A bag of tricks for computing the hadronic vacuum polarisation in lattice QCD

Lattice Practices 2012

Desy Zeuthen

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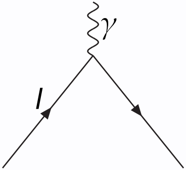
- there is a very recent surge of interest in computing it in lattice QCD
- brute force doesn't help you - one needs to develop new field theoretical techniques and state-of-the art lattice-simulations in order to have an impact $\rightarrow$ many very recent new ideas in this lecture

Lepton anomalous magnetic moment

- leptons ($l = e, \mu, \tau$) exhibit a magnetic dipole moment



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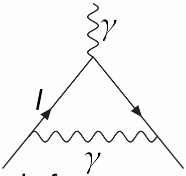

$$\vec{\mu}_l = g_l Q \frac{\vec{\sigma}}{2}$$

- Dirac's (classical) prediction for the gyromagnetic factor: $g_l = 2$.
- quantum corrections generate deviations from this value $\rightarrow$ anomalous magnetic lepton moment

$$a_l = \frac{g_l - 2}{2}$$

Outline

- setting the scene
- leading hadronic contribution
  - status
  - some preparatory calculus
  - problems of current lattice simulations
  - how to solve the problems
- light-by-light

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the LHC proton-source

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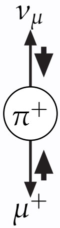
- produce muons: pions from $p \to$ target and then decay $\pi^+ \to \mu^+ + \nu_\mu$ muons are polarized by their production process:
- π^+ is spin-0 and ν_μ is left-handed, μ^+ must be left-handed by angular momentum conservation



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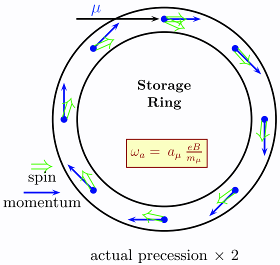
- π^+ is spin-0 and ν_μ is left-handed, μ^+ must be left-handed by angular momentum conservation
- inject μ^+ into storage ring
- the μ^+ decays after a few rounds in the ring and the resulting e^+ keeps the helicity



Jegerlehner, Nyffeler, PR 477 (2009)

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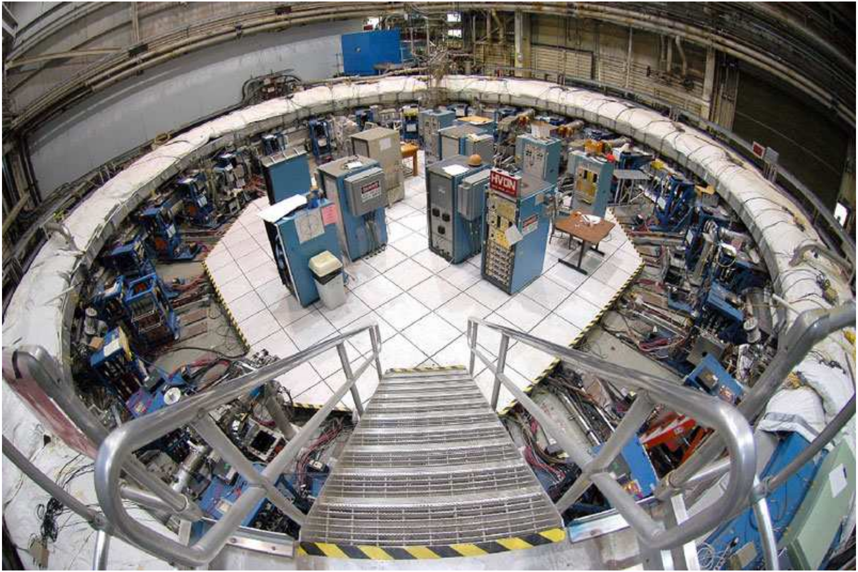

The BNL $g-2$ experiment (illustration: Fermilab Today)

history of experiments

exp.	year	result
CERN I	1961	11 450 000(220000)
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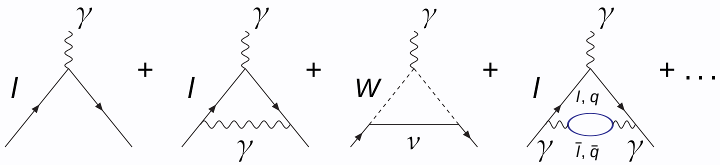
experiment is going to improve: new muon $g-2$ experiment at FNAL aiming at a 4-fold improvement over current result
http://gm2.fnal.gov

if all goes well first data-taking in 2016?

a_l in theory

$a_l \propto$



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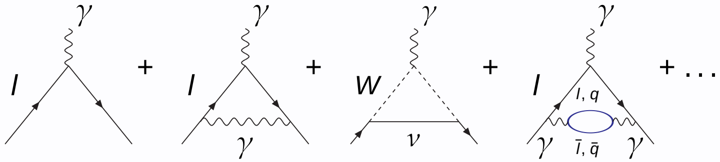

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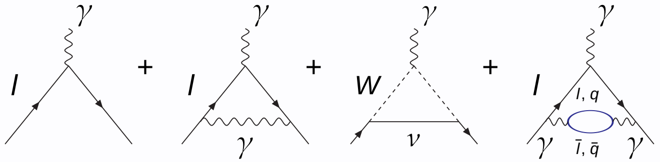
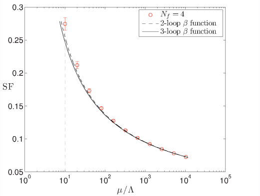
$a_l \propto$ 

- QED and weak: perturbation theory OK
- the QCD coupling constant becomes too large for a reliable perturbative expansion at low energies



Lattice computation of the N_f=4 Schrödinger Functional coupling by ALPHA arXiv:1011.2332

a_l status

- muon: non-perturbative effects relevant

$$a_\mu^{\rm exp} = 1.16592080(63)\times 10^{-3}$$

$$a_\mu^{\rm SM} = 1.16591753(53)\times 10^{-3}$$

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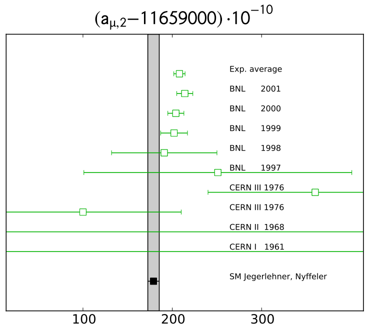
Benayoun, Eur. Phys. J. C (2012) 72:1848



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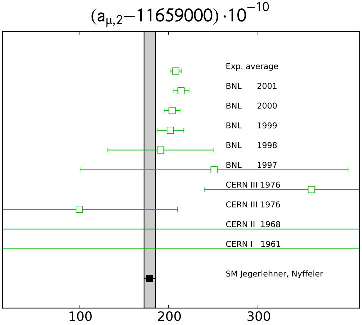


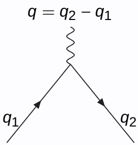
- just for completeness:
  electron: non-perturbative effects negligible

$$a_e^{\mathrm{exp}} = 1.15965218073(28) \times 10^{-3}$$ *Fogwell, Gabrielse, PRL 100 (2008)*

$$a_e^{\mathrm{SM}} = 1.15965217299(930) \times 10^{-3}$$ *Jegerlehner, Nyffeler, PR 477 (2009)*

one of the most precise tests of quantum field theory/SM

The leading hadronic contribution



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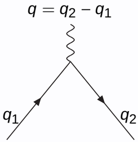

$$\bar{u}_2 M_\mu u_1 = \bar{u}_2 \left[F_1^{(1)}(q^2)\gamma_\mu - F_2^{(1)}(q^2)\frac{i\sigma^{\mu\nu}q_\nu}{2m} \right] u_1$$

- $q = (p_2 - p_1)$
- $F_1(q^2)$ charge form factor (so $F_1(0) = 1$)
- $F_2(q^2)$ magnetic form factor

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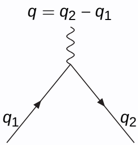

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after a couple of steps

Levine, Roskies, Remiddi, Adv.Ser.Direct.High Energy Phys. 7 (1990) 162-217

(non-relativistic approximation for small momenta)

$\vec{\mu} = g\left(\frac{e}{2m}\right)\vec{S}$ where:

- Landé/gyromagnetic factor $g = 2 + 2F_2(0)$
- anomalous magnetic moment $a \equiv \frac{1}{2}(g-2) = F_2(0)$

New physics?

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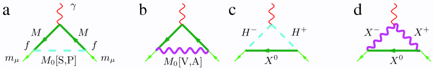
- sensitivity?
  - typical scale for loop-induced new physics effects is M_{NP}
  - $\delta a_\mu \propto \frac{m_\mu^2}{M^2}$

    $(m_e = 0.5\text{MeV}, m_\mu = 106\text{MeV}, m_\tau = 1777\text{MeV})$

New physics?

- new heavy states



- extra dimensions
- super symmetry
- ...

What's limiting the precision in the theory prediction?

	$a_\mu/10^{-11}$	
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QED incl 4-loops+LO 5-loops	116584718.1	0.2
weak 2-loop	153.2	1.8
lead. had. VP (experimentally e^+e^-, τ)	6877.2	46.3
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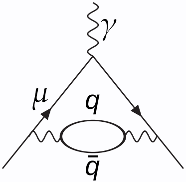

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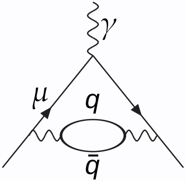
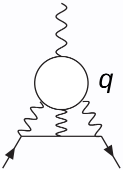
- lead. had. VP light-by-light scattering





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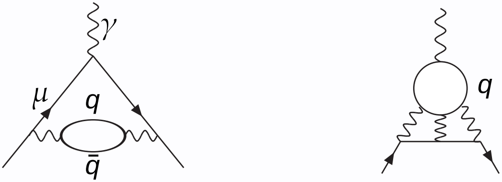
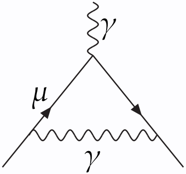


- hadronic uncertainties dominate the overall uncertainty of the SM prediction.

The leading contribution: back to the basics



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- how the NLO contribution in QED (Schwinger) is computed

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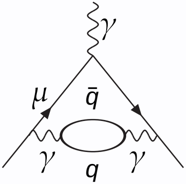

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- how the current prediction is being done (non-lattice)
- how to predict it in lattice QCD

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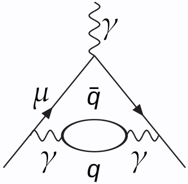

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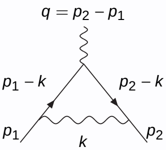
- how to predict it in lattice QCD
- this involves some derivations. . .

  Levine, Roskies, Remiddi, Adv.Ser.Direct.High Energy Phys. 7 (1990) 162-217
  Peskin & Schroeder, . . .

1-loop in QED



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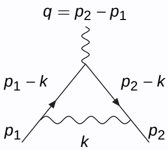

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$$= ie^2 \int \frac{d^4k}{(2\pi)^4}\gamma_\nu \frac{-i\not{p}_2+i\not{k}+m}{(p_2-k)^2+m^2-i\epsilon}\gamma_\mu \frac{-i\not{p}_1+i\not{k}+m}{(p_1-k)^2+m^2-i\epsilon}\gamma_\nu \frac{1}{k^2-i\epsilon}$$

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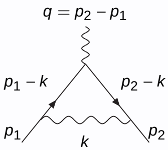

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- use projector to isolate F_2
  $\left(N^\mu = \left(-i\not{p}_1 + m\right)\left[g_1\gamma_\mu + \frac{i}{m}g_2 q_\mu\right](-i\not{p}_2 + m)\right)$

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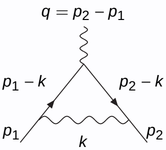

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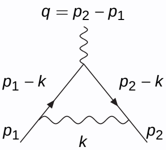
- after a bit of algebra:
  $$\left(\frac{\alpha}{\pi}\right)a_\mu^{(2)} = ie^2 \int \frac{d^4k}{(2\pi)^4}\frac{1}{((p-k)^2+m^2-i\epsilon)^2}\frac{4}{k^2}\left(\frac{4(p\cdot k)^2}{3m^2} + \frac{1}{3}k^2 + (p\cdot k)\right)$$
  evaluated on mass-shell

1-loop in QED



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  evaluated on mass-shell
- IR finite
- UV divergencies as well absent

1-loop in QED

- from previous slide:

$$\left(\frac{\alpha}{\pi}\right) a_\mu^{(2)} = ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{((p-k)^2+m^2-i\epsilon)^2} \frac{4}{k^2} \left(\frac{4(p\cdot k)^2}{3m^2} + \frac{1}{3}k^2 + (p\cdot k) \right)$$

- analytical continuation to Euclidean region with space-like loop and external momenta:

$$\left(\frac{\alpha}{\pi}\right) a_\mu^{(2)} = -e^2 \int \frac{d^4K}{(2\pi)^4} \frac{1}{((P-K)^2+m^2)^2} \frac{1}{K^2} \left(\frac{16(P\cdot K)^2}{3m^2} + \frac{4}{3}K^2 + 4(P\cdot K) \right)$$

1-loop in QED

- how to get rid of angular integrals?
  (hyperspherical formalism)

$$\frac{1}{((K-P)^2+m^2)^2} = \frac{Z_{KP}^2}{K^2P^2}\frac{1}{1-Z_{KP}^2}\sum_{n=0}^{\infty}(n+1)Z_{KP}^n C_n(\hat{K}\cdot\hat{P})$$

where

$$Z_{KP} \equiv \frac{K^2+P^2+m^2-\sqrt{(K^2+P^2+m^2)^2+4K^2P^2}}{2KP}$$

and $C_n(x)$ Gegenbauer polynomials with

$$\int C_n(\hat{a}\cdot\hat{b})C_m(\hat{b}\cdot\hat{c})\frac{d\Omega_b}{2\pi^2} = \frac{\delta_{m,n}}{n+1}C_n(\hat{a}\cdot\hat{c})$$

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$$Z_{KP} \equiv \frac{K^2+P^2+m^2-\sqrt{(K^2+P^2+m^2)^2+4K^2P^2}}{2KP}$$

and $C_n(x)$ Gegenbauer polynomials with

$$\int C_n(\hat{a}\cdot\hat{b})C_m(\hat{b}\cdot\hat{c})\frac{d\Omega_b}{2\pi^2} = \frac{\delta_{m,n}}{n+1}C_n(\hat{a}\cdot\hat{c})$$

- numerator:

$$P\cdot K+\tfrac{1}{3}K^2+\tfrac{4}{3}(P\cdot K)^2 = \tfrac{1}{2}PKC_1(\hat{P}\cdot\hat{K})+\tfrac{1}{3}K^2+\tfrac{1}{3}P^2K^2C_2(\hat{P}\cdot\hat{K})+\tfrac{1}{3}P^2K^2$$

1-loop in QED

$$\frac{\alpha}{\pi}a_\mu^{(2)} = -\frac{4e^2}{(2\pi)^4}\int d^4K\left\{\frac{\frac{1}{2}PKC_1(\hat{P}\cdot\hat{K})+\frac{1}{3}K^2+\frac{1}{3}P^2K^2C_2(\hat{P}\cdot\hat{K})+\frac{1}{3}P^2K^2}{K^2}\right.$$
$$\left.\frac{Z_{PK}^2}{K^2P^2}\frac{1}{1-Z_{PK}^2}\sum_{n=0}^{\infty}(n+1)Z_{PK}^nC_n(\hat{P}\cdot\hat{K})\right\}$$

orthogonality relation for Gegenbauers and mass-shell for muons:

$$\frac{\alpha}{\pi}a_\mu^{(2)} = \frac{\alpha}{\pi}\int_0^\infty dK^2\frac{m^2K^2\tilde{Z}_{PK}^3}{1+m^2K^2\tilde{Z}_{PK}}(1-K^2\tilde{Z}_{PK}^2) \quad \text{where } \tilde{Z}_{PK} = \frac{Z_{PK}}{PK}$$
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$$\left. \frac{Z_{PK}^2}{K^2P^2} \frac{1}{1-Z_{PK}^2} \sum_{n=0}^{\infty} (n+1) Z_{PK}^n C_n(\hat{P}\cdot\hat{K}) \right\}$$

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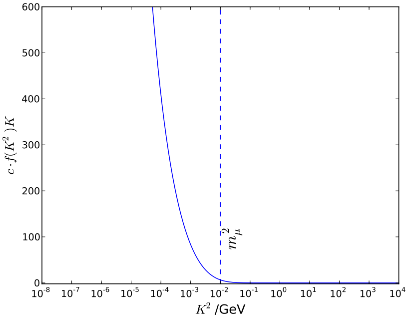
$$\equiv \frac{\alpha}{\pi} \int_0^\infty dK^2 f(K^2)$$

- integral over Euclidean momenta K^2
- can be solved exactly: $\frac{\alpha}{\pi} a_\mu^{(2)} = \frac{\alpha}{2\pi} \approx 1.1617 \cdot 10^{-3}$

(cf. full SM prediction as to date $1.16591753(53) \cdot 10^{-3}$)

The function $f(K^2)$

$f(K^2) \propto \frac{1}{\sqrt{K^2}}$



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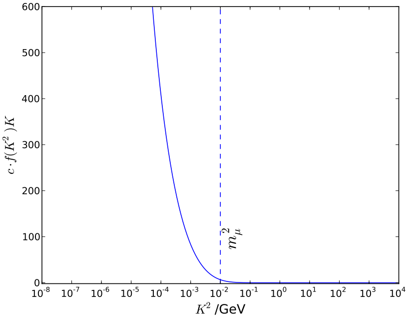
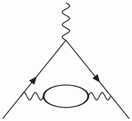


Let's now see how to implement the leading QCD contribution

The leading hadronic contribution



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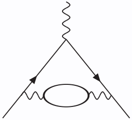

quark loop does not affect rest of integral and depends on K^2 only, simply insert it into $K^2 = q^2$-integration

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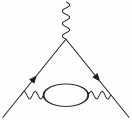

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vacuum polarisation tensor

$$\mu \quad \nu = \Pi_{\mu\nu}(q) = (\delta_{\mu\nu}q^2 - q_\mu q_\nu)\Pi(q^2)$$
$$= \int d^4x e^{iq(x-y)} \langle j_\mu^{\mathrm{elm.}}(y) j_\nu^{\mathrm{elm.}}(x)\rangle_{\mathrm{QCD}}$$

with $j_\mu^{\mathrm{elm.}} = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d + \frac{2}{3}\bar{s}\gamma_\mu s + \cdots$

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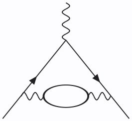

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vacuum polarisation tensor

μ  ν

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combine with QED:

$$\left(\frac{\alpha}{\pi}\right) a_\mu^{LH} = \left(\frac{\alpha}{\pi}\right)^2 \int dq^2 f(q^2)\hat{\Pi}(q^2)$$

$$\hat{\Pi}(q^2) = 4\pi^2 \sum_i Q_i^2 [\Pi_i(q^2) - \Pi_i(0)]$$

Current SM-prediction from experiment

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- analyticity: $\Pi(q^2) = \Pi(0) + \frac{q^2}{\pi}\Phi(q^2)$ where

$$\Phi(q^2) = \int_0^\infty \frac{ds\,\mathrm{Im}\Pi(s)}{s(s-q^2-i\epsilon)} \qquad s = \sqrt{E}$$

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$$\mathrm{Im}\Pi(s) = \frac{s}{4\pi\alpha(s)}\sigma_{\mathrm{tot}}(e^+e^- \to \mathrm{hadrons})$$

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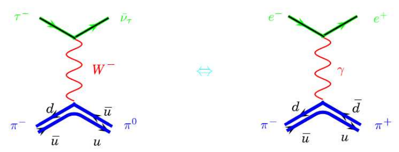
- current *SM-prediction* for vacuum polarisation is actually an experimental measurement of $e^+e^- \to$ hadrons and $\tau^\pm \to \pi^\pm\pi^0\nu$-decays



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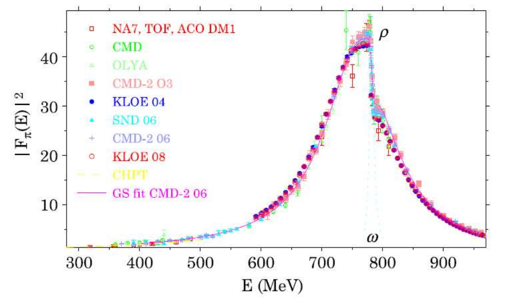

And now in lattice QCD...

- a run through the Mainz group's simulation as an example
- discussion of issues arising, systematic effects
- a bag full of tricks for how to remove or control systematic effects

Definition of the vector two-point function

conserved Vector current *Bochicchio et al. Nucl. Phys. B262 (1985) 331-355*

$$V_\mu^f(x) = \frac{1}{2}\Big(\bar{\psi}_f(x+a\hat{\mu})(1+\gamma_\mu)U_\mu^\dagger(x)\,\psi_f(x) - \bar{\psi}_f(x)(1-\gamma_\mu)U_\mu(x)\,\psi_f(x+a\hat{\mu})\Big)$$

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vector two-point function

$$\Pi_{\mu\nu}^{(N_\mathrm{f})}(x) = a^6\Big\langle \sum_{f=1}^{N_\mathrm{f}}(Q_f V_\mu^f(x)) \sum_{f'=1}^{N_\mathrm{f}}(Q_{f'} V_\nu^{f'}(0))\Big\rangle$$

$$\Pi_{\mu\nu}^{(N_\mathrm{f})}(\hat q) = \sum_x e^{iq(x+a\hat\mu/2-a\hat\nu/2)}\Pi_{\mu\nu}^{(N_\mathrm{f})}(x)$$

$q_\mu = \frac{2\pi}{L}n_\mu$ and $\hat q_\mu = \frac{2}{a}\sin\left(\frac{aq_\mu}{2}\right)$ with $n_\mu \in 0,1,\dots L/a-1$

Definition of the vector two-point function

Ward identity:

$$\hat{q}_\mu \Pi^{(N_\mathrm{f})}_{\mu\nu}(\hat{q}) = \Pi^{(N_\mathrm{f})}_{\mu\nu}(\hat{q})\,\hat{q}_\nu = 0$$

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$\rightarrow$ also for the lattice VP tensor:

$$\Pi_{\mu\nu}(\hat{q}) = (\delta_{\mu\nu}\hat{q}^2 - \hat{q}_\mu\hat{q}_\nu)\Pi(\hat{q}^2)$$

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UKQCD use a two-point function of conserved and local vector currents and redefine the WI, also OK and numerically cheaper

Boyle, Del Debbio, Kerrane, Zanotti arXiv:1107.1497

A run through the Mainz-group simulations and problems encountered along the way

Della Morte et al. JHEP 1203 (2012) 055

In principle computing $\langle j_\mu^{\mathrm{elm.}} j_\nu^{\mathrm{elm.}} \rangle$ in lattice QCD is a well-defined problem. . .

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  - m_π ((pion mass)$^2 \propto m_q$)
  - L (lattice-volume)
  - a (lattice spacing)
  - $\hat{q}$ (Euclidean momenta)

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- structure of VP tensor doesn't allow to compute $\Pi(0)$
- fit $\hat{q}^2$-dependence of $\Pi(0)$ and extrapolate to $\hat{q}^2 = 0$
- integrate fitted function

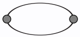
$$a_\mu^{\mathrm{LHV}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty d\hat{q}^2 f(\hat{q}^2)\Big(\Pi(\hat{q}^2) - \Pi(0)\Big)$$

Issues

contributions of quark-disconnected diagrams

$$\left\langle j_\mu^{qq}(y) j_\nu^{qq}(x) \right\rangle = \left\langle \bar{q}\gamma_\mu q(y)\, \bar{q}\gamma_\nu q(x) \right\rangle$$

$$= \left\langle \mathrm{Tr}\left\{ S_q(y,x)\gamma_\mu S_q(x,y)\gamma_\nu \right\} \right\rangle$$



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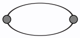

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disconnected - challenging and bad signal/noise, so far no detailed study available and currently neglected

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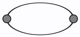

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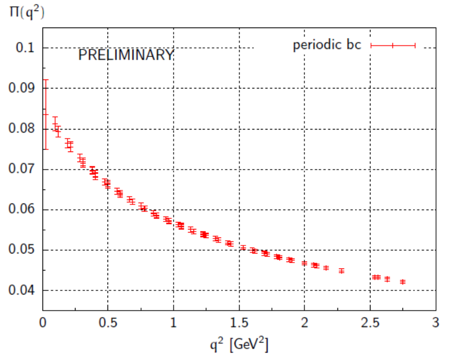
- ETM did exploratory numerical study and concluded that disc. contrib. is small *Feng et al. Phys.Rev.Lett. 107 (2011) 081802*
- rough estimate in chiral perturbation theroy: connected is -10% of the disconnected contribution, mildly varying with the momentum transfer *Della Morte, AJ JHEP 1011 (2010) 154*

Example: Mainz data *Della Morte, Jäger, AJ, Wittig, JHEP 1203 (2012) 055*

compute $\Pi(q^2) = \frac{1}{\delta^2_{\mu\nu} q^2 - q_\mu q_\nu} \Pi_{\mu\nu}(q^2)$ (no summation)

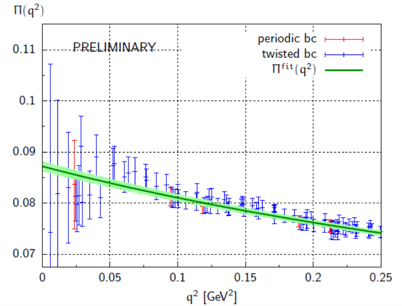


plots by Benni Jäger

- no result at $q^2 = 0$
- discrete momenta

Example: Mainz group *Della Morte et al., JHEP 1203 (2012) 055*

fit q^2-dependence and extrapolate



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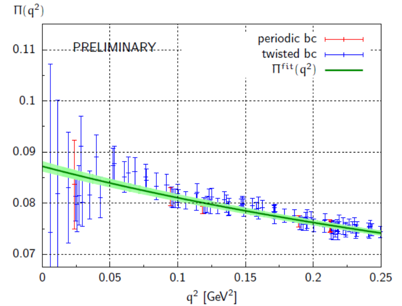

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issues:
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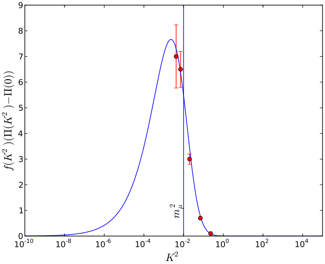

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Example: Mainz group *Della Morte et al., JHEP 1203 (2012) 055*

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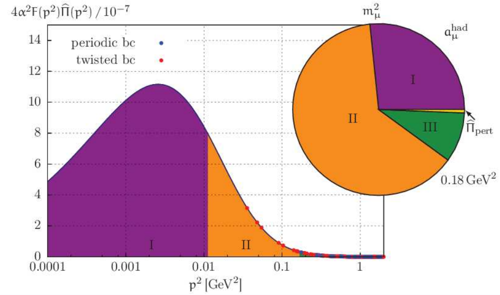

plots by Benni Jäger

issues:
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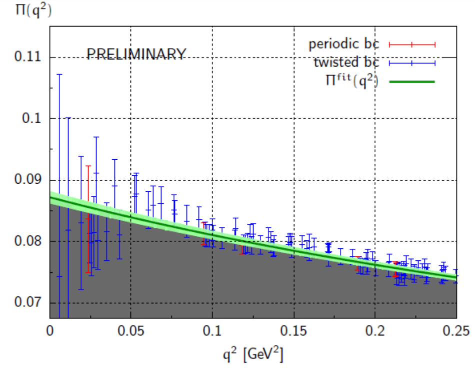
Example: Mainz group *Della Morte et al., JHEP 1203 (2012) 055*

integrate over phase space (and match to PT at large q^2)

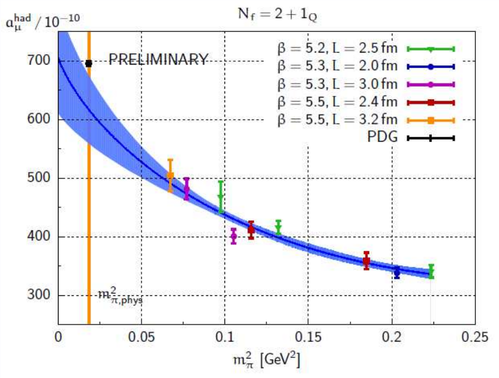
$$a_\mu^{\rm LHV} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dq^2 f(q^2)\left(\Pi(q^2) - \Pi(0)\right)$$



plots by Benni Jäger

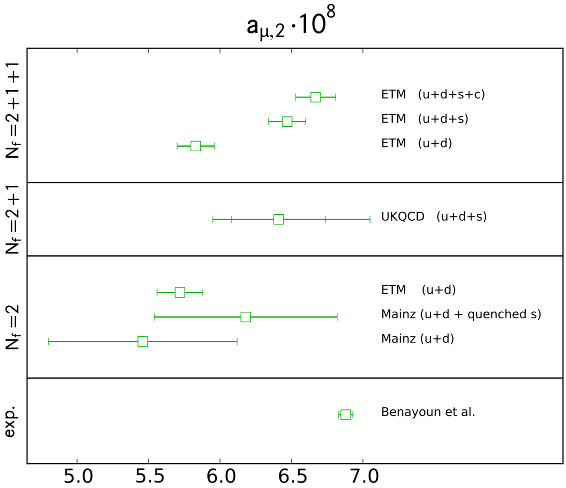
Example: Mainz group *Della Morte et al., JHEP 1203 (2012) 055*

repeat lattice simulation and analysis for several parameter choices a, L, m_q



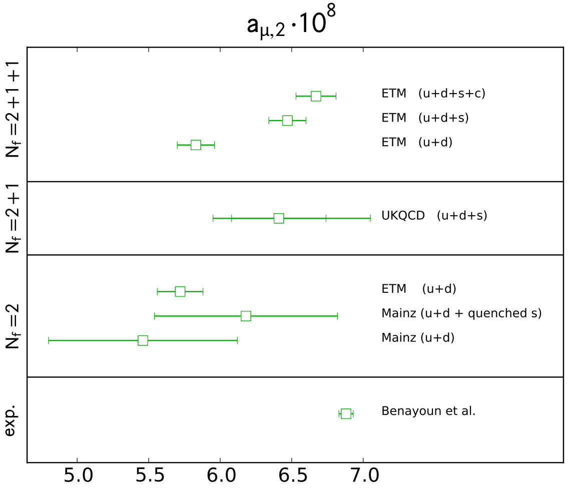
plots by Benni Jäger

Status



$a_{\mu,2} \cdot 10^8$
$N_f=2+1+1$
ETM (u+d+s+c)
ETM (u+d+s)
ETM (u+d)
$N_f=2+1$
UKQCD (u+d+s)
$N_f=2$
ETM (u+d)
Mainz (u+d + quenched s)
Mainz (u+d)
exp.
Benayoun et al.
5.0
5.5
6.0
6.5
7.0

Status



Looks like there is still a long way to go before we can do better than experiment. . .

A bag full of tricks:

low momenta	part. twisted bcs	*Della Morte, Jäger, AJ, Wittig, JHEP 1203 (2012) 055*
	result directly at $q^2 = 0$	*de Divitiis, Petronzio, Tantalo, arXiv:1208.5914*
q^2-dependence	Padé approximants	*Aubin, Blum, Golterman, Peris, Phys.Rev. D86 (2012) 054509*
m_q dependence	physical point or improved extrapols.	*phys. pt. not quite but close: $m_\pi = 166$MeV RBC/UKQCD Phys.Rev. D85 (2012) 074504, idea for new extrapolation ETM Feng et al. PRL 107, 081802 (2011)*
stat. error	all-mode-averaging	*Blum, Izubuchi, Shintani, arXiv:1208.4349*
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Partially twisted boundary conditions

periodic bc's

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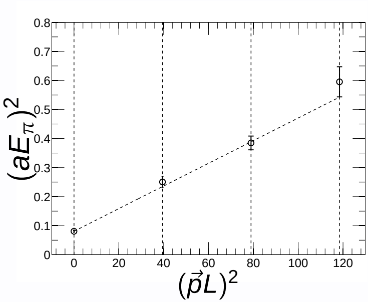

Partially twisted boundary conditions

periodic bc's | twisted bcs

P.F. Bedaque PLB539(2004)

$$\tilde{q}(x_i+L)=\tilde{q}(x_i) \qquad q(x_i+L)=e^{i\theta_i}q(x_i)$$

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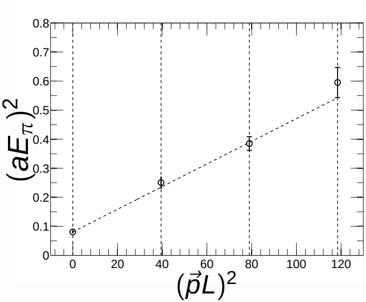

Twisting in the free theory

- periodic boundary conditions ($\tilde{q}(x_i + L) = \tilde{q}(x_i)$):

$$\mathcal{L} = \bar{\tilde{q}}(x)(\not{D} + M)\tilde{q}(x)$$

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$$S(x) = \langle q(x)\bar{q}(0)\rangle = \int \frac{dk_4}{2\pi}\frac{1}{L^3}\sum_{\vec{k}} \frac{e^{i(k+B)\cdot x}}{i(\not{k} + \not{B}) + M}$$

Twisted boundary conditions in the interaction theory?

Sachrajda, Villadoro, Phys.Lett. B609 (2005)

- effective theory in finite volume

  $\mathcal{L}^{(2)} = \frac{F^2}{4}\mathrm{Tr}\left\{\partial_\mu U^\dagger \partial^\mu U\right\} - \frac{BF^2}{2}\mathrm{Tr}\left\{MU^\dagger + UM^\dagger\right\}$

  with boundary conditions:

$$U(x+\hat{i}L) = V_i U(x) V_i{}^\dagger \qquad V_i = exp(i\theta_i^a T^a)$$

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- poles of charged effective d.o.f.'s gets shifted:

$$\tilde{D}_\mu \tilde{U} = \partial_\mu \tilde{U} + i[B_\mu, \tilde{U}] \to [B_\mu, \sigma^\pm] = \pm\frac{\theta_{q_1,i} - \theta_{q_2,i}}{L}\sigma^\pm$$

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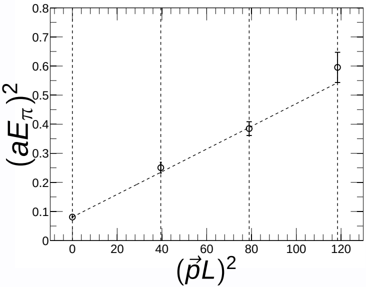

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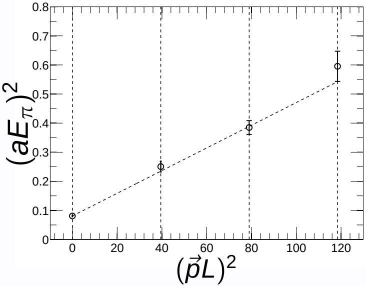
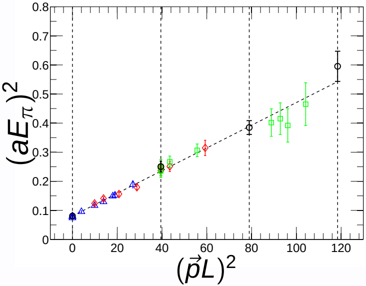




Flynn, AJ, Sachrajda JHEP 2005

Partially twisted boundary conditions in the effective theory

implementation: multiply gauge-links by $U(1)$-phase

$\bar{\psi}(x)(\not{D}+M)\psi(x) \rightarrow$

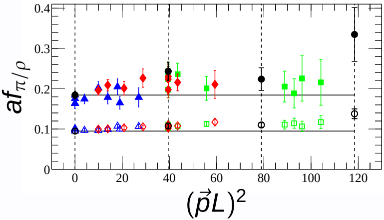
$$\bar{\tilde{\psi}}(x)\Big[e^{i\frac{a\theta_i}{L}}U_i(x)(1-\gamma_i)\tilde{\psi}(x+\hat{i})+e^{-i\frac{a\theta_i}{L}}U_i^\dagger(x-\hat{i})(1+\gamma_i)\tilde{\psi}(x-\hat{i})\Big]$$

one inversion for every choice of twist angle

Partially twisted boundary conditions in the effective theory

Sachrajda, Villadoro, Phys.Lett. B609 (2005)

- for matrix elements with maximally one hadron in the initial and/or final state twisting is effectively an exponentially supressed finite volume effect



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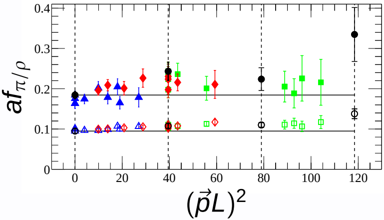

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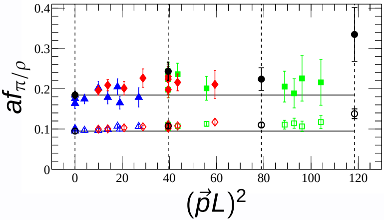

- remains valid for partially twisted boundary conditions
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- many applications in lattice QCD:
  - computation of hadron form factors (e.g. $K \to \pi l \nu$, $\pi \to \pi$)
  - meson, baryon structure functions
  - applications in Schrödinger functional
  - renormalisation scale in non-perturbative renormalisation
  - vacuum polarisation ...

How does this work for the VP?

- in 2pt functions of flavour-diagonal currents the twist cancels between quark anti-quark

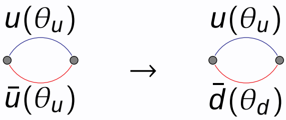
How does this work for the VP?

- in 2pt functions of flavour-diagonal currents the twist cancels between quark anti-quark
- trick:

$$\langle j_\mu^{uu} j_\nu^{uu}\rangle \stackrel{\text{iso-spin}}{=} \langle j_\mu^{ud} j_\nu^{du}\rangle + \langle j_\mu^{uu} j_\nu^{dd}\rangle$$

the decomposition into connected and disconnected part leads to a flavour off-diagonal 2pt-function at least for the connected part *Della Morte, AJ JHEP 1011 (2010) 154*

$\rightarrow$ one can induce momentum at least into connected part



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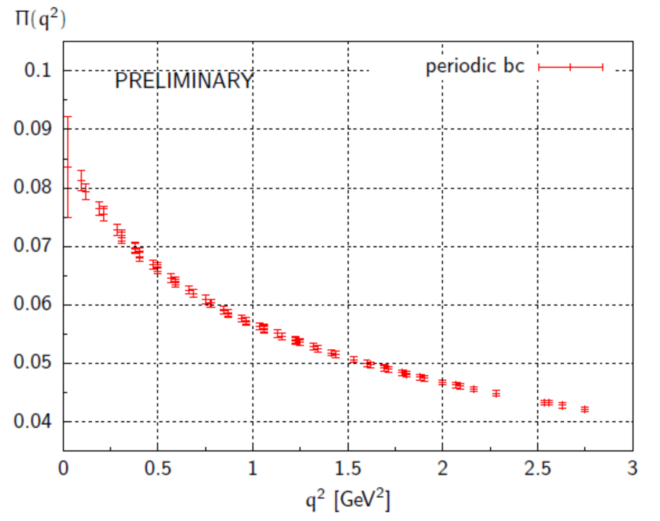

Mainz group

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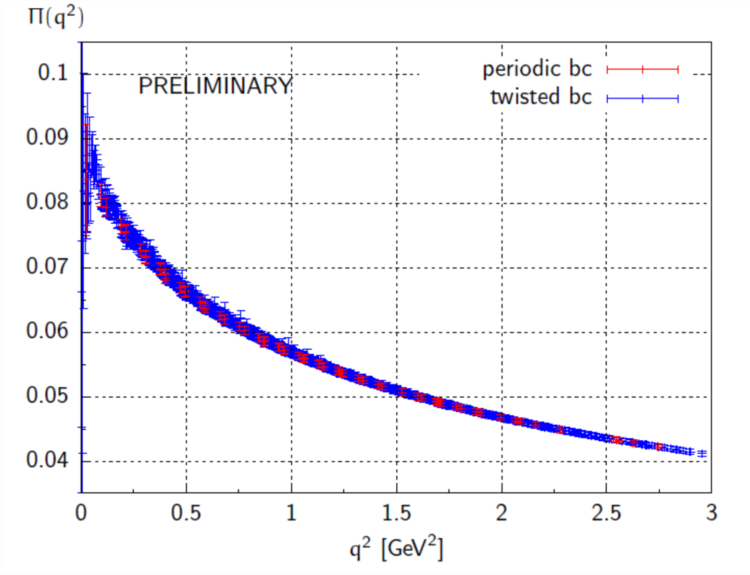

Mainz group

A bag full of tricks:

low momenta	part. twisted bcs	*Della Morte, Jäger, AJ, Wittig, JHEP 1203 (2012) 055*
	result directly at $q^2 = 0$	*de Divitiis, Petronzio, Tantalo, arXiv:1208.5914*
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Model-independent parameterisation of $\Pi(q^2)$

Aubin, Blum, Golterman, Peris, Phys.Rev. D86 (2012) 054509

- once-subtracted dispersion relation

$$\Pi(q^2) \quad = \quad \Pi(0) - q^2\Phi(q^2) \quad \text{where} \quad \Phi(q^2) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} dt\, \frac{\mathrm{Im}\Pi(t)}{t(t+q^2)}$$

- $\mathrm{Im}\Pi(t) = \rho(t) \geq 0$ for $t \geq 4m_\pi^2$
- $\Phi(q^2)$ is analytic for $t > -4m_\pi^2$

Model-independent parameterisation of $\Pi(K^2)$

can one find/motivate a model-independent param. of $\Pi(K^2)$?

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- consider a series of ordered kinematical points $0 < K_1^2 < K_2^2 < \cdots < K_P^2$
- start with approximation $\Phi(K^2) = \frac{\Phi(K_1^2)}{1+(K^2-K_1^2)\Psi_1(K^2)}$
  - $\Psi_1(K^2)$ is positive on $K^2 \in [-4m_\pi^2, \infty)$
  - upper bound to avoid singularity in range $K^2 > -4m_\pi^2$: $\Psi_1(K^2) \leq \Psi_1(-4m_\pi^2) \leq \frac{1}{4m_\pi^2+K_1^2}$

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- since $\Psi_1(K^2)$ same properties as Φ, recurse:

$$\Phi(K^2) = \cfrac{\Phi(K_1^2)}{1 + \cfrac{(K^2-K_1^2)\Psi_1(K_2^2)}{1+ \quad \ddots \quad \cfrac{(K^2-K_{P-1}^2)\Psi_{P-1}(K_P^2)}{1+(K^2-K_K^2)\Psi_P(K^2)}}}$$

where $\Psi_i(K_i^2)$ are the exact values of the function $\Phi(K_i^2)$

Model-independent parameterisation of $\Pi(K^2)$

- upper and lower bound for all coefficients
  $0 \leq \Psi_{i-1}(K^2) \leq \frac{1}{4m_\pi^2+K^2}\left(1 - \frac{\Psi_{i-1}(K_i^2)}{\Psi_{i-1}(-4m_\pi^2)}\right)$
- $[M,N]$ Padé approximation

$$R_M^N(K^2) = \frac{\sum\limits_{n=0}^{N} a_n K^{2n}}{\sum\limits_{n=0}^{M-1} b_n K^{2n} + K^{2M}}$$

results from saturating

- lower bound for $\Psi_P(Q^2) \rightarrow [(P-1)/2, P/2]$ Padé
- upper bound for $\Psi_P(Q^2) \rightarrow [P/2, (P+1)/2]$ Padé

Model-independent parameterisation of $\Pi(K^2)$

- in a practical application the Padé is approximated by

$$R_{P/2}^{(P-1)/2}(K^2) = a_0 + \sum_{n=1}^{P/2} \frac{a_n}{b_n + K^2}$$

with P even and

$$a_0 = 0, \qquad \text{if } P \text{ even} a_n > 0, \qquad n \in \{1, \cdots, P/2\}$$
$$b_{P/2} > b_{P/2-1} > \cdots > b_1 \leq 4m_\pi^2$$

- their suggested fit strategy:

$$\Pi(K^2) = \Pi(0) - K^2 \left(a_0 + \sum_{n=1}^{N} \frac{a_n}{b_n + K^2} \right)$$

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Aubin, Blum, Golterman, Peris, Phys.Rev. D86 (2012) 054509

- both types of Padé should be fit to the data in a sequence with increasing N
- both Padés provide upper and lower bound for Φ; convergence can thus be studied

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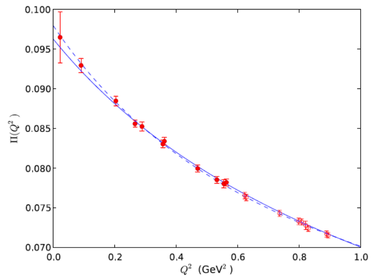
- both types of Padé should be fit to the data in a sequence with increasing N
- both Padés provide upper and lower bound for Φ; convergence can thus be studied
- poles are free parameters and there is NO a priori reason for identifying them with the mass of vector resonances
- Padés have previously been used in fits but the above work provides the argument that it allows for a model-independent description of the data



VMD solid vs. Padé with $N = 1$

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Direct computation of $\Pi(0)$

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$$\frac{\partial^2}{\partial q_\mu \partial q_\nu}\Pi_{\mu\nu}(q)|_{\mu\neq\nu,q^2=0} = -\frac{\partial^2}{\partial q_\mu \partial q_\nu}\left(q_\mu q_\nu \Pi(q^2)\right)|_{\mu\neq\nu,q^2=0} = -\Pi(0)$$

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How can this possibly work in practice?

Rome approach

- two-point function in terms of path-integral

$$
\begin{aligned}
C_2(q) &= \frac{1}{L^3}\sum_{\vec{x}} e^{-i\vec{x}\cdot\vec{q}}\frac{1}{\mathcal{Z}}\int D[U]P[U]C_2(\vec{x};U) \\
&= \frac{1}{L^3}\frac{1}{\mathcal{Z}}\int D[U]P[U]\tilde{C}_2(\vec{q};U)
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- remember quark propagator with twisted boundary conditions has inherent momentum dependence through choice of boundary condition B
- $\Pi_{12}(q) = \sum_{x,y}\mathrm{Tr}\left\{S[y,x;U,0]\Gamma_{V,1}(x,\frac{\vec{B}}{2})S[x,y;U,B]\Gamma_{V,2}(y,\frac{\vec{B}}{2})\right\}$
  - one quark carries momentum
  - currents being point-split/conserved contain phase-information

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- derivative with respect to external momentum involves product rules and relations like
  - $\frac{\partial S}{\partial p_k} = -S\frac{\partial D}{\partial p_k}S$
  - $\bar{\psi}(x)\frac{\partial D}{\partial p_k}\psi(x) = iV^k(x)$

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- $\Pi(0) = -\frac{\partial^2\Pi_{12}(q^2)}{\partial q_1 \partial_k 2}|_{K^2=0}$, i.e. 2nd derivative of the VP-tensor at vanishing K^2

$$\Pi(0) = \frac{1}{(TL^3)^2}\sum_{x,y}\langle \quad \mathrm{Tr}\left[S\Gamma_V^1 \frac{\partial^2 S}{\partial p_1 \partial p_2}\Gamma_V^2\right] - \frac{1}{4}\mathrm{Tr}\left[S\Gamma_T^1 S\Gamma_T^2\right]$$
$$-\frac{i}{2}\mathrm{Tr}\left[S\Gamma_T^1 \frac{\partial S}{\partial p_2}\Gamma_V^2\right] - \frac{i}{2}\mathrm{Tr}\left[S\Gamma_V^1 \frac{\partial S}{\partial p_1}\Gamma_T^2\right]\rangle$$

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- $\Pi_{12}(q) = \sum_{x,y} \mathrm{Tr}\left\{ S[y,x;U,0]\Gamma_{V,1}(x,\frac{\vec{B}}{2}) S[x,y;U,B]\Gamma_{V,2}(y,\frac{\vec{B}}{2})\right\}$
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$$\Pi(0) = \frac{1}{(TL^3)^2}\sum_{x,y}\langle \quad \mathrm{Tr}\left[S\Gamma_V^1 \frac{\partial^2 S}{\partial p_1 \partial p_2}\Gamma_V^2\right] - \frac{1}{4}\mathrm{Tr}\left[S\Gamma_T^1 S \Gamma_T^2\right]$$
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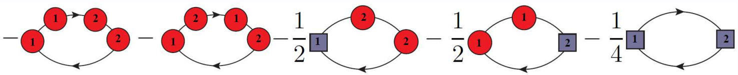

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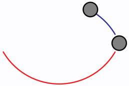

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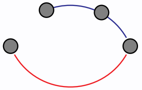

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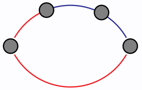

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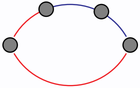
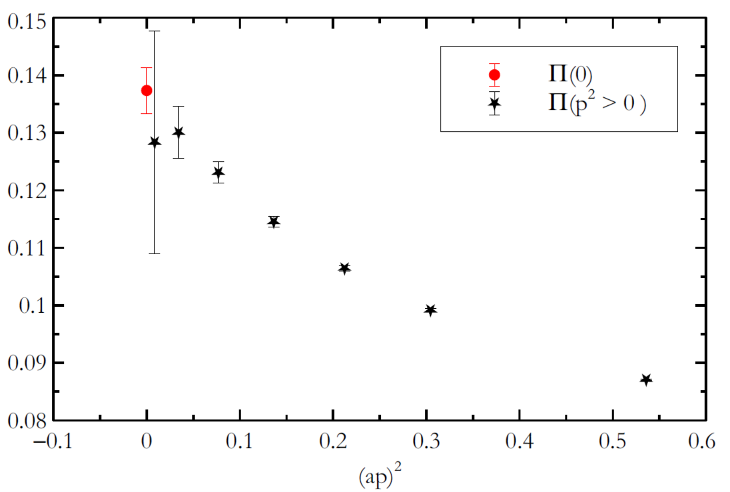


- at the cost of a handful of additional propagator-inversions get $\Pi(0)$
- I would naively have assumed that this produced only noise

How to compute the 2nd derivative of the VP tensor?



A bag full of tricks:

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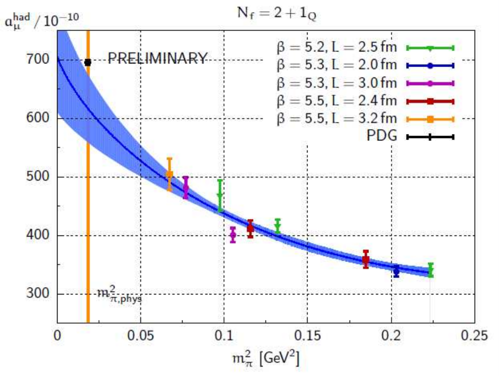

Mainz result

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    - for choice $h = \frac{m_\rho^2}{m_V^2(m_\pi)}$ the dependence of the pion mass cancels

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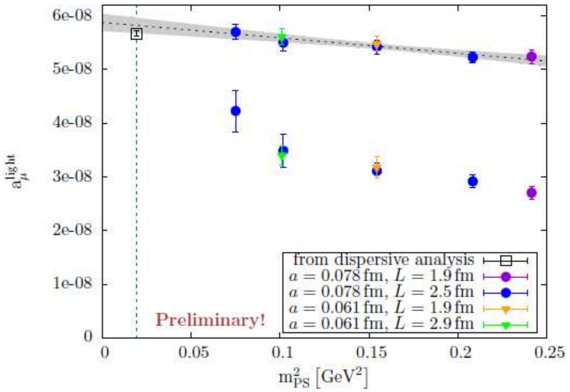

Hotzel (ETM), talk at Lattice 2012

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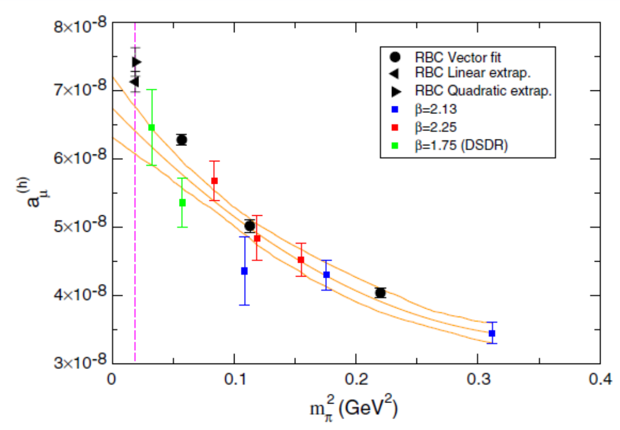

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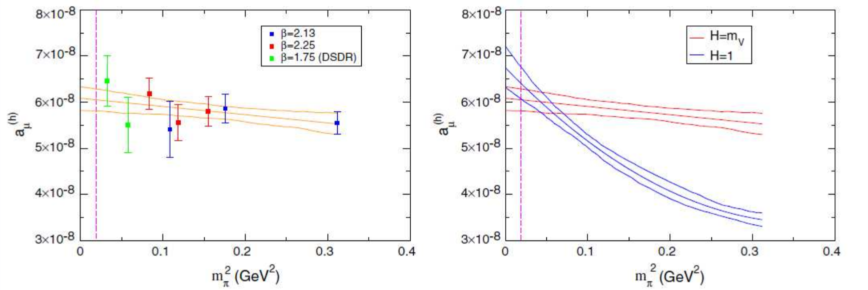

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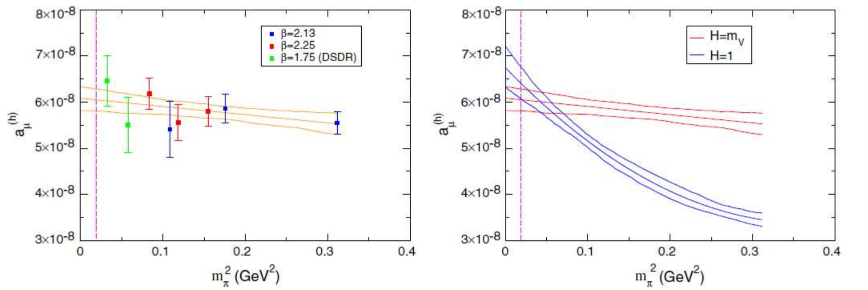

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still need to be careful since vector dominance is a model, extrapolation might still contain structure besides linear term

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A closer look at the vector 2pt function

elm. current $j_\mu(x) \equiv \frac{2}{3} j_\mu^{uu}(x) - \frac{1}{3} j_\mu^{dd}(x) - \frac{1}{3} j_\mu^{ss}(x)$

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$\Pi_{\mu\nu}(q) =$ μ  ν

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Della Morte, AJ JHEP 1011 (2010) 154

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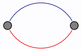

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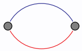

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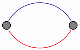

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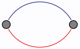

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adding a valence quark

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- the partition function provides for a field theoretic frame work that allows to define observables containing r- and $\bar{r}$-valence quarks
- a low-energy effective theory can be defined; idea: find expression for disconnected contribution in chiral effective theory

Graded flavour symmetry

describe VP (full, conn, disc) correlators in $SU(4|1)$ partially quenched chiral perturbation theory

Gasser & Leutwyler Ann. Phys. 158 (1984), Nucl. Phys. B250 (1985), Sharpe & Shoresh Phys. Rev. D62 (2000) Bernard & Golterman Phys. Rev. D46 (1992)

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$$SU(N_s + N_v|N_v)_L \times SU(N_s + N_v|N_v)_R$$

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- going to the effective theory the elements of $SU(N_s + N_v|N_v)$ have the form

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A contains $(N_s + N_v) \times (N_s + N_v)$ commuting elements, D contains $N_v \times N_v$ commuting elements and B and C are made of anti-commuting numbers

Graded flavour symmetry

- the flavour-trace is replaced by the super-trace
$$STr(U) = Tr(A) - Tr(D)$$
- the graded Lie-algebra takes the form
$$T_a T_b - (-1)^{\eta_a \eta_b} T_b T_a = i \sum_c C_{ab}^c T_c$$
where the *grading* $\eta_a = 1$ if T_a mixes the valence or sea sector with the ghost sector (it's 0 otherwise)
- for $SU(4|1)$ there are 24 generators

$SU(4|1)$ chiral Lagrangian vor VP

$$\mathcal{L}^{(2)} = \frac{F^2}{4}\mathrm{STr}\left\{D_\mu U^\dagger D^\mu U\right\} - \frac{BF^2}{2}\mathrm{STr}\left\{MU^\dagger + M^\dagger U\right\}$$

$$U = e^{-i\frac{2\Phi}{F}} \qquad D_\mu U = \partial_\mu U + iv_\mu U - iUv_\mu$$

- $\mathcal{L}^{(2)}$ - maintains original form (Tr $\rightarrow$ STr)

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- $\mathcal{L}^{(2)}$ - maintains original form (Tr $\rightarrow$ STr)
- $\mathcal{L}^{(4)}$ - needs to be modified

$$\mathcal{L}^{(4)} = \bar{L}\,\mathrm{Str}\left\{\hat{v}_{\mu\nu}\hat{v}_{\mu\nu}\right\} + H_s\,\mathrm{Str}\left\{v_{\mu\nu}\right\}\mathrm{Str}\left\{v_{\mu\nu}\right\}$$

  - $\bar{L}$ is conventional $SU(N)$ low-energy coupling
  - $v_{\mu\nu}$ is field strength of external vector current source
  - H_s needed for $T_0 \propto 1$ source

PQχPT

- express $\langle j^{ss} j^{rr} \rangle$ in terms of the $SU(4|1)$ generators:

$$\psi = (u, d, s, r, \tilde{r})^T$$

$$j^{rr}_\mu(x) = \sqrt{\tfrac{2}{3}}\bar{\psi}(x)\gamma_\mu\left(T^0 - \tfrac{1}{2}T^{24} - \tfrac{3}{2}T^{15}\right)\psi(x)$$

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PQχPT

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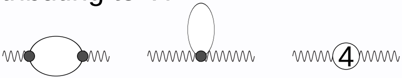
- construction of corresponding currents in the effective theory
- diagrams contributing to VP



apart from some peculiarities due to the graded symmetry group the computation of the vacuum polarisation in the effective theory is straight forward

Predictions

- result for $N_f = 2+1$ Della Morte, AJ JHEP 1011 (2010) 154 :

$$\Pi^{(4|1)}_{\rm Full}(q^2) = \bar{L} - 4i\Big(\bar{B}_{21}(q^2, M_\pi^2) \quad + \bar{B}_{21}(q^2, M_K^2)\Big)$$

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- conn and disc receive unphysical contribution from a “meson” containing two strange quarks which cancel in the sum
- absence of disconnected diagrams in $SU(3)$-limit is reproduced
- $SU(3)$-symmetry does not allow for LECs to contribute to the disconnected diagram $\rightarrow$ parameter-free prediction

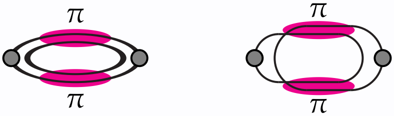
Interpretation

- quark level

Interpretation

- quark level

- intuitive picture for effective degrees of freedom in terms of *quark flow diagrams*



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i.e. only the difference $\hat{\Pi}(q^2) \equiv \Pi(q^2) - \Pi(0)$ is relevant for us

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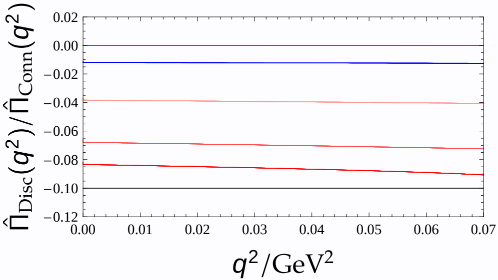
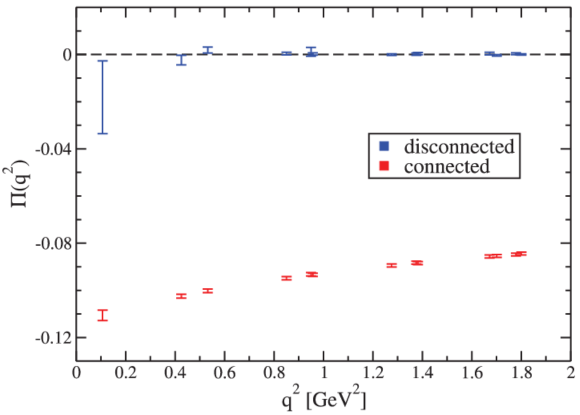


Discussion

- argument shows that disconnected contribution is indeed a small but non-negligible
- relying on this NLO prediction one way to proceed is to to compute the connected diagram in lattice QCD and to predict the (subdominant) disconnected diagram in the eff. theory
- alternatively we can estimate how precisely we would like to know the quark-disconnected contribution
- vector-d.o.f.s may dominate and modify the expressions found

Discussion

There are some first numerical results for the quark-disconnected contribution by ETM Renner Lat10 :



$(m_\pi = 420\text{MeV})$

turn out to be small also numerically, needs to be studied in more detail

Discussion

Comments:

- quark-disconnected diagrams can be predicted for arbitrary hadronic n-point functions for which an effective theory description exists (in particular all possible 2-point and 3-point mesonic and baryonic correlators)

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- applicable to many interesting physical processes and nice example for lattice QCD $\leftrightarrow$ effective field theory beyond just extrapolating data
- expected to work well
  - for observables without vector contributions
  - for small goldstone boson mass; predictions should improve as $m_\pi \to m_\pi^{\text{phys}}$ (where computation of disc. diagrams becomes more and more expensive)

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- $\to$ disconnected contribution to the pion's scalar form factor

AJ, JHEP 1201 (2012) 007

Status of $g-2$

- leading hadronic VP previous lattice efforts compared to $e^+e^- \to$ hadrons

Benayoun, Eur. Phys. J. C (2012) 72:1848	$688(05)\times 10^{-10}$		

lattice QCD:

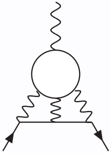
ETM Lattice 2012	$583(13)\times 10^{-10}$	$N_f = 2+1+1$	u,d
ETM Lattice 2012	$647(13)\times 10^{-10}$	$N_f = 2+1+1$	u,d,s
ETM Lattice 2012	$667(14)\times 10^{-10}$	$N_f = 2+1+1$	u,d,s,c
UKQCD Phys.Rev. D85 (2012) 074504	$641(46)\times 10^{-10}$	$N_f = 2+1$	u,d,s
Mainz JHEP 1203 (2012) 055	$618(64)\times 10^{-10}$	$N_f = 2$	u,d,s
Mainz JHEP 1203 (2012) 055	$546(66)\times 10^{-10}$	$N_f = 2$	u,d
ETM PRL 107, 081802 (2011)	$572(16)\times 10^{-10}$	$N_f = 2$	u,d

- increased recent interest in lattice computations
- in particular Feng, Jansen, Petschlies, Renner have reached surprisingly high precision indicating that lattice QCD can have decisive impact

Outline

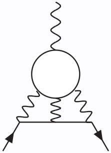
- setting the scene
- leading contribution
  - status
  - some preparatory calculus
  - current lattice calculations
  - recent developments
- light-by-light
  - challenges
  - current directions

light-by-light



- current model estimates $\approx 25 - 30\%$ uncertainty
- for now this is not about precision but about feasibility

light-by-light



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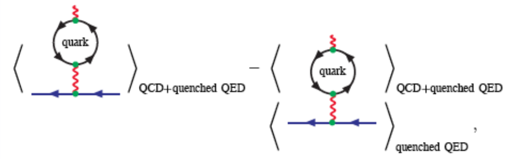
- one lattice method: brute force 4pt function computation and pert. treatment of QED *QCDSF, Rakow*

light-by-light

or *Hayakawa, et al. hep-lat/0509016; Blum et al. PoS LATTICE2008:251,2008, Blum Lattice 2012*



light-by-light

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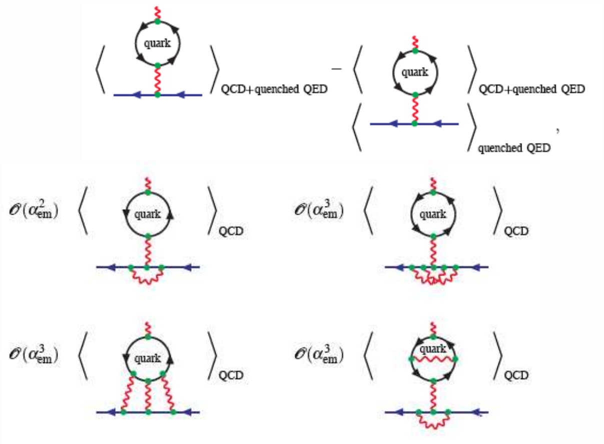

light-by-light

- Blum et al. are now getting a signal ($\approx$ 15% stat error) for unphysical parameters (heavy muon, heavy quark) and are now studying systematics
  for now strong parameter-dependence but optimistic that results will lie in the expected ballpark
- further diagrams involving quark-disconnected diagrams need to be studied
- this is really exciting (Blum's plenary at Lattice 2012) but so far nothing published

Summary

- in particular in view of new results coming in from the LHC experiments the anomalous magnetic moment of the muon remains a phenomenologically highly relevant quantity
- lattice determinations can potentially play an important role in producing a solid SM prediction
- recent surge of interest has produced a long list of new ideas and we can expect real progress in the near future
    - twisted boundary conditions for better momentum resolution
    - theory for model-independent fit
    - quark-disconnected diagrams in effective theory
    - computation of $\Pi(0)$
    - improved mass-extrapolations / physical point simulations
    - ...
- so far no published results for light-by-light but it will hopefully not take too long
- relatively young field, so plenty of things to discover, develop and do...