scale setting in lattice QCD

Gregorio Herdoíza

IFT, UAM

Lattice Practices 2012

DESY Zeuthen, Oct. 11, 2012

outline

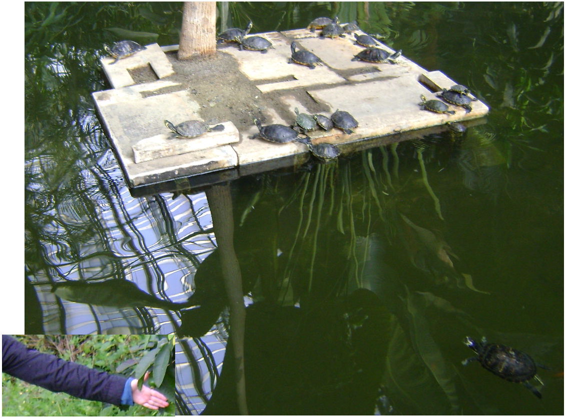
scale setting

- general considerations
- examples of intermediate reference scales
- examples of scale setting observables
- comparison

scale setting

scale setting

scale setting



QCD

action :

$$S[\psi, \bar{\psi}, A] = \frac{1}{g^2} \int \mathrm{d}^4x \, \mathrm{tr} \, [F_{\mu\nu}(x) F_{\mu\nu}(x)] \; + \sum_{f=1}^{\mathrm{N_f}} \int \mathrm{d}^4x \, \bar{\psi}^{(f)}(x) (\, \gamma_\mu \mathrm{D}_\mu + m^{(f)} \,) \psi^{(f)}(x)$$

parameters : $g, \; m^{(f)}$

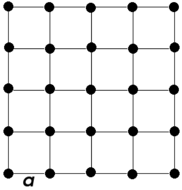
- determined from matching the theory to experimental measurements
- . . . applies more generally to the Standard Model

renormalisable theory :

- a finite number of experimental measurements is sufficient
  to set the parameters and to make predictions . . .

Lattice QCD

regularisation of QCD on a lattice :



- lattice spacing a : a^{-1} is a UV cutoff
- $x = a\,n\,, \quad n_i = 0, \ldots, N\,, \quad L = Na$

a lattice action :

$$S[\psi, \bar{\psi}, U] = \frac{\beta}{3} \sum_n \mathrm{Re\,tr}\,[1 - U_{\mu\nu}(n)] \; + \; a^4 \sum_{f=1}^{\mathrm{N_f}} \sum_n \bar{\psi}^{(f)}(n)(\,\mathrm{D} + m_0^{(f)}\,)\psi^{(f)}(n)$$

$\beta = \frac{6}{g_0^2}$

$N = L/a$

bare (input) parameters : $g_0\,,\; am_0^{(f)}$

- only dimensionless combinations appear
- ... in particular, the value of a in physical units is not known a priori ...
- a is a function of the bare parameters

Lattice QCD

$$S[\psi,\bar\psi,U] = \frac{\beta}{3}\sum_n \mathrm{Re\,tr}\,[1 - U_{\mu\nu}(n)] \; + \; a^4 \sum_{f=1}^{N_f} \sum_n \bar\psi^{(f)}(n)(\,\mathrm{D} + m_0^{(f)}\,)\psi^{(f)}(n)$$

bare (input) parameters : g_0, $am_0^{(f)}$ $N = L/a$

renormalisable theory :

- a finite number of experimental measurements is sufficient to set the parameters and to make predictions . . .
- remove the cutoff : $a \to 0$

how is this done?

notation

- O is any observable
- S is an observable used in the scale setting

  e.g. M_Ω , f_π , f_K , ...
- ρ is an observable used to relate different lattice spacings

  reference scale, intermediate scale, scaling variable

  e.g. r_0, t_0, w_0, ...
- R is a ratio of observables
- O^{phys} refers to O at the physical point
- O^{exp} refers to the experimental value of O
- O^{ref} refers to O at some reference point
- for simplicity, assume $[S] = [\rho] = mass$

  e.g. $\rho = 1/r_0$ and $a \times \rho = (r_0/a)^{-1}$

continuum limit scaling

- fix the "physical situation" at a reference point:

  i.e. for every value of g_0, fix $(L\rho)|_{\rm ref}$, $(m_R^{(f)}/\rho)|_{\rm ref}$

- study the dependence of $\frac{O}{\rho}$ on the lattice spacing via $a\,\rho$

continuum limit scaling

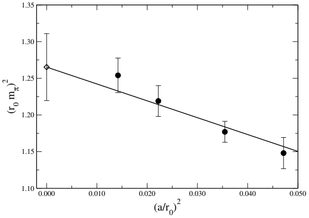
- fix the "physical situation" at a reference point:

  i.e. for every value of g_0, fix $(L\rho)|_{\mathrm{ref}}$, $(m_R^{(f)}/\rho)|_{\mathrm{ref}}$

- study the dependence of $\frac{O}{\rho}$ on the lattice spacing via $a\,\rho$

example :

- $N_\mathrm{f} = 2$ sea quarks $\rightsquigarrow$ $m_\ell = m_u = m_d$
- $\beta = 3.80,\ 3.90,\ 4.05,\ 4.20$
- scaling variable : $\rho = r_0^{-1}$
- measurements of $aO = am_\pi$ and r_0/a
- reference point : $L\rho = L/r_0 \approx 4.5$
  $m_\ell^\mathrm{R}/\rho = m_\ell^\mathrm{R} r_0 \approx 0.11$



[ETMC, 1010.3659]

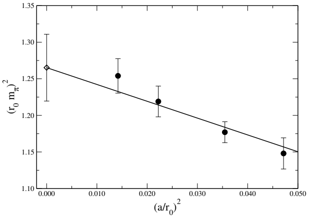
continuum limit scaling

- fix the "physical situation" at a reference point:

  i.e. for every value of g_0, fix $(L\rho)|_{\text{ref}}$, $(m_R^{(f)}/\rho)|_{\text{ref}}$

- study the dependence of $\frac{O}{\rho}$ on the lattice spacing via $a\,\rho$

- remove the cutoff : the continuum limit result should be "universal"
- observed dependence of a on $\beta = 6/g_0^2$ in order to keep the "physical situation fixed" $\rightsquigarrow$ $g_0(a)$ and $m_0(a)$
- scaling violations depend on the choice of the reference point
- so far, we did not use the actual value of a ...



[ETMC, 1010.3659]

contact with experiment

We have seen

- that the bare parameters depend on a
- how to remove the cutoff a^{-1}

but we also want to make contact with experiments, make predictions ...

- a finite number of experimental measurements is sufficient
  to set the parameters and to make predictions . . .
- we need N_f experimental inputs to set the quark masses in physical units
- and one more for the lattice spacing a

  $\rightsquigarrow$ "setting the scale"
- what about the coupling?

renormalisation group

In the scaling analysis of the continuum limit,
the "physical situation" is fixed
when changing the lattice spacing

$\rightsquigarrow$ imposed via a Callan-Symanzik equation

illustration : β function in terms of bare quantities

$$\beta(g_0) \equiv \frac{\partial g_0(a)}{\partial \log(a)}$$

perturbative expansion of $\beta(g_0)$ around $g_0 = 0$

$$\beta(g_0) = -g_0^3\,(b_0 + b_1 g_0^2 + \ldots)$$

solution is given by

$$a = \frac{1}{\Lambda_{\rm lat}}\,\left(b_0\, g_0^2(a)\right)^{-\frac{b_1}{2b_0^2}}\,\exp\left(-\frac{1}{2b_0\, g_0^2(a)}\right)\,\left(1 + \mathcal{O}(g_0^2)\right)$$

$\rightsquigarrow$ illustrates the connection between " setting the scale" and setting the coupling

we want to establish this connection non-perturbatively

scale setting

- the determination of the lattice scale a allows the conversion of O into physical units

  How is this done for lattice QCD?

(i) take any dimensionful observable S

(ii) bring S to the point where

$$\frac{m_q}{S(am_q)} = \left(\frac{m_q}{S}\right)^{\text{phys}}$$

$$[aS(am_q)] \rightarrow [aS]^{\text{phys}}$$

(iii) match S to its experimental value

$$[aS]^{\text{latt,phys}} \equiv a \times S^{\text{exp}}$$

scale setting

- conceptually rather simple but not easy in practice
- remarks on step (iii)

  (iii) match S to its experimental value:

$$[aS]^{\mathrm{latt,phys}} \equiv a \times S^{\mathrm{exp}}$$

  - S in l.h.s and r.h.s described by the same theory
  - r.h.s known accurately
  - l.h.s at the physical point depend little on m_q
  - l.h.s can be computed accurately
  - cutoff effects on l.h.s
  - mass-independent scale setting : $a(g_0)$

    for all ensembles at a given g_0, the lattice spacing in physical units is the same

- remarks on step (ii)
  - control of statistical and systematic uncertainties

precision in step (ii)

- systematic uncertainties
  - number of dynamical flavours (u,d,s,c,... quarks) — $N_{\rm f} = 0;\ 2;\ 2+1;\ 2+1+1$
    partial quenching
  - cutoff effects: lattice spacing a — $O(a)$ improvement, continuum limit
    broken symmetries at $a \neq 0$
    $m_q \ll 1/a$
  - range of quarks masses : simulation/physics
    applicability of χPT, HQET, NRQCD, ...
  - finite size effects : lattice size L — $m_{\rm PS} L \gg 1$
  - renormalisation — non-perturbative
  - isospin breaking : $m_u \neq m_d$, electromagnetism
- statistical errors
  - autocorrelations

more effects become relevant with increasing precision

choice of S

remark on (i) : any S . . .

- S can suffer from practical or conceptual imperfections
- try to avoid introducing large uncertainties into observables of interest via the scale setting

$\rightsquigarrow$ choose S carefully . . . and make cross-checks

implications

- once a is known : obtain dimensionful observables in physical units
- useful at the tuning stage : fix the bare parameters

reference scale ρ

$$r_0, r_1, t_0, w_0, \ldots$$

reference scale ρ

express lattice results (in lattice units)

in terms of a reference scale ρ

- compare data from different lattice spacing
- perform the continuum extrapolation
- useful when ρ can be determined accurately

Sommer scale : r_0

[R. Sommer, hep-lat/9310022]

- hadronic length scale
- determined from the force $F(r)$ between static quarks

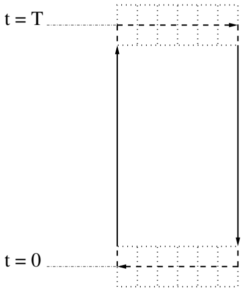
$$r^2 F(r)\Big|_{r=r_0} = 1.65 \qquad \text{where} \qquad F(r) = V'(r)$$

- potential $V(r)$ between a static (*infinitely massive*) quark and anti-quark pair separated by distance r
- determined from Wilson loops $\langle W(r,T)\rangle$

$$\langle W(r,T)\rangle \overset{N_t\to\infty}{\sim} \sum_n c_n c_n^* \, \mathrm{e}^{-V_n(r)(T-2a)}$$

with

$$V_0(r) = E_0^{(q\bar{q})}(r) - E_0^{(0)}$$



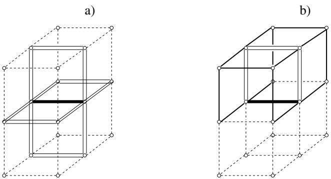
Sommer scale : r_0

- signal : falls exponentially with the area of the loop (eventually the string breaks)
  variance : $\sim$ constant
- as r increases : exponential decrease of signal-to-noise ratio
- smearing:
  - time : modification of the static action
  - space : basis of operators $\rightsquigarrow$ variational analysis

smoothing of gauge fields

- smoothing or smearing : reduce the short-distance roughness of gauge fields

- HYP smearing
- iterations



[Hasenfratz & Knechtli, hep-lat/0103029]

- stout smearing

[Morningstar & Peardon, hep-lat/0311018]

$$U'_\mu(x) = e^{iQ_\mu(x,\rho)}\, U_\mu(x)$$

- $Q_\mu(x,\rho)$ built from staples

  traceless, Hermitian
- differentiable $\rightsquigarrow$ HMC

r_0 : properties

- does not require extrapolations in r
- $\mathcal{O}(a)$ improved (for improved sea quarks)
- r_0/a can be calculated with good statistical precision [$\sim$ 1%]
- no direct connection of r_0 with experiment, only with phenomenological potential models

  $\rightsquigarrow$ $r_0 \approx 0.5\,\mathrm{fm}$

- How is r_0/a computed?

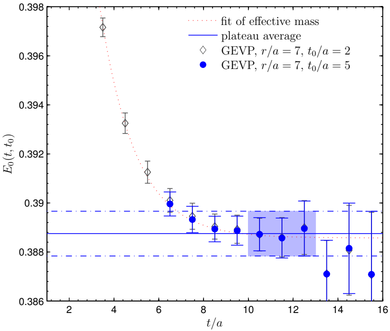
(i) determine the static potential at some distance r/a

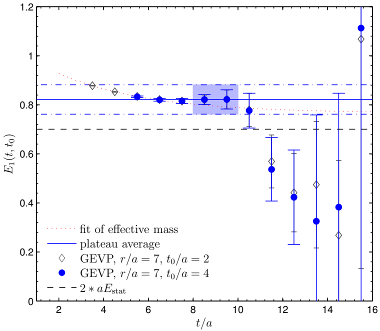
(ii) compute the force

(iii) interpolate in r

- note on the string tension : requires extrapolation in r and ill-defined in unquenched QCD

static potential : $N_f = 2$

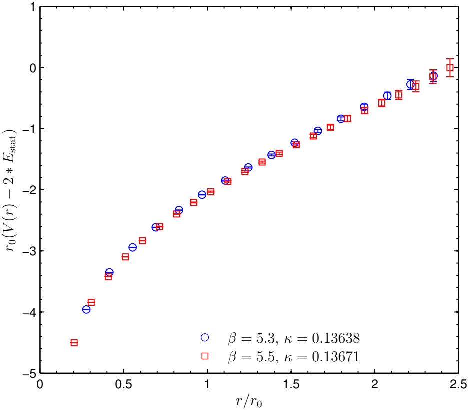




[ALPHA, 1012.3037]

- $E_{\rm stat}$: binding energy of the static-light meson
- it is expected that the first excited state is related to a meson–anti-meson state : $V_1 \approx 2E_{\rm stat}$
- ... would require the use of suitable operators

static potential : $N_f = 2$



[ALPHA, 1112.1246]

$$r^2\, F(r)\Big|_{r=r_0} = 1.65 \qquad \text{where} \qquad F(r) = V'(r)$$

$$r^2\, F(r)\Big|_{r=r_1} = 1.0 \qquad\qquad r_1/r_0 \sim 0.65$$

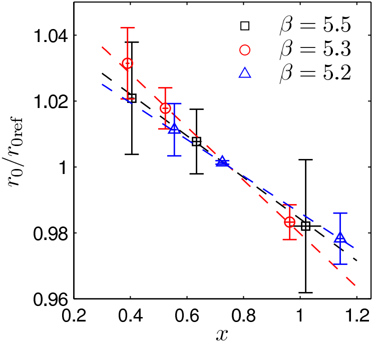
[C. Bernard et al., hep-lat/0002028]

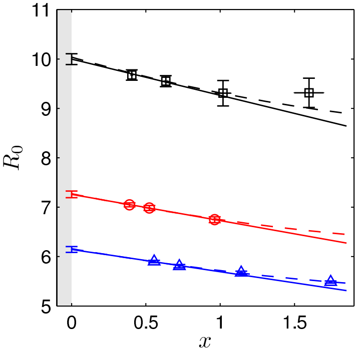
r_0/a : $N_{\rm f} = 2$

- light quark mass dependence
- discretisation effects in mass-dependence

[ALPHA, 1205.5380]

$x = (r_0 m_{\rm PS})^2$ $\qquad x_{\rm ref} \equiv (r_0 m_{\rm PS})^2|_{\rm ref} = 0.75$ $\qquad R_0 = r_0/a$

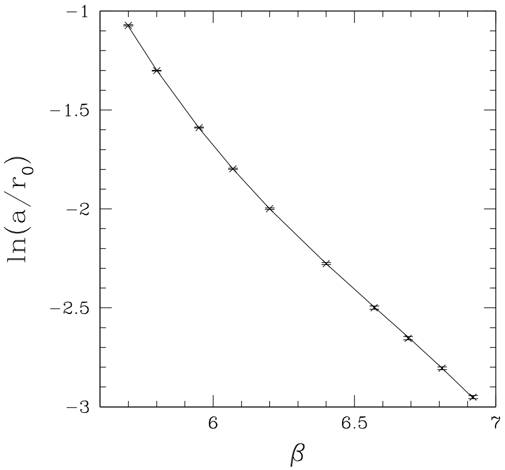




$$\frac{r_0}{r_{0\rm ref}}(x) = 1 + s(a/r_{0\rm ref}) \cdot (x - x_{\rm ref})$$

$$R_0(\beta, x) = R_{0c}(\beta)\left(1 + S_1\, x + S_2\, x^2\right)$$

r_0/a : quenched



$\beta = 6/g_0^2$

[S. Necco & R. Sommer, hep-lat/0108008]

Wilson flow

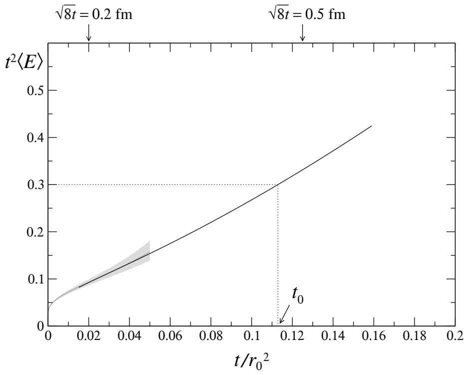
[M. Lüscher, 1006.4518]

- flow $B_\mu(t,x)$, for $t > 0$ and $B_\mu(0,x) = A_\mu(x)$
  - flow equation : $\partial_t B_\mu = D_\nu G_{\nu\mu}$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$$

  - the flow smooths the gauge field : range $\sqrt{8t}$
  - infinitesimal stout-smearing steps
- on the lattice : $V_t(x,\mu)$ with $V_0(x,\mu) = U(x,\mu)$
  - $\partial_t V_t(x,\mu) = -g_0^2 \{\partial_{x,\mu} S_{\rm w}(V_t)\} V_t(x,\mu)$
  - observables : $E(t) = \frac{1}{2} G^a_{\nu\mu} G^a_{\nu\mu}$
  - $\langle E(t) \rangle$ does not need renormalisation

Wilson flow : t_0



pure gauge , $a = 0.05$ fm

[M. Lüscher, 1006.4518]

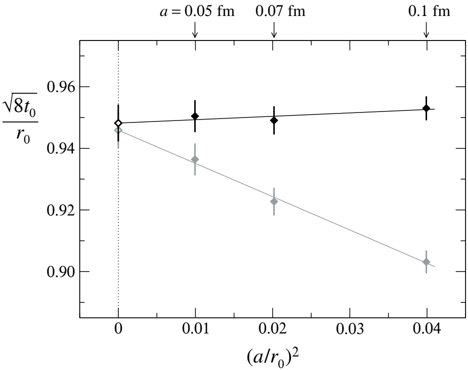
simple and precise

$$t^2 \langle E(t) \rangle \Big|_{t=t_0} = 0.3$$

Wilson flow : t_0

continuum limit scaling

pure gauge



[M. Lüscher, 1006.4518]

$$t^2 \langle E(t) \rangle \Big|_{t=t_0} = 0.3$$

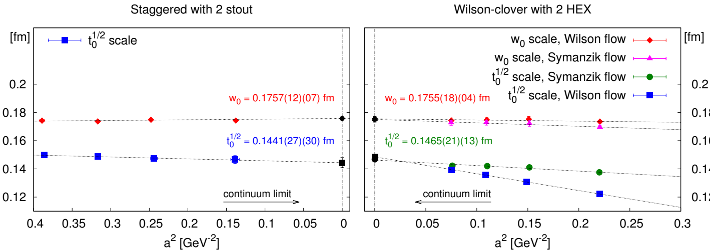
recent proposal :

$$t \frac{\mathrm{d}}{\mathrm{d}t} \left[t^2 \langle E(t) \rangle \right] \Big|_{t={w_0}^2} = 0.3$$

[BMW, 1203.4469]

w_0

continuum limit scaling $N_f = 2 + 1$



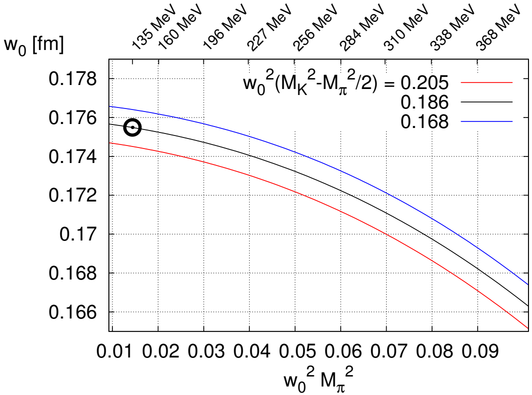
[BMW, 1203.4469]

$$t \frac{\mathrm{d}}{\mathrm{d}t} \left[t^2 \langle E(t) \rangle \right] \Big|_{t=w_0^2} = 0.3$$

$$w_0 = 0.1715(18)(04)\ \mathrm{fm} \quad [1\%]$$

scale from M_Ω : largest contribution to the error of w_0

w_0 away from the physical point

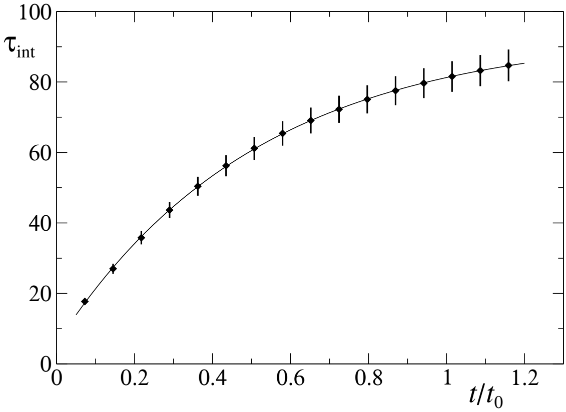


$N_f = 2 + 1$

[BMW, 1203.4469]

Wilson flow observables : autocorrelation

$\tau_{\rm int}(\langle E\rangle)$



pure gauge, open BC, $a = 0.05\,\mathrm{fm}$, L/a=32, [M. Lüscher & S. Schaefer, 1105.4749]

reference scale with meson masses

define a reference scale within the available data range : $\rho = M_K^{\rm ref}$

(i) $M_K/M_{K^*} = (M_K/M_{K^*})^{\rm exp} = 0.554$

$N_f = 2$ [CERN, hep-lat/0610059]

$\rightsquigarrow$ m_s at a given m_ℓ

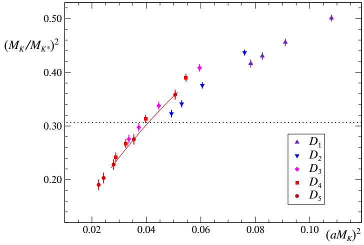
- observation:

  in this point, aM_K does not significantly depend in m_ℓ (in particular for small m_ℓ)

(ii) pick m_ℓ such that

$(M_\pi/M_K)^{\rm ref} = 0.85$ $\qquad (M_\pi/M_K)^{\rm exp} \approx 0.28$

$\rightsquigarrow$ $a\rho = aM_K^{\rm ref}$

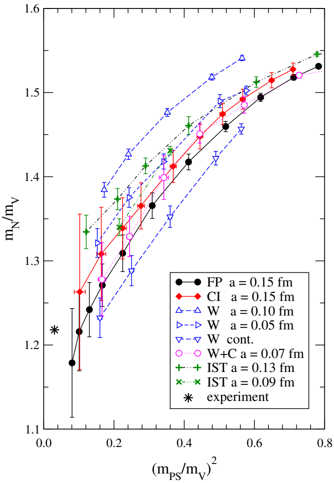


- useful for comparing results from different lattice spacings
- why $M_\pi/M_K = 0.85$?
  - long extrapolation to $(M_{K^*}/M_K)^{\rm exp}$
  - K^* is a resonance :

    the extrapolation would have to go through the kinematical threshold

quenched era

APE plot : ratios



[BGR, hep-lat/0307013]

- in the quenched case : $r_0 = 0.5\,\mathrm{fm}$, ρ-meson, ϕ, f_K, M_N, ...
- $\sim$10% ambiguity between setting the scale with f_K and M_N

[ALPHA-UKQCD, hep-lat/9906013]

scale setting : pseudoscalar meson decay constants

$$f_\pi , \quad f_K$$

scale setting with f_π , f_K

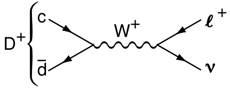
- pseudoscalar mesons :
  - do not decay in QCD
  - signal/noise ratio
  - $am_q \ll 1$
  - excited state contamination
- light-quark mass dependence :
  - guidance from chiral perturbation theory (χPT) where applicable
  - f_K milder log effects than in f_π
  - comparison of χPT and polynomial fits
- finite volume effects : χPT
- f_K : requires m_s
- renormalisation
- experimental situation

pseudoscalar meson decay constants : $f_{\rm PS}$

$$f_\pi \, p_\mu = \langle 0 | \, \bar{u} \, \gamma_\mu \gamma_5 d \, | \, \pi(p) \rangle$$

$\pi^\pm$, $K^\pm$, $D^\pm$, . . .

decay width of charged pseudoscalar meson P into leptons $\rightsquigarrow$ $f_{\rm PS}$



$$\Gamma(P \to \ell\nu) = \frac{G_F^2}{8\pi} \, f_{\rm PS}^2 \, m_\ell^2 \, M_{\rm PS} \left(1 - \frac{m_\ell^2}{M_{\rm PS}^2} \right) |V_{q_1 q_2}|$$

f_π : experimental input

$$\Gamma(P \to \ell\nu) = \frac{G_F^2}{8\pi} f_{\rm PS}^2 \, m_\ell^2 \, M_{\rm PS} \left(1 - \frac{m_\ell^2}{M_{\rm PS}^2}\right) |V_{q_1 q_2}|$$

- π^- : $(\bar{u}d)$; $|V_{ud}|$; $\pi^- \to \mu^- + \bar{\nu}\,(\gamma)$

  $f_{\pi^-} = (130.4 \pm 0.2)\,\text{MeV}$ [0.2%] [PDG, 2012]

  note : $f_{\pi^-} = (130.7 \pm 0.4)\,\text{MeV}$ [0.3%] [PDG, 2007]

- π^0 : $(\bar{u}u + \bar{d}d)$; $\pi^0 \to \gamma\gamma$ $\rightsquigarrow$ $f_{\pi^0}^{\rm exp}$ [4%]

- $M_{\pi^\pm} = 139.6\,\text{MeV}$; $M_{\pi^0} = 135.0\,\text{MeV}$; $M_{\pi^\pm} - M_{\pi^0} = 4.6\,\text{MeV}$ (i.e $\sim 3\%$)

  $m_u = m_d$ & w/out QED : $M_\pi = 134.8(3)$ MeV [FLAG, 1011.4408]

- $m_u \neq m_d$: $(f_{\pi^\pm} - f_{\pi^0})/f_{\pi^0} \propto (m_d - m_u)^2 \sim 10^{-4}$ i.e. small

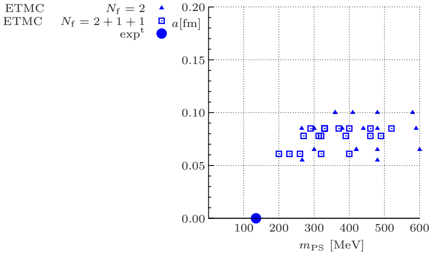
  idem for QED (not unique way to split QED and QCD)

  [Gasser & Zarnauskas, 1008.3479]

- PDG uses M_{π^0} $\rightsquigarrow$ $\delta f_{\pi^-} = +0.4\,\text{MeV}$

$N_f = 2$ & $2+1+1$ ETMC ensembles

- Wilson twisted mass
  [ALPHA, Frezzotti et al.,2001; Frezzotti & Rossi,2003]
- tlSym gauge action [$N_f = 2$]
- Iwasaki gauge action [$N_f = 2+1+1$]



$N_f = 2+1+1$

- $a = \{0.06,\ 0.08,\ 0.09\}$ fm
- $L = \{1.9,\ 2.7\}$ fm , $M_{PS} L \gtrsim 3.5$
- $m_\pi \in \{200,\ 520\}$ MeV
- HMC + PHMC
- 5000 thermalised traj. , $\tau = 1$

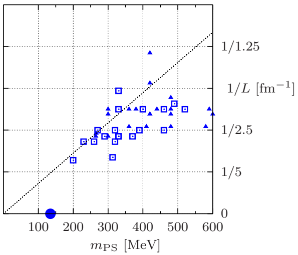
$N_f = 2$

- $a = \{0.06,\ 0.07,\ 0.08,\ 0.10\}$ fm
- scale setting :

$$(i) \quad (f_\pi r_0)^{\text{phys,latt}} = f_\pi^{\text{exp}} \times r_0^{\text{phys}}$$

$$(ii) \quad a = \frac{r_0^{\text{phys}}}{\left(\frac{r_0}{a}[\beta]\right)^{\text{phys}}}$$

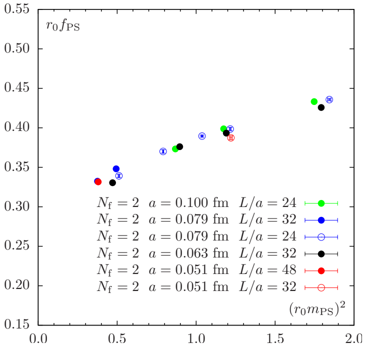
- M_N is also used [ETMC, 0910.2419, 1012.3861]

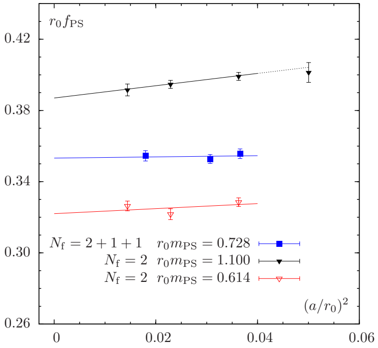


f_π : discretisation effects

pion decay constant

$$f_{\rm PS} = \frac{2\mu_\ell}{M_{\rm PS}^2} |\langle 0|P^1(0)|\pi^\pm\rangle|$$





chiral perturbation theory (χPT) $m_\pi,\ f_\pi$

- SU(2) χPT

$$M^2_{\mathrm{PS}}(L) = 2\widehat{B}_0\mu_R \left[1 + \xi \ln(\chi_\mu/\Lambda_3^2) + T_m^{\mathrm{NNLO}} + a^2 D_m\right] \cdot \left(K_m^{\mathrm{CDH}}(L)\right)^2$$

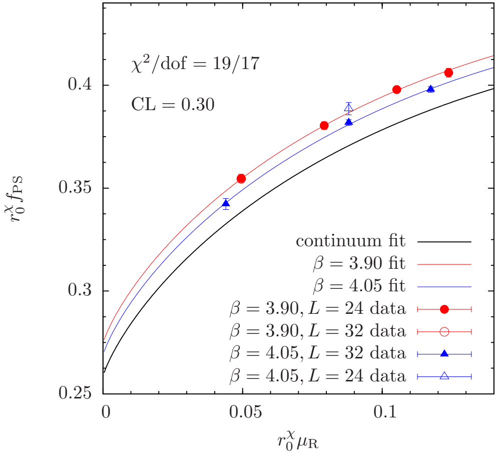
$$f_{\mathrm{PS}}(L) = f_0 \left[1 - 2\xi \ln(\chi_\mu/\Lambda_4^2) + T_f^{\mathrm{NNLO}} + a^2 D_f\right] \cdot K_f^{\mathrm{CDH}}(L)$$

where $\chi_\mu = 2\widehat{B}_0\mu_R\,, \quad \mu_R = 1/Z_{\mathrm{P}}\,\mu_\ell\,, \quad \xi = \chi_\mu/(4\pi f_0)^2$

- derived quantities : $m_{u,d}$, $\langle\bar{q}q\rangle$, low-energy constants : $\bar{l}_{3,4} \equiv \log(\Lambda^2_{3,4}/M^2_{\pi^\pm})$
- finite size corrections [Colangelo et al.(CDH), 2005]
- mass dependence NLO and NNLO (extra parameters : $r_0\Lambda_{1,2}, k_M, k_F$)
- include $O(a^2)$ terms in the fits

f_π : χPT $N_f = 2$

SU(2) χPT at NLO including $O(a^2)$ terms



[ETMC, 0911.5061]

systematics : χPT fits $N_f = 2$

$(f_\pi r_0)^{\text{phys,latt}} = f_\pi^{\text{exp}} \times r_0^{\text{phys}}$ $\quad a = \dfrac{r_0^{\text{phys}}}{(\frac{r_0}{a}[\beta])^{\text{phys}}}$ $\quad f_\pi^{\text{exp}} = 130.7\ \text{MeV}$

	$\beta = 3.8, 3.9, 4.05$	$\beta = 3.9, 4.05$	$\beta = 3.9, 4.05, 4.2$
r_0 [fm]	0.446(9)	0.420(14)	0.429(8)
$a(\beta = 3.90)$ [fm]	0.0847(15)	0.0790(26)	0.0801(14)
$a(\beta = 4.05)$ [fm]	0.0672(12)	0.0630(20)	0.0638(10)
$\lvert\Sigma\rvert^{1/3}$ [MeV]	262.2(4.0)	269.9(6.5)	268.4(6.6)
$\bar{\ell}_3$	3.32(21)	3.50(31)	3.49(9)
$\bar{\ell}_4$	4.69(17)	4.66(33)	4.63(4)
f_π / f_0	1.0742(81)	1.0755(94)	1.0750(8)
$m_{u,d}$ [MeV]	3.84(18)	3.54(26)	3.58(26)

$\lvert\Sigma\rvert^{1/3}$ and $m_{u,d}$ are given in $\overline{\text{MS}}$ at 2 GeV [ETMC, 0911.5061]

note : latest $N_f = 2$ analysis [ETMC, 1010.3659]

- four values of a , updated determinations of Z_P
- fits to tmWχPT [Colangelo et al., 1003.0847] & [O. Bär, 1008.0784]

$r_0^{\text{phys}} = 0.450(15)$ fm [3.3%] $\qquad m_{u,d} = 3.6(2)$ MeV

f_K

K^- : $(\bar{s}d)$; $|V_{us}|$; $K^- \to \mu^- + \bar{\nu}\,(\gamma)$

unitarity of the CKM matrix: first row

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 + O\left(\frac{M_W^2}{\Lambda_{\rm NP}^2}\right)$$

Relative contributions

- $|V_{ud}| \approx 0.974$: rel. error $\delta \sim 0.02\%$ — nuclear β decays
- $|V_{us}| \approx 0.225$: $\delta \sim 0.50\% \div 1\%$ — $K_{\ell 3}$ and $K_{\ell 2}$ decays; $\tau \to$ hadrons
- $|V_{ub}| \approx 0.004$: small

f_K

K^- : $(\bar{s}d)$; $|V_{us}|$; $K^- \to \mu^- + \bar{\nu}\,(\gamma)$

unitarity of the CKM matrix: first row

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 + O\left(\frac{M_W^2}{\Lambda_{\rm NP}^2}\right)$$

Relative contributions

- $|V_{ud}| \approx 0.974$: rel. error $\delta \sim 0.02\%$ nuclear β decays
- $|V_{us}| \approx 0.225$: $\delta \sim 0.50\% \div 1\%$ $K_{\ell 3}$ and $K_{\ell 2}$ decays; $\tau \to$ hadrons
- $|V_{ub}| \approx 0.004$: small

Determinations of $|V_{us}|$

- semileptonic $K_{\ell 3}$ decays: $K \to \pi \ell \nu$

$$\Gamma(K_{\ell 3(\gamma)}) \propto |V_{us}|^2\, f_+(0)^2$$

  - $\delta(|V_{us}| f_+(0)) \sim 0.20\%$

f_K

K^- : $(\bar{s}d)$; $|V_{us}|$; $K^- \to \mu^- + \bar{\nu}\,(\gamma)$

unitarity of the CKM matrix: first row

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 + O\left(\frac{M_W^2}{\Lambda_{\mathrm{NP}}^2}\right)$$

Relative contributions

- $|V_{ud}| \approx 0.974$: rel. error $\delta \sim 0.02\%$ — nuclear β decays
- $|V_{us}| \approx 0.225$: $\delta \sim 0.50\% \div 1\%$ — $K_{\ell 3}$ and $K_{\ell 2}$ decays; $\tau \to$ hadrons
- $|V_{ub}| \approx 0.004$: small

Determinations of $|V_{us}|$

- semileptonic $K_{\ell 3}$ decays: $K \to \pi \ell \nu$

$$\Gamma(K_{\ell 3(\gamma)}) \propto |V_{us}|^2\, f_+(0)^2$$

  - $\delta(|V_{us}| f_+(0)) \sim 0.20\%$

- leptonic $K_{\ell 2}$ decays: $K \to \ell \nu$

$$\frac{\Gamma(K^{\pm}_{\ell 2(\gamma)})}{\Gamma(\pi^{\pm}_{\ell 2(\gamma)})} \propto \left|\frac{V_{us}}{V_{ud}}\right|^2 \left(\frac{f_K}{f_\pi}\right)^2$$

  - $\delta(V_{us}/V_{ud} \times f_K/f_\pi) \sim 0.20\%$

f_K : experimental input

f_K [MeV]	method
156.1 ± 0.9 [0.5%]	$K^- \to \mu^- \bar{\nu}\,(\gamma)$ & $f_+(0)$ from lattice [PDG]
155.6	lattice $N_f = 2+1$
157.8	lattice $N_f = 2$
155.5	lattice $N_f = 2+1$ + unitarity
155.7	lattice $N_f = 2$ + unitarity
158.7	$\lvert V_{us} \rvert$ from τ decays

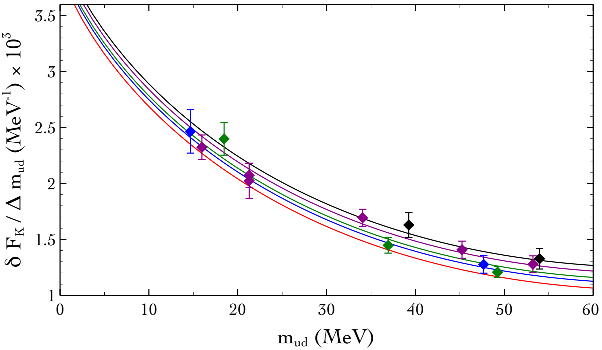
note : this is not a thorough comparison of f_K

[FLAG, 1011.4408] : does not quote f_K, only f_K/f_π ; warnings about τ decays

$$f_\pi^{\rm exp} = 130.4 \pm 0.2 \ \ [0.2\%]$$

f_K : isospin breaking

$$\delta F_K \equiv \left.\frac{F_{K^0} - F_{K^+}}{F_K}\right|_{\rm QCD}$$



[RM123, 1110.6294]

input : $\left[M^2_{K^0} - M^2_{K^+}\right]^{QCD} = 6.05(63) \times 10^3\ \mathrm{MeV}^2$ exp^t + χPT + lattice [FLAG, 1011.4408]

$$\left[\frac{F_{K^0} - F_{K^+}}{F_K}\right]^{QCD} = 0.0078(8)$$

$N_f = 2$ ALPHA

- Wilson fermions : $O(a)$ improved with non-perturbative c_{SW}

  [CLS; ALPHA, 1205.5380]

- Wilson plaquette gauge action

- $N_f = 2$
- $a = \{0.08,\ 0.07,\ 0.05\}$ fm
- $L = \{2.1,\ 3.2\}$ fm , $M_{PS} L > 4$
- $m_\pi \in \{270,\ 630\}$ MeV

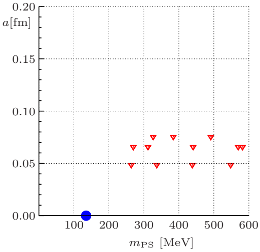
- DD-HMC and MP-HMC
- effective statistics :

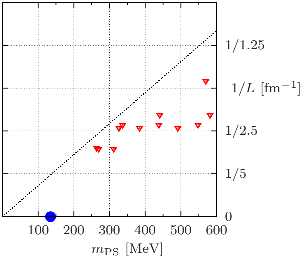
  $1200 \div 2{\times}6000$ thermalised traj. , $\tau = 1$

- scale setting:

$$a = \frac{(\, a f_K[\beta]\,)^{\rm phys,latt}}{f_K^{\rm exp}}$$

- M_Ω is also used [ALPHA, 1110.6365]





f_K

[ALPHA, 1205.5380]

physical input

$$m_{\pi,\mathrm{phys}} = 134.8\ \mathrm{MeV}\,, \qquad m_{\mathrm{K,phys}} = 494.2\ \mathrm{MeV}\,, \qquad f_{\mathrm{K,phys}} = 155\ \mathrm{MeV}$$

subtracted isospin breaking effects [FLAG, 1011.4408]

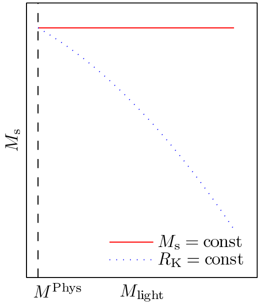
$$R_\mathrm{K} = \frac{m_\mathrm{K}^2}{f_\mathrm{K}^2}\,, \qquad R_\pi = \frac{m_\pi^2}{f_\mathrm{K}^2}$$

- reaching the physical point

  (i) $R_\mathrm{K}(\kappa_1, \kappa_3) = R_\mathrm{K}^{\mathrm{phys}} = \mathrm{const}$

  see also [QCDSF-UKQCD, 1102.5300]

  (ii) $M_s = M_S^{\mathrm{phys}} = \mathrm{const}$



$N_\mathrm{f} = 2$: partial quenching ; Z_A

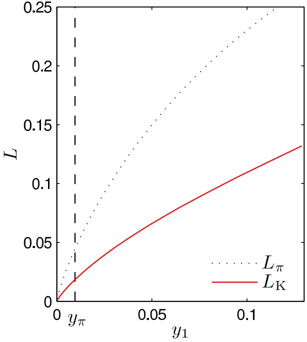
f_K

$$R_K = \frac{m_K^2}{f_K^2}\,, \qquad R_\pi = \frac{m_\pi^2}{f_K^2}$$

[ALPHA, 1205.5380]

$$y_1 = \frac{m_\pi^2(\kappa_1)}{8\pi^2 f_K^2(\kappa_1, h(\kappa_1))} = \frac{1}{8\pi^2} R_\pi(\kappa_1)$$

$$y_\pi = \frac{m_{\pi,\mathrm{phys}}^2}{8\pi^2 f_{K,\mathrm{phys}}^2} = 0.00958\,, \qquad y_K = \frac{m_{K,\mathrm{phys}}^2}{8\pi^2 f_{K,\mathrm{phys}}^2} = 0.12875$$



(i) $R_K(\kappa_1, \kappa_3 = h(\kappa_1)) = R_K^{\mathrm{phys}} = \mathrm{const}$

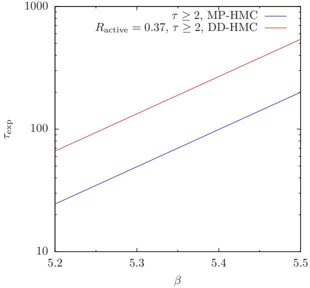
SU(3) PQχPT at NLO:

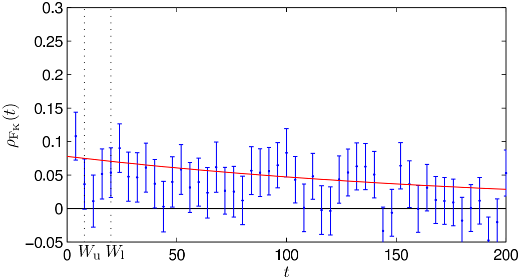
$$f_K(\kappa_1, h(\kappa_1)) = f_{K,\mathrm{phys}} \left[1 + \bar{L}_K(y_1, y_K) + (\alpha_4 - \frac{1}{4})(y_1 - y_\pi) + \mathrm{O}(y^2)\right]$$

$$\bar{L}_K(y_1, y_K) = L_K(y_1, y_K) - L_K(y_\pi, y_K)$$

$$L_K(y_1, y_K) = -\frac{1}{2} y_1 \log(y_1) - \frac{1}{8} y_1 \log(2y_K/y_1 - 1)$$

f_K : autocorrelations

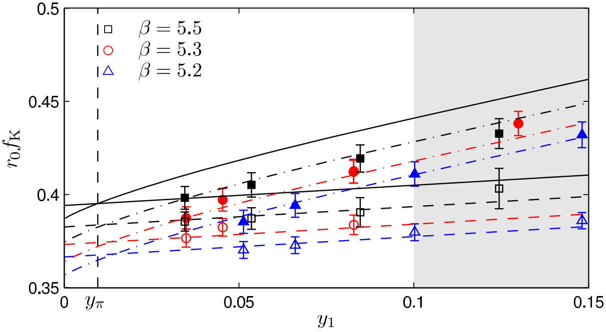




[ALPHA, 1205.5380]

- $a \approx 0.05\,\text{fm}$, $M_\pi \approx 270\,\text{MeV}$, $M_\pi L = 4.2$, 4000 MD units
- including the tail : factor of two increase in the error

f_K : cutoff effects



open symbols: r_0 / a at finite mass

filled symbols: r_0 / a at the chiral limit

[ALPHA, 1205.5380]

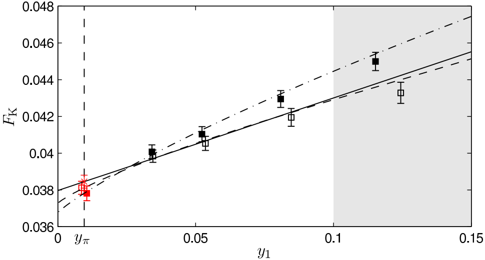
$$y_1 = \frac{m_\pi^2(\kappa_1)}{8\pi^2 f_K^2(\kappa_1, h(\kappa_1))}$$

- $(r_0 f_K)^{\rm phys} = 0.3951(62)$ [1.6%]

$r_0 = 0.503(10)$ fm [2.0%]

- note : r_0 is not used in the scale setting

f_K : scale setting



open symbols: strategy (i)
filled symbols: strategy (ii)

$F_K = af_K$ $\beta = 5.5$

[ALPHA, 1205.5380]

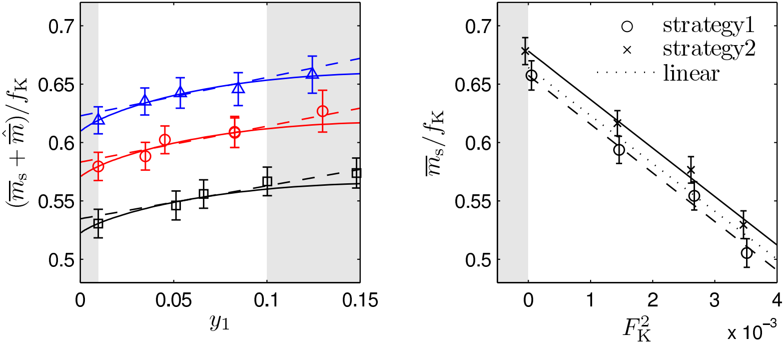
$$a = \frac{(af_K)^{\rm phys,latt}}{f_K^{\rm exp}} \qquad f_K^{\rm exp} = 155\,\text{MeV}$$

β	$(af_K)^{\rm phys,latt}$	a [fm]
5.2	0.0593(7)(6)	0.0755(9)(7) [1.5%]
5.3	0.0517(6)(6)	0.0658(7)(7) [1.5%]
5.5	0.0382(4)(3)	0.0486(4)(5) [1.3%]

note : another possibility $(r_0 f_K)^{\rm latt,phys}$ $\rightsquigarrow$ $r_0^{\rm phys}$ $\rightsquigarrow$ a by using $[r_0/a](\beta)$

scale setting with f_K $\rightsquigarrow$ m_s

strange quark mass :



$F_K = af_K$ [ALPHA, 1205.5380]

RGI mass : $M_s = 138(3)(1)$ MeV

$\overline{\text{MS}}$ mass : $m_s[\overline{\text{MS}}, 2\text{GeV}] = 102(3)(1)$ MeV

scale setting : baryon masses

$$M_\Omega,\ M_\Xi,\ M_N$$

scale setting with M_Ω , M_Ξ , M_N

- M_Ω , M_Ξ , M_N baryons :
  - do not decay in QCD
  - signal/noise ratio : exponential decrease with t
  - excited state contamination
  - octet/decuplet : variance
  - $am_q \ll 1$
- light-quark mass dependence :
  - convergence of baryon χPT
  - $M_\Omega(sss)$ milder dependence than M_N
- M_Ω : requires m_s
- well known experimentally

$N_f = 2+1$ BMW ensembles

- Wilson fermions with tree-level c_{SW}
  6-stout [BMW, 0906.3599]
  2-HEX [BMW, 1011.2711]
- tlSym gauge action

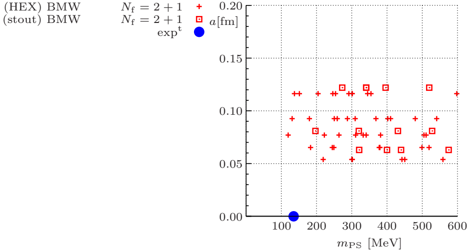
6-stout

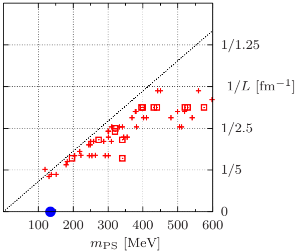
- $a = \{0.065, 0, 085, 0.125\}$ fm
- $M_\pi \in \{190, \ 580\}$ MeV

- HMC + RHMC
- 1000 ÷ 10000 thermalised traj.
- scale setting :

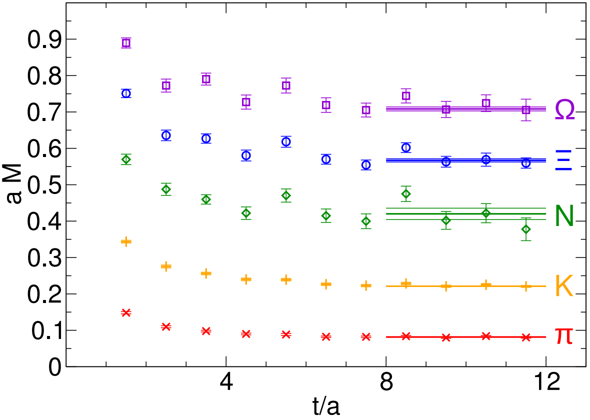
$$a = \frac{(\,aM_\Omega[\beta]\,)^{\text{phys,latt}}}{M_\Omega^{\text{exp}}}$$

- M_Ξ $(S=-2, \text{octet})$ is also used





effective masses

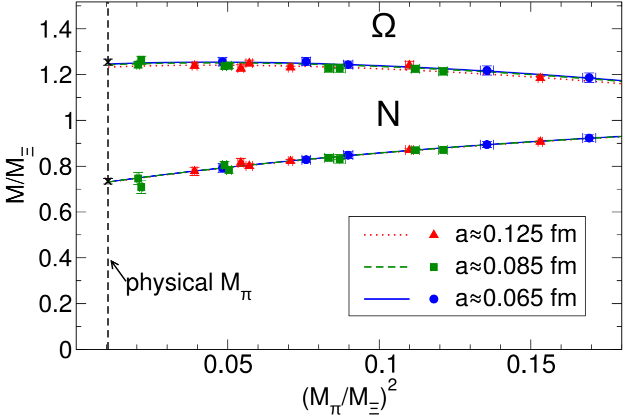


6-stout, $M_\pi \approx 190\,\text{MeV}$, $a \approx 0.085\,\text{fm}$

[BMW, 0906.3599]

ratio plot

light-quark mass dependence and cutoff effects



6-stout, $(2M_K^2 - M_\pi^2)^{\mathrm{phys}}$ [BMW, 0906.3599]

analysis in terms in ratios

scale setting : M_Ω and M_Ξ

X	Exp.	M_X [Ξ]	M_X [Ω]
Ω	1.672	1.676(20)(15)	1.672
Ξ	1.318	1.318	1.317(16)(13)
ρ	0.775	0.775(29)(13)	0.778(30)(33)
N	0.939	0.936(25)(22)	0.953(29)(19)
Δ	1.232	1.248(97)(61)	1.234(82)(81)

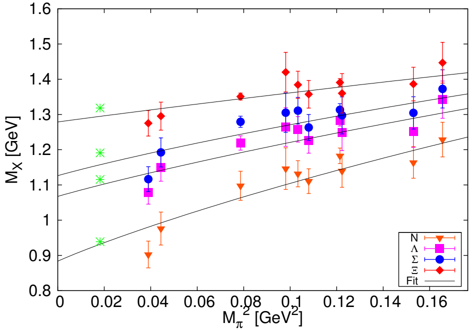
6-stout [BMW, 0906.3599]

σ-terms

- scale from M_Ω
- Feynman-Hellman theorem

$$\sigma_{\pi N} \approx M_\pi^2 \frac{\partial M_N}{\partial M_\pi^2}$$

$$\sigma_{\pi N} = 39(4) \binom{+18}{-7} \ \mathrm{MeV} \quad [35\%]$$



[BMW, 1109.4265]

m_{ud} : systematics

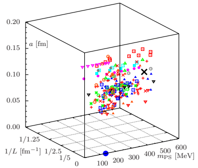
cut	m_{ud}	σ_{stat}	σ_{syst}	plateau	scale	fit form	mass cut	renorm.	cont.
120 MeV	3.503	0.048	0.049	0.330	0.034	0.030	0.157	0.080	0.926
200 MeV	3.523	0.057	0.063	0.354	0.078	0.470	0.236	0.087	0.765
240 MeV	3.484	0.079	0.131	0.316	0.092	0.807	0.341	0.046	0.349

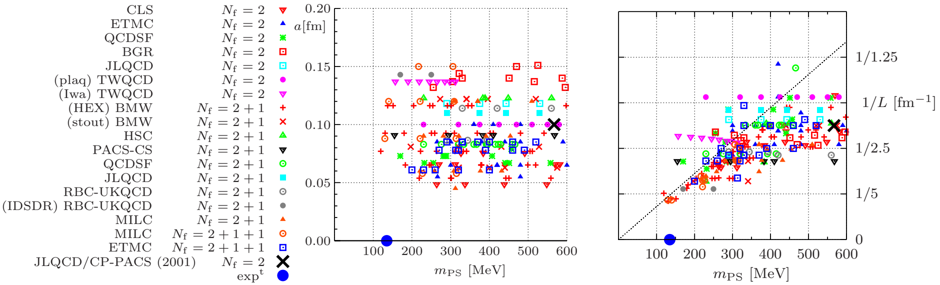
2-HEX : scale from M_Ω [BMW, 1011.2711]

masses in MeV, RI/MOM at $\mu = 4\,\mathrm{GeV}$ source : [H. Wittig, lat11, 1201.4774]

dynamical simulations : parameter landscape

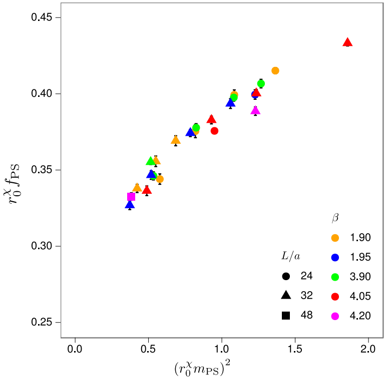
- number of flavours : N_f
- lattice spacing : a
- lattice size : L
- pion masses : M_{PS}

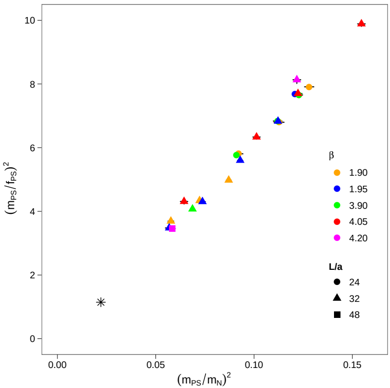




caveat in plots : no information on systematic effects (cut-off effects, FSE, scale setting, ...), m_s, m_c, ...

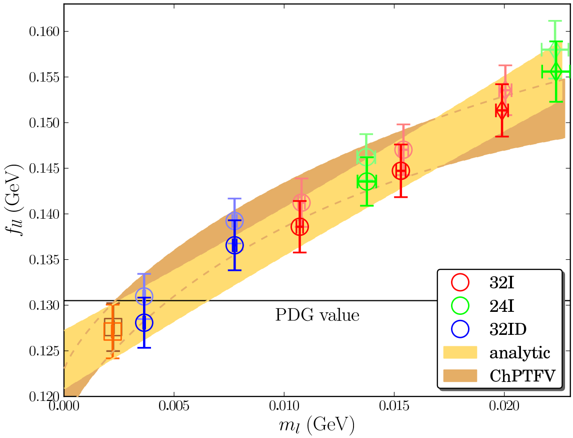
ratio plots





$N_f = 2 + 1 + 1$ and comparison to $N_f = 2$ $a = 0.08,\ 0.09$ fm [ETMC, 1004.5284]

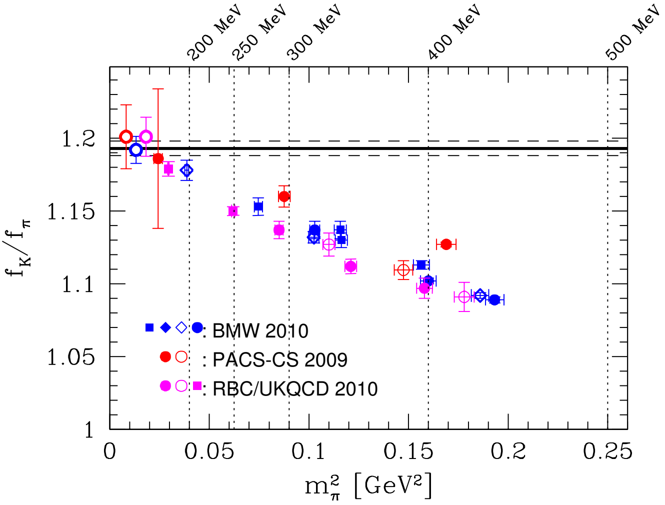
f_π



scale from M_Ω

[RBC-UKQCD, 1201.0706]

f_K / f_π

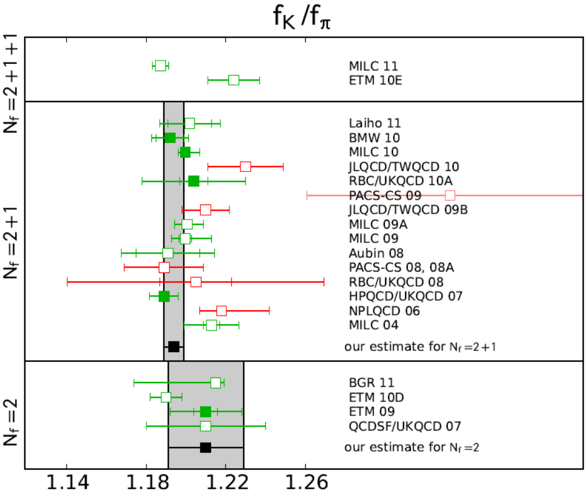


light-quark mass dependence

[H. Wittig, lat11, 1201.4774]

SU(3) breaking

f_K / f_π



[FLAG, lat12]

UKQCD-QCDSF: singlet quark mass $N_f = 2 + 1$

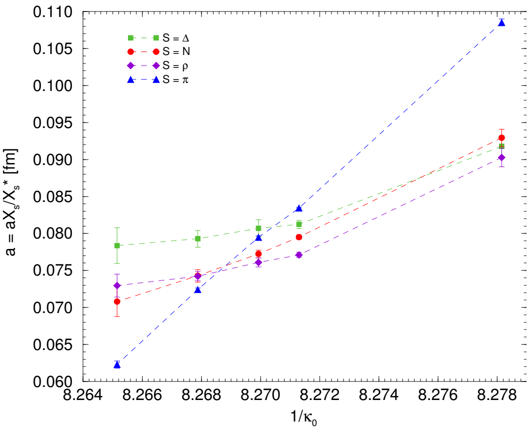
$$\overline{m}^R = \frac{1}{3}(m_u^R + m_d^R + m_s^R)$$

$$X_\pi^2 = \frac{1}{3}(2M_K^2 + M_\pi^2)$$

$$X_\rho = \frac{1}{3}(2M_{K^*} + M_\rho)$$

$$X_N = \frac{1}{3}(M_N + M_\Sigma + M_\Xi)$$

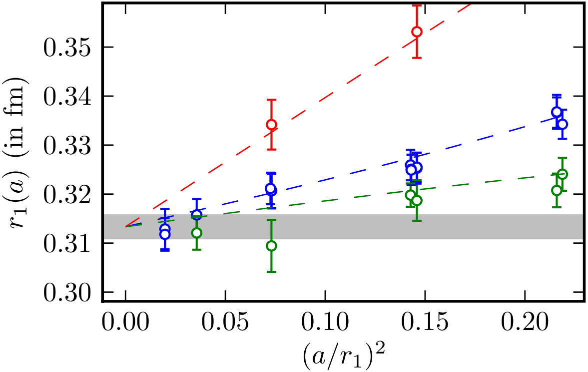
$$X_\Delta = \frac{1}{3}(2M_\Delta + M_\Omega)$$



$\beta = 5.5, \quad L/a = 32$

[UKQCD-QCDSF, 1102.5300]

r_1 : HPQCD $N_f = 2 + 1$



[HPQCD, 0910.1229]

red (top) symbols: $M_{D_s} - M_{\eta_C}/2$

blue (middle) symbols: f_{η_s}

green (bottom) symbols: $M_{\Upsilon'} - M_\Upsilon$

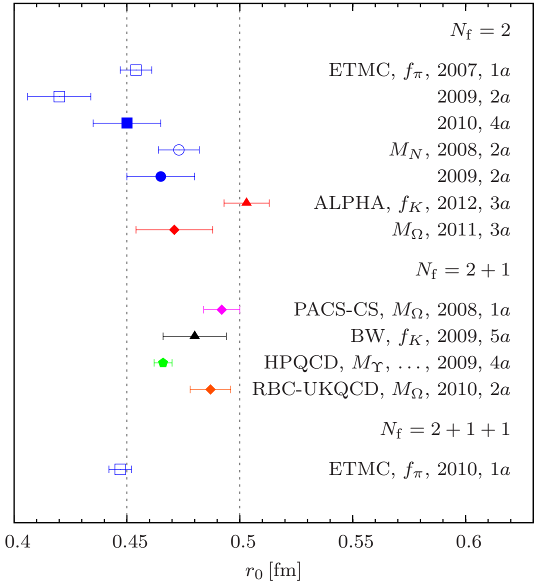
$r_1^{\rm phys} = 0.313(2)\,\mathrm{fm}$ [0.7%]

r_0 : comparison

group	N_f	r_0 [fm]	scale	# a	ref.
ETMC	2	0.454(07)	f_π	1	hep-lat/0701012
		0.420(14)		2	0911.5061
		0.450(15)		4	1010.3659
		0.473(09)	M_N	2	0803.3190
		0.465(15)		2	0910.2419
ALPHA		0.503(10)	f_K	3	1205.5380
		0.471(17)	M_Ω	3	1110.6365
PACS-CS	2+1	$0.492(^{10}_{6})$	M_Ω	1	0807.1661
Budapest-Wuppertal		0.480(14)	f_K	5	0903.4155
HPQCD		0.466(04)	$M_{D_s} - M_{\eta_c}/2,\ f_{\eta_s},\ M_{\Upsilon'} - M_\Upsilon$	4	0910.1229
RBC-UKQCD		0.487(09)	M_Ω	2	1011.0892
ETMC	2+1+1	0.447(05)	f_π	1	1004.5284

$r_0 \in [0.45\,,\ 0.50]\,\mathrm{fm} \ \rightsquigarrow\ \sim 10\%$ rel. variation

r_0 : comparison



r_1 : comparison

group	N_f	r_1 [fm]	scale	# a	ref.
MILC	2+1	0.311(08)	f_π	5	0903.3598
HPQCD		0.321(05)	$M_{\Upsilon'} - M_\Upsilon$	3	0706.1726
		0.313(02)	$M_{D_s} - M_{\eta_c}/2,\ f_{\eta_s},\ M_{\Upsilon'} - M_\Upsilon$	4	0910.1229
Fermilab-MILC		0.312(02)	f_π + av.	-	1112.3051
RBC-UKQCD		0.333(09)	M_Ω	2	1011.0892
HPQCD	2+1+1	0.321(03)	$f_{\eta_s},\ M_{\Upsilon'} - M_\Upsilon$	3	1110.6887

$r_1 \in [\,0.31\,,\ 0.33\,]$ fm $\rightsquigarrow$ $\sim$ 7% rel. variation

conclusions

scale setting

- can introduce large uncertainties in all dimensionful observables
- careful choice of the observable and of the procedure to reach the physical point
- cross-checks : vary ρ and S in different sectors
- worth the effort $\rightsquigarrow$ check of universality