- Higgs partial widths in the SM4 are calculated with HDECAY, which includes the higher order corrections from [Denner et al.; arXiv:1111.6395]
- We use post-ICHE2012 signal strengths ($H \to \gamma\gamma, ZZ, WW, \tau\tau$ from LHC and $H \to b\bar{b}$ from Tevatron).
- Quark masses were allowed to float between 600 and 800 GeV.

Signal Strength Deviations



"Standard" Likelihood Ratio Tests

- Consider a "full" theory F with parameters $x_1, \ldots, x_n$ and chi-square function $\chi^2_F(x_1, \ldots, x_n)$.

- Consider a "constrained" theory C obtained from F by fixing the last k parameters ($k < n$). The chi-square function is

$$\chi^2_C(x_1, \ldots, x_{n-k}) = \chi^2_F(x_1, \ldots, x_{n-k}, 0, \ldots, 0) \quad .$$

- Minimize both chi-square functions and compute

$$\Delta\chi^2 = \chi^2_{C,\text{min}} - \chi^2_{F,\text{min}} \quad .$$

- The statistical significance (p-value) is (Wilk's theorem)

$$p = 1 - \underbrace{P_{k/2}(\tfrac{1}{2}\Delta\chi^2)}_{\text{normalised lower incomplete Gamma function}} = 1 - \text{Prob}(k, \Delta\chi^2) \quad .$$

"Standard" Likelihood Ratio Tests

This does not work for SM3 vs. SM4.

You cannot fix the parameters of the SM4 so that you re-obtain the SM3.

"General" Likelihood Ratio Tests

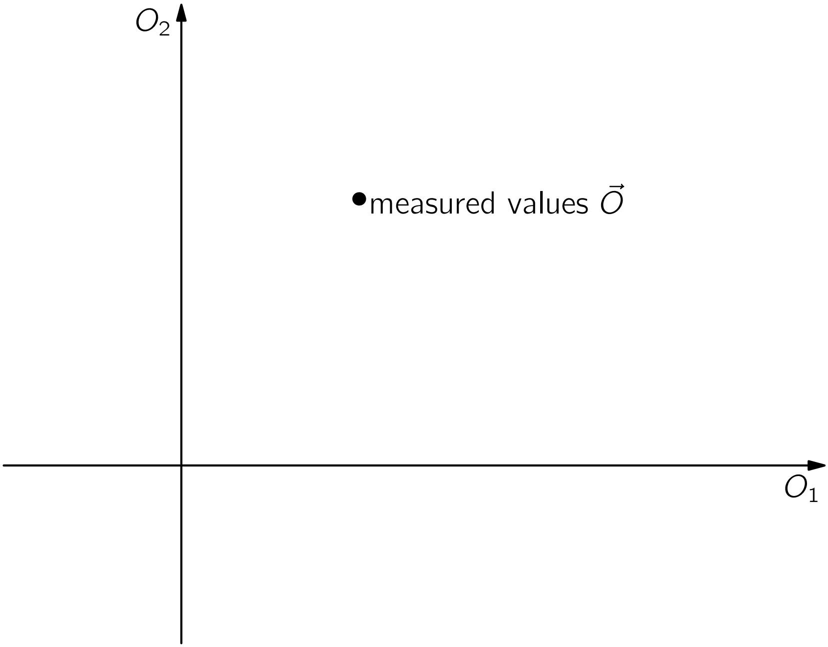
- Consider two unrelated Theories A and B with chi-square functions χ^2_A and χ^2_B.
- Fit both Theories to the measured observables $\vec{O}$ and compute
$$\Delta\chi^2(\vec{O}) = \chi^2_{A,\min}(\vec{O}) - \chi^2_{B,\min}(\vec{O}) \quad .$$
- Generate a large sample of toy measurements $\vec{O}'_i$ distributed about the best-fit prediction of theory A (the null hypothesis) according to their errors.
- Fit both theories for each set of toy measurements and compute
$$\Delta\chi^2(\vec{O}'_i) = \chi^2_{A,\min}(\vec{O}'_i) - \chi^2_{B,\min}(\vec{O}'_i) \quad .$$
- The statistical significance (of theory A) is the fraction of toy measurements with $\Delta\chi^2(\vec{O}'_i) > \Delta\chi^2(\vec{O})$.

"General" Likelihood Ratio Tests

Drawback: for small p-values you have to do a lot of fits.

$\Rightarrow$ Can this method be improved?

Derivation of Wilk's Theorem



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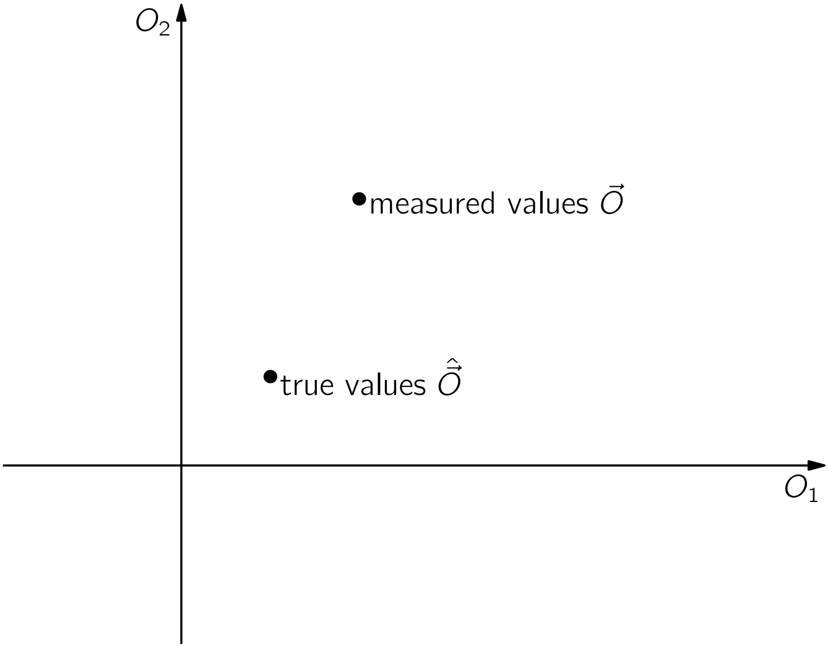

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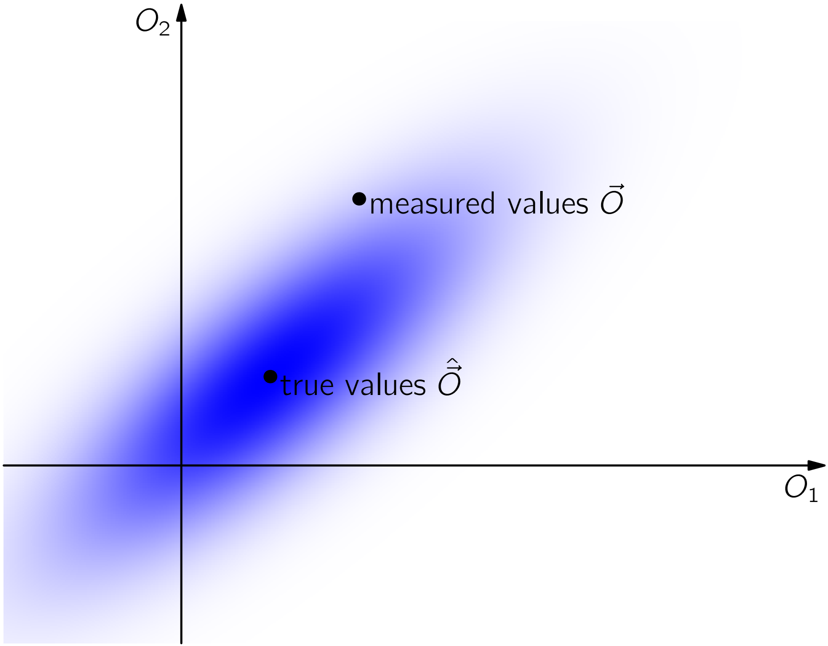

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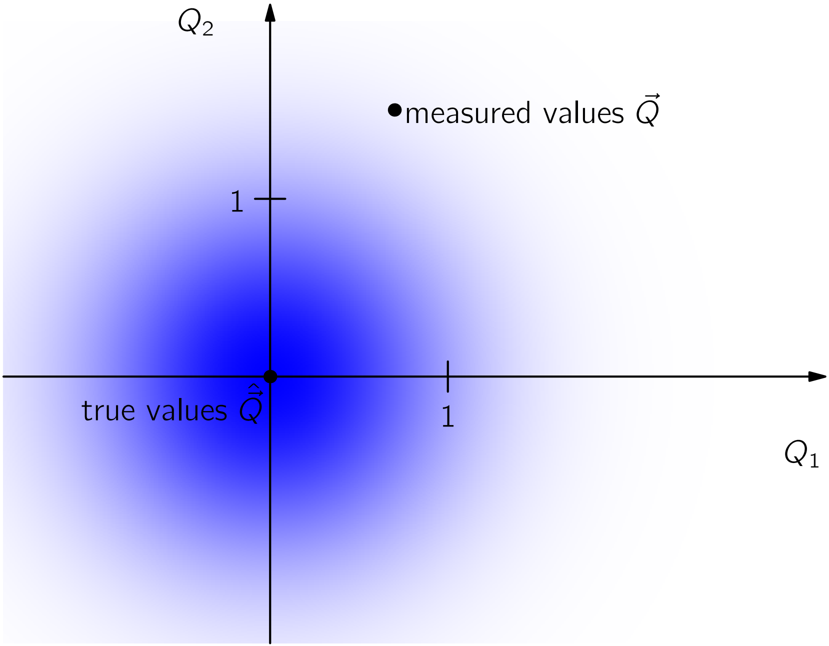

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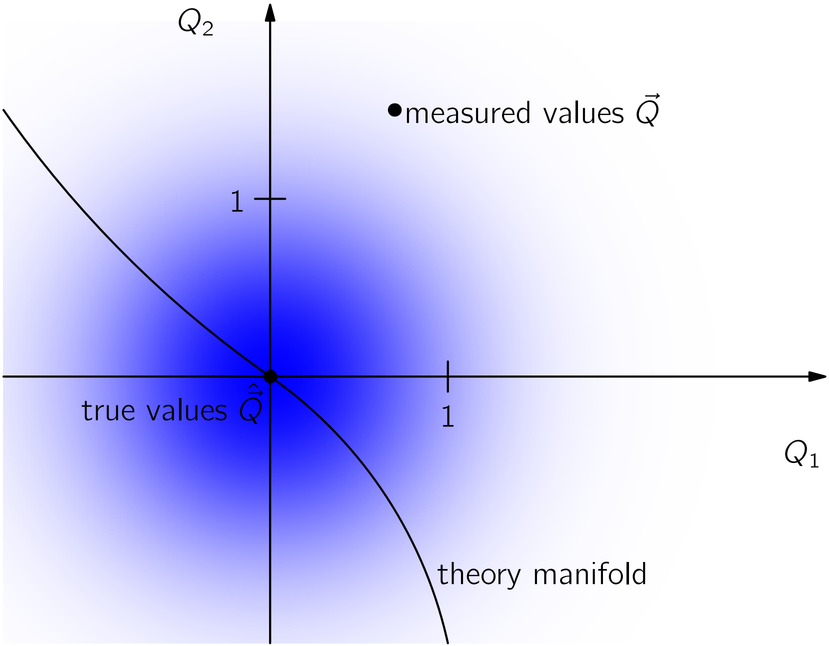

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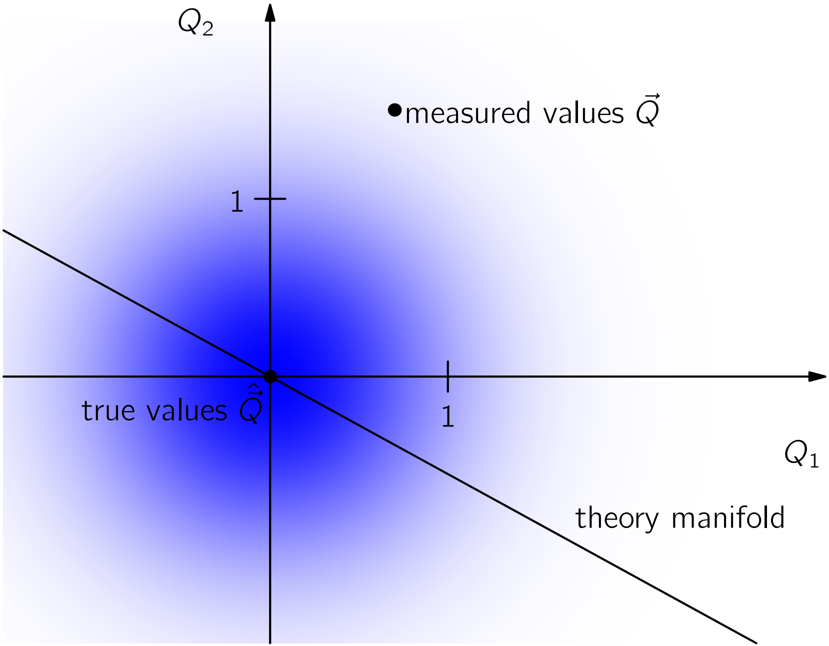

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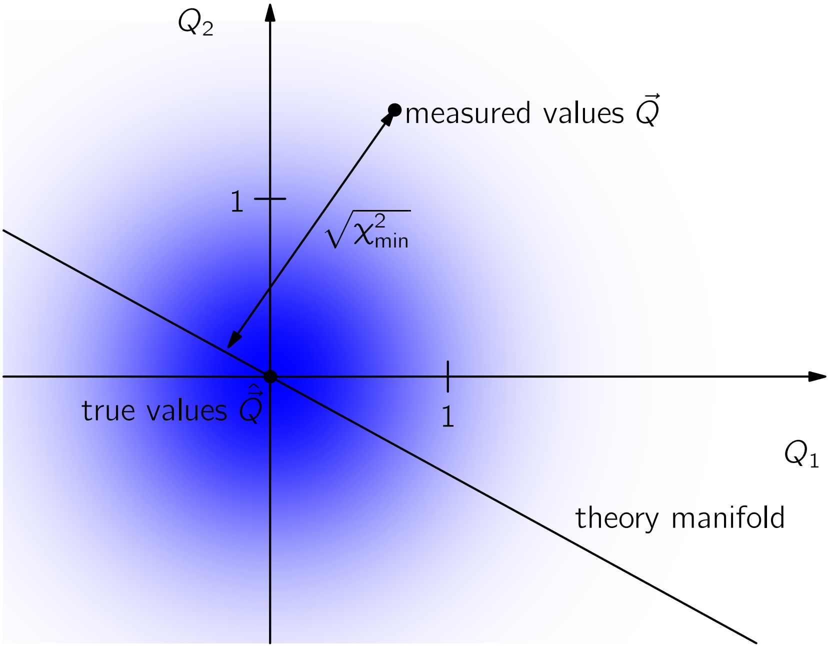

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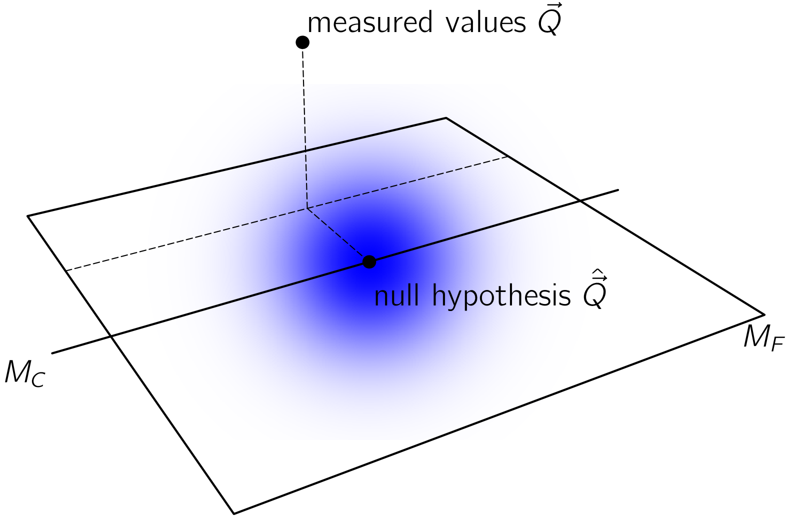

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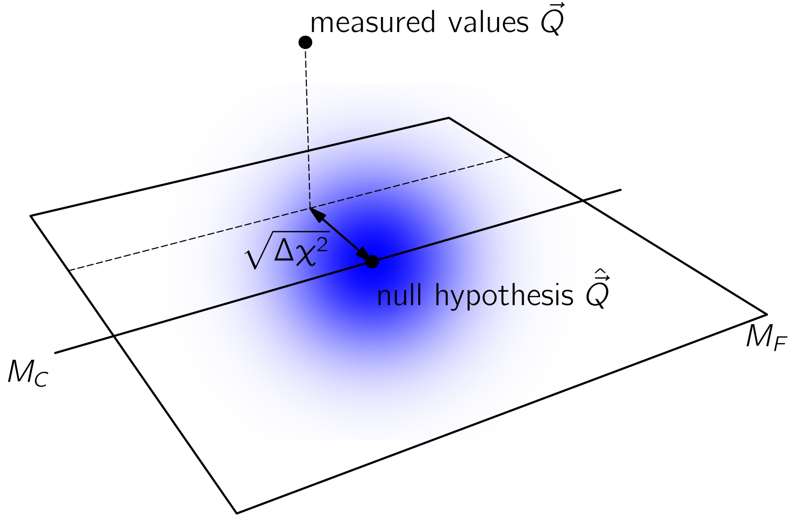

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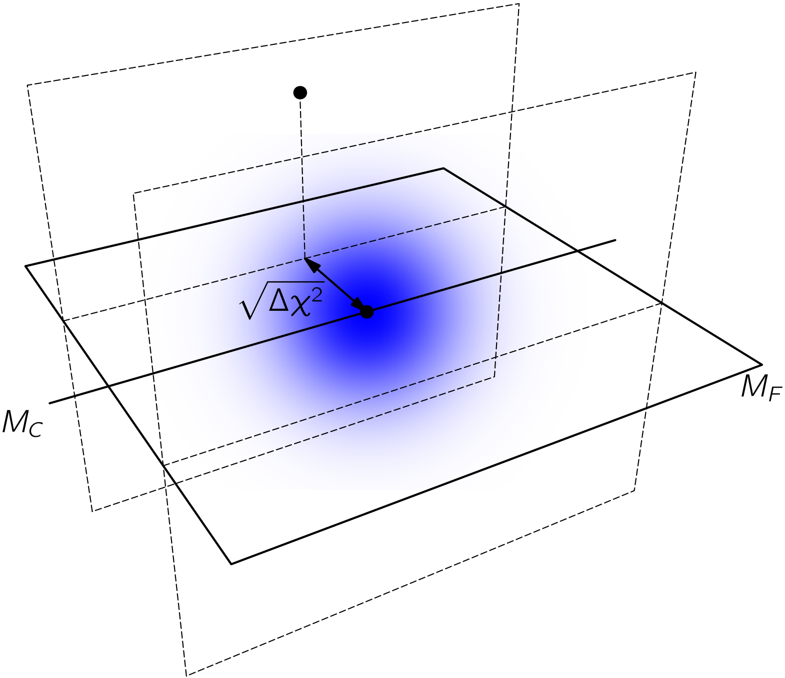

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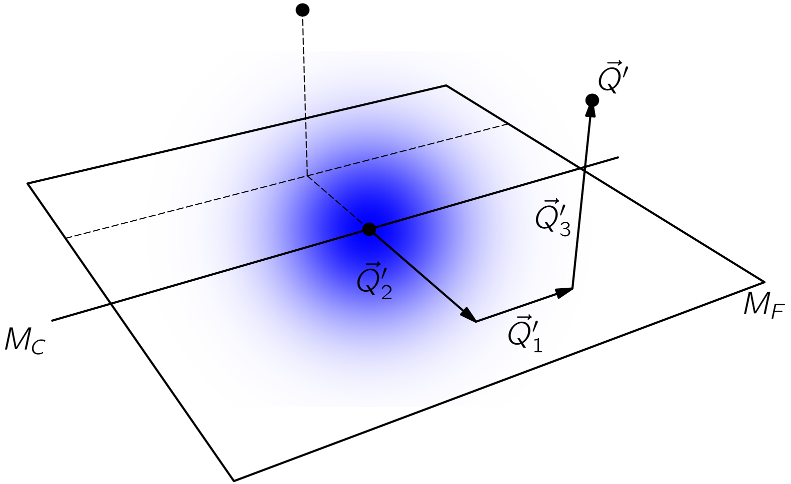

Applicability of Wilk's Theorem

Wilk's theorem applies if

- the errors are gaussian
- the theory manifolds are
  - nested
  - approximately flat
  - unbounded

Otherwise, we need numerical simulations.

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Otherwise, we need numerical simulations.

Strategy: optimise numerical simulations for the case where Wilk's theorem applies.

Monte Carlo Method

We need the integral

$$p = \int d^n\vec{Q}'\ f(\vec{Q}') \quad , \quad f(\vec{Q}') = \pi(\vec{Q}')\theta(\Delta\chi^2(\vec{Q}') - \Delta\chi^2(\vec{Q})) \quad .$$

(For gaussian errors: $\pi(\vec{Q}') = (2\pi)^{-n/2} e^{-|\vec{Q}'|^2/2}$)

- Choose a probability density function ρ which is as similar to f as possible.
- Generate N random sample points $\vec{Q}'_i$ distributed according to ρ.
- The integral is

$$p \approx \frac{1}{N} \sum_{i=1}^{N} \frac{f(\vec{Q}'_i)}{\rho(\vec{Q}'_i)} \quad .$$

How to choose ρ

- Simple choice

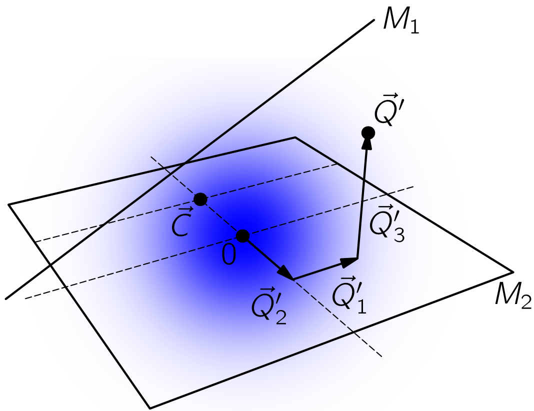
$$\rho(\vec{Q}') = \pi(\vec{Q}') = (2\pi)^{-n/2} e^{-|\vec{Q}'|^2/2} \quad .$$

- Better choice which avoids the "inner region":

$$\rho(\vec{Q}') = e^{-\frac{1}{2}|\vec{Q}'_1+\vec{Q}'_3|^2} \begin{cases} a|\vec{Q}'_2|^\alpha & , \quad |\vec{Q}'_2|^2 < \Delta\chi^2(\vec{Q}) \\ be^{-\frac{1}{2}|\vec{Q}'_2|^2} & , \quad |\vec{Q}'_2|^2 \geq \Delta\chi^2(\vec{Q}) \end{cases} \quad .$$

- Speedup of a factor 100 to 1000 in realistic situations.
- For further details see [M.W.; arXiv:1207.1446].

Non-nested Models



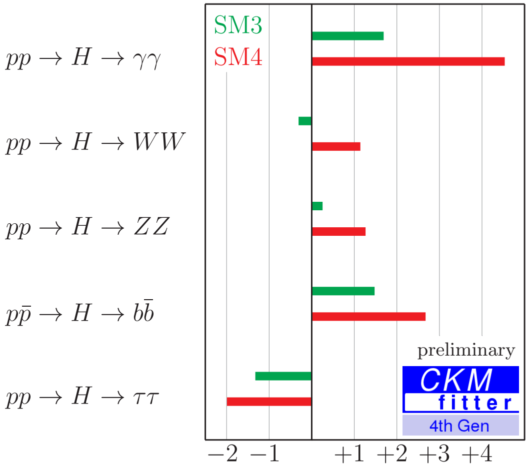
$$\rho(\vec{Q}') = e^{-\frac{1}{2}|\vec{Q}'_1+\vec{Q}'_2|^2} \begin{cases} a|\vec{Q}'_3|^\alpha & , \quad |\vec{Q}'_3|^2 < \Delta\chi^2(\vec{Q}) + |\vec{C} - \vec{Q}'_2|^2 \\ be^{-\frac{1}{2}|\vec{Q}'_3|^2} & , \quad |\vec{Q}'_3|^2 \geq \Delta\chi^2(\vec{Q}) + |\vec{C} - \vec{Q}'_2|^2 \end{cases}$$

Introducing *my*Fitter

These strategies for numerical computations of p-values were implemented in the public code *my*Fitter.

- It is a C++ class library.
- It allows implementation of arbitrary models and likelihood functions (via polymorphism).
- It supports parallel adaptive Monte Carlo integration by linking to the Dvegas/OmniComp package (N. Kauer).
- It comes with complete documentation and uses the GNU build system.
- The implementation is explained in [M.W.; arXiv:1207.1446].
- It is available at `http://myfitter.hepforge.org`.

Application to the SM4



⇒ The SM4 is ruled out with a p-value of $5.7 \cdot 10^{-8}$ ($5.4\,\sigma$).

Conclusions

- The SM4 struggles to produce the Higgs signal strengths measured at LHC and Tevatron.
- To compute a p-value for the SM4 one has to perform a likelihood ratio test for non-nested models (because of the non-decoupling nature of SM4 fermions).
- I presented a general method for numerical computations of p-values in likelihood ratio tests for nested and non-nested models.
- The method has been implemented in the public code *my*Fitter (`http://myfitter.hepforge.org`).
- Using post-ICHEP2012 signal strengths, $H \to b\bar{b}$ from Tevatron and $m_{t',b'} > 600\,\text{GeV}$ the SM4 is ruled out at $5.4\,\sigma$.

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