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Lattice Practice 2012
October 10^{th}–12^{th}

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Lattice Practices

Solvers II – Preconditioning and Deflation

October 12^{th} 2012 – DESY Zeuthen

The scope of this excercise is to explore and play around with some options of preconditioning. Again demos for each Task can be found in the `octave` folder for this exercise. The questions given on this sheet are meant to be discussed with your fellow lattice practicioners while inspecting the demo.

Task 1 *Preconditioned Conjugate Gradients*

1. The first and most simple preconditioning one can use is diagonal scaling (a.k.a. Jacobi) preconditioning. In here

$$S = D^{-1}, \quad D = \mathrm{diag}(A).$$

   - Why does this preconditioning idea fail miserably (It does not help at all!)? (*Hint:* Inspect the diagonal of A.)
   - * Try modifying the matrix by first randomly scaling the matrix A:

$$A = D_z A D_z, \quad D_z = \mathrm{diag}(z_1, \ldots, z_n).$$

     Compare the results with the ones obtained for the original matrix.

2. Preconditioning by *SSOR* (symmetric successive over-relaxation) reduces the condition number significantly. You can modify the over-relaxation parameter $\omega \in [0, 2)$ and look at the impact on preconditioning efficiency.

3. Compare the spectrum of the SSOR preconditioned matrix with the one you obtained in task 1 of the first exercise.

4. One of the black-box preconditioners available are the thresholded ILU methods.

- Take a look at the fill-in created by ILU, comparing the number of non-zeros in A and $\tilde{L}, \tilde{U}$. Play around with the threshold and see what happens.

5. The finite difference discretization of the Poisson operator used in this exercise has next-neighbor structure on a 2-dimensional grid. Hence we can use the odd-even preconditioner. In order to make it a preconditioner rather than another way of solving the linear system of equations, we solve
$$\hat{A}_{\text{ee}} x_e = \tilde{b}_e$$
by again applying the CG method to a certain accuracy ρ.
   - Play around with the accuracy of the inner CG method. What happens?
   - Is the odd-even preconditioner still a stationary pre-conditioner using CG to solve $\hat{A}_{\text{ee}} x_e = \tilde{b}_e$?

6. **Bonus***: Consider the situation, where the spectrum of A (hermitian positive definite) has the following structure. All the eigenvalues but one of A are contained in an interval $[a, b]$, the remaining eigenvalue is located at $c \gg b$ (or $0 < c \ll a$). Hence the condition number κ is given by
$$\kappa = \frac{c}{a} \quad (\text{or } \kappa = \frac{b}{c}).$$
Why do expect the CG method to converge much faster than predicted by the convergence theory? (*Hint:* Think about the interpretation of CG as approximating A^{-1} on the spectrum of A by a polynomial!)
   - Can you come up with a simple linear system of $Ax = b$ to test the situation? (*Hint:* Prescribe the eigenvalues!)
   - Especially if using diverging preconitioners situations like the one described can occur, why? Assume that the preconditioner only diverges on a small subspace of eigenmodes.

Task 2 *Preconditioned GMRES*
In order to show properties of the GMRES iteration we consider an example from Lattice QCD. The system matrix A is given by the Wilson discretization of the Dirac equation on a 4^4 lattice at $\beta = 6$ with an additive mass shift. The system matrix is non-hermitian with its eigenvalues in the right half-plane.

1. The first preconditioner to try for this problem is a domain decomposition approach with 2^4 blocks (including all 12 variables on each lattice site).

2. Again we can apply the ILU preconditioner with a set threshold. Change the threshold and observe how the fill-in and the preformance of the preconditioner changes.

3. The last preconditioner to be used is the odd-even preconditioner. In here we solve $A_{\mathrm{ee}} x_e = \tilde{b}_e$ by BiCGstab with a fixed accuracy. The method does not converge for an accuracy of 10^{-1}. What happened? What do we have to do to solve the problem?

Task 3 *Preonditioned BiCGstab* To conclude we demonstrate the typical convergence behaviour of preconditioned BiCGstab, applied again to the 4^4 Lattice Dirac Wilson operator. We use the same preconditioner as for the GMRES method. Again, play around with the thresholds and settings and observe the effectivity of the preconditioners.