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MC methods in pricing and risk assessment

February 20, 2013

Gabor Molnar-Saska
Morgan Stanley

Session outline

- What is pricing
- Pricing by replication
- Binomial tree
- Risk-Neutral pricing
- Monte Carlo simulation
- Risks
- An example in commodity
- American type options
- LSM algorithm and the bias correction

What is and what isn't derivative pricing

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The fair price of an instrument is the price

- which rules out arbitrage
- allows a static or dynamic replication portfolio with the same payoff
- is an inter- / extrapolation of market prices of related products.

Assumptions

- No transaction costs
- We can borrow and lend arbitrary amounts of money at the same rate
- Underlying products are infinitely divisible (e.g. we can buy $\sin(e/\pi)$ number of shares)
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$\Rightarrow$ Owning an instrument or owning its current market value in cash is equivalent in value.

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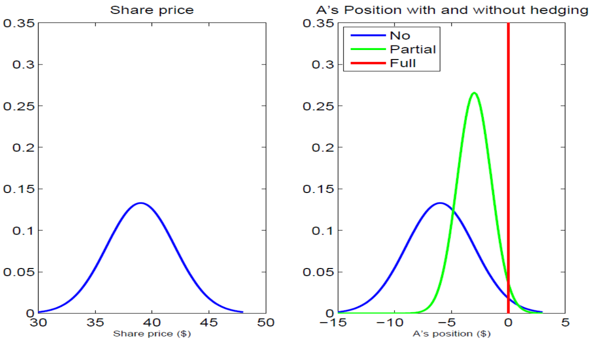
Eliminating risk by hedging

1. Enter into the same (similar) contract from the other side
   - Common practice, market makers charge some fee on top of the fair price.
2. Perform the opposite replication strategy
   - Hedging may be partial



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- Present value of 45 cents paid in 6 months time is ~ 42.86 cents

Call option – binomial model

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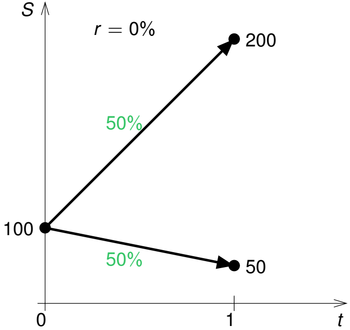

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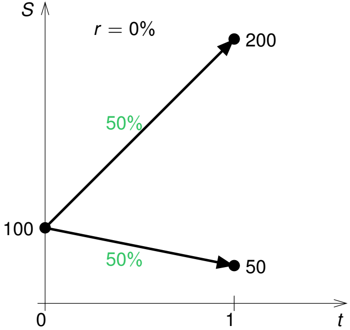

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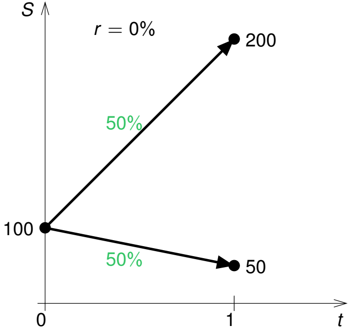

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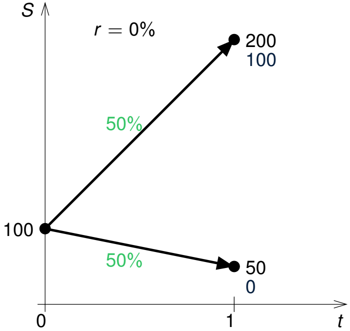

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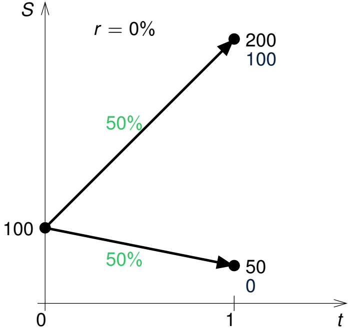

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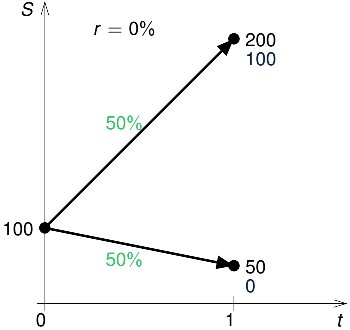

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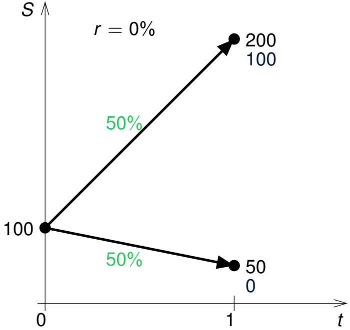

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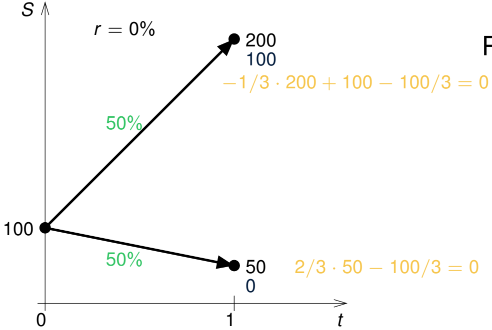

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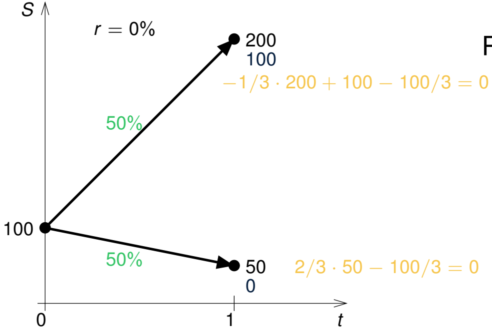

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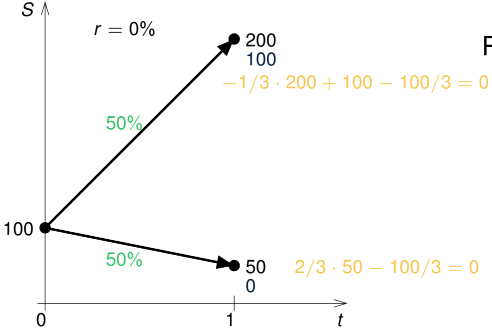

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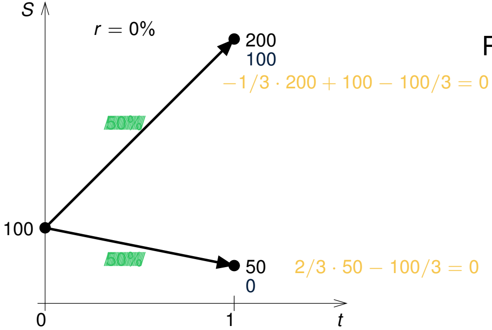

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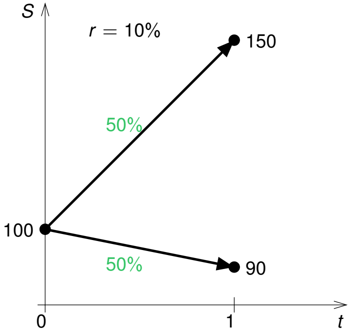
  - We get:

$$V_0 = \frac{S_0 - S_2}{S_1 - S_2} V_1 + \frac{S_1 - S_0}{S_1 - S_2} V_2$$

    This is an expectation under a new measure!

Risk neutral measure



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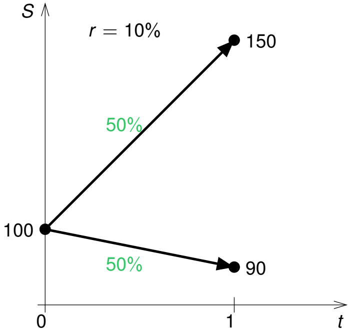

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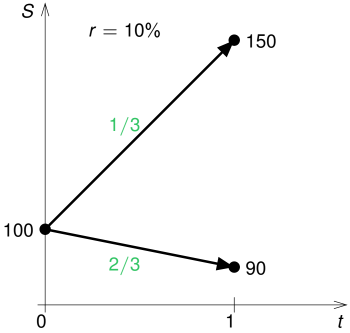

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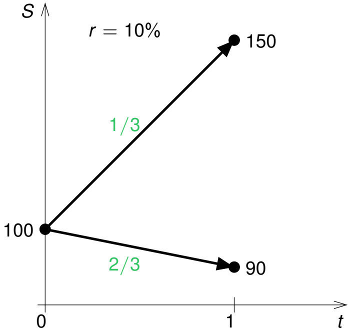

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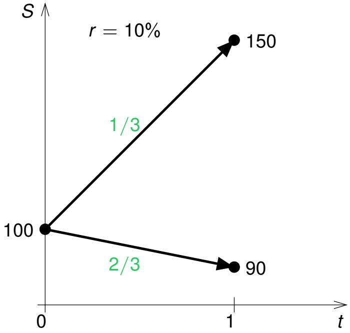

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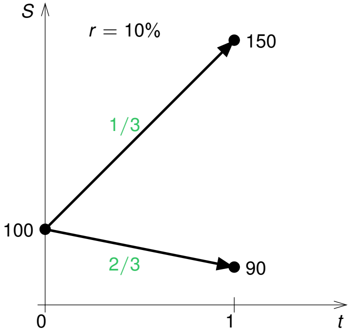

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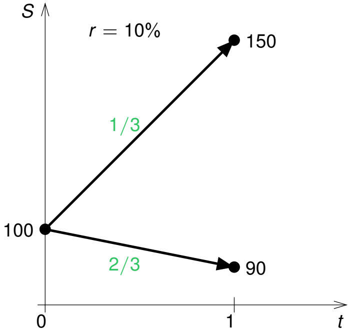

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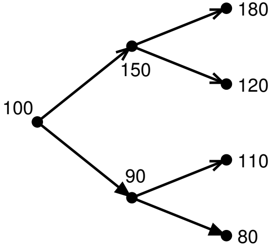
Derivative payoff is replicated by a combination of stock and cash.
Under risk-neutral measure both grow in average at the risk-free rate, so their combination, which is the option value also grows at the risk-free rate.

Multi-period binomial model

What is the price of a call option (strike = 100) expiring at $T = 2$?
($r = 10\%$)



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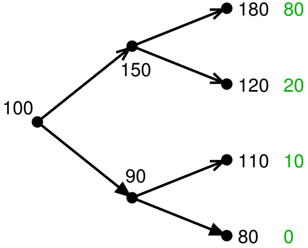

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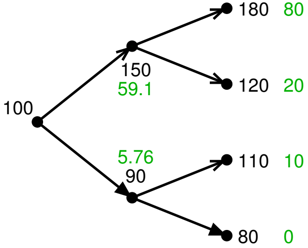

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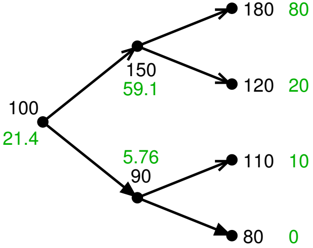

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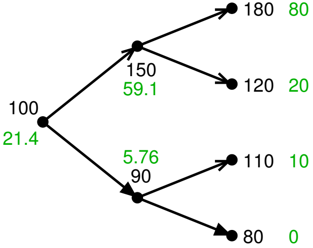

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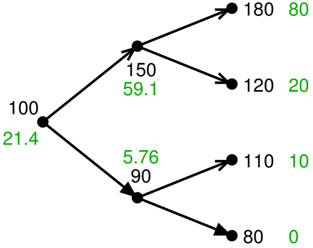

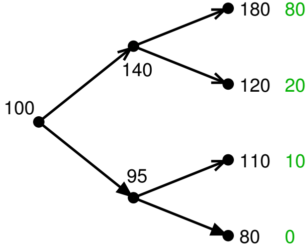

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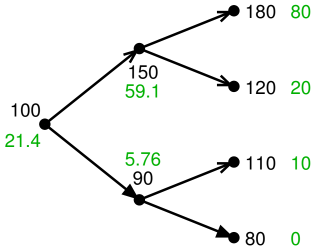

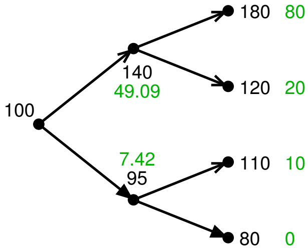

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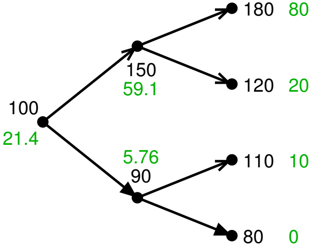

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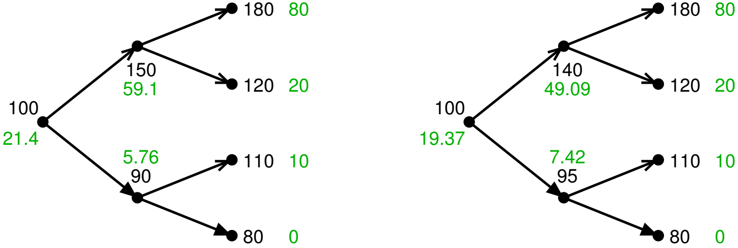

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- At $t = 1$ replication portfolio has to be rebalanced based on outcome.
- Portfolio is self-financing: no need to add money/no money is gained at $t = 1$.
- Price depends on set of available paths

Using $V(0)\mathbb{E}[V(T)]D(0,T)$ to price derivatives

Pricing a forward

$$V_{\mathrm{FW}}(0)=\mathbb{E}[S(T)-K]D(0,T)=\mathbb{E}[S(T)]D(0,T)-KD(0,T)=S(0)-KD(0,T)$$

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- Need to have some assumptions, a MODEL $\Rightarrow$Model for the possible trajectories!

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Continuous-time models

$$\begin{aligned} B_t &= \exp(rt) \\ dS_t &= S_t\mu(t, S_t)dt + S_t\sigma(t, S_t)dW_t \end{aligned}$$

where r is the riskless interest rate, σ is the volatility, and μ is the drift of the stock. Both instruments are freely and instantaneously tradable either long or short at the price quoted.

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We need MONTE CARLO simulation!

Morgan Stanley

Monte Carlo simulation

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- Generate Monte Carlo paths for the underlying

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But with the same random source.

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Can we avoid regenerating and reevaluating paths to compute partial derivatives?

The answer is yes!

Likelihood ratio weighting

Assume we know the joint distribution function of the underlyings with a density function

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The price is

$$\hat{V} = \frac{1}{N_\omega} \sum_{k=1}^{N_\omega} C(p_1(\omega_k), \ldots, p_L(\omega_k))$$

Likelihood ratio weighting

Since

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- If dynamics is simple (e.g. lognormal), then weights can be computed fast
- We have to evaluate function C only once!

American option

Option pricing

- European option
  - a contract that provides the right but not the obligation to engage in transaction on an asset at a reference price at maturity
- American option
  - a contract that provides the right but not the obligation to engage in transaction on an asset at a reference price any time before or at maturity
- Many American type option in practice
  - Bermudian option
  - Callable bond
  - Loan

Monte Carlo simulation

Stochastic calculus

- Determining the underlying dynamics
  - Black-Scholes dynamics: $dS(t) = r \cdot S(t)dt + \sigma \cdot S(t)dW(t)$
  - the discounted stock price dynamics under the risk neutral measure: $d\tilde{S}(t) = \sigma \cdot \tilde{S}(t)d\tilde{W}(t)$
- No arbitrage asset pricing theory
  - the true value of an asset is the expectation of all future discounted cash flows with respect to the risk neutral measure

Brute-Force algorithm

A natural approach to pricing

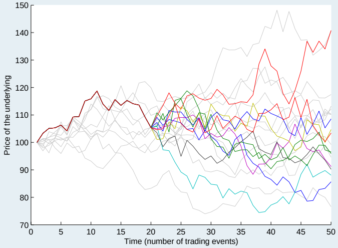

Figure: Brute-Force algorithm

- Pros: simple, flexible and transparent
- Cons: computationally intractable

LSM algorithm

The least squares method by Longstaff and Schwartz (2001)

- Objective
  - determine the exercise policy which maximizes the value of the option
- The algorithm works backwards
  - determines the expected value of continuation by regression
  - regression: *spot price* $(S) \rightarrow$ *cash flow* (y)
  - exercise if intrinsic value is equal or greater than the expected value of continuation
  - average for all ω paths

Least squares regression

Analysis of the regression model

- Using a set of basis functions those form a basis
- Regression value: $\hat{Y} = X\hat{\beta} = X(X^T X)^{-1} Xy = Vy$
- The theoretical value of the regression Y
- Errors are $\epsilon = y - Y \sim N(0, \sigma^2 I)$
- Residuals are $r = y - \hat{Y} \sim N(0, \sigma^2 (I - V))$
- Paths removed from the bulk of the cases are biased

Implications for LSM algorithm

At-the money American option values

LSM algorithm and independent path method respectively

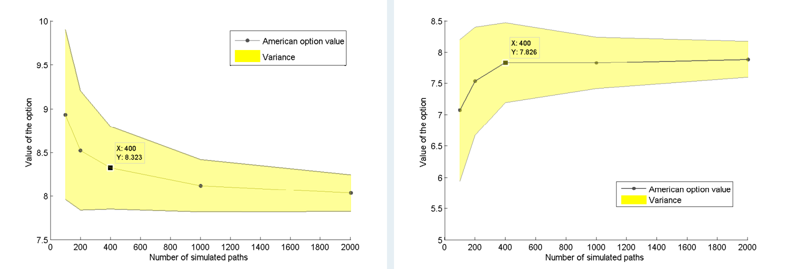

Figure: Biased and unbiased approximations

Theoretical solution

At-the money American option values

Theoretical solution is to increase the number of paths to infinity

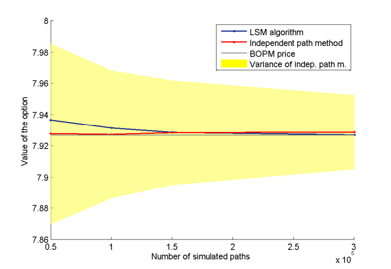

Figure: Convergence of the methods

The true value of the option at a given time t

Fries (2006) suggests to consider:

$$E\left(max((K - S_i)^+, Y_i)|\mathcal{F}(\mathrm{t})\right) = E(max((K - S_i)^+, \hat{Y}_i + e_i)|\mathcal{F}(\mathrm{t}))$$

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Thus the bias of the algorithm equals to:

$$b \doteq \left((K-S_i)^+ - \hat{Y}_i\right) \cdot \Phi\left(\frac{(K-S_i)^+ - \hat{Y}_i}{\delta_i}\right) + \delta_i \cdot \varphi\left(\frac{(K-S_i)^+ - \hat{Y}_i}{\delta_i}\right)$$

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Kovacs - MSG (2012)

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- $\begin{cases} exercise & if\ (K - S_i)^+ \geq \hat{Y}_i + b_i \\ no\ exercise & else \end{cases}$

Test results

At-the-money American option values

The strike is 100 and volatility is 0.2 and 0.4 respectively

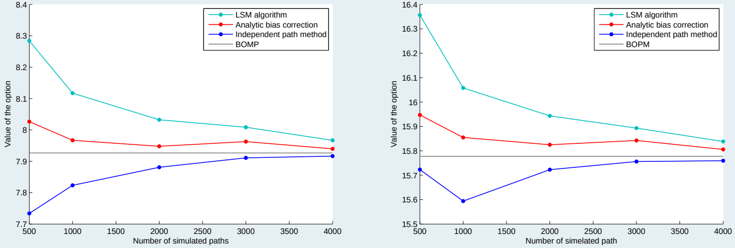

Figure: Comparison of the four introduced methods

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- Questions?